COMBINATORIAL RIGIDITY

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1. Introduction

Caution. This blueprint is part of an ongoing *experiment* in autoformalization using LLMs. The human author (Bryan) has not yet fully vetted the prose or the Lean code, so take everything you read here with a grain of salt.

This blueprint accompanies the `CombinatorialRigidity` Lean 4 / mathlib4 formalization. It begins with **Laman's theorem** [PG27, Lam70]: for $n \geq 2$, a graph G on n vertices is generically rigid in the plane if and only if it has a spanning subgraph H with $2n - 3$ edges in which every set S of at least two vertices spans at most $2|S| - 3$ edges. Such a graph H is a *Laman graph*. In an equivalent form, the theorem says that the Laman graphs are exactly the graphs that are generically minimally rigid in the plane. Pollaczek-Geiringer [PG27] stated and proved it in this form in 1927 (pp. 60 and 64), and Laman [Lam70] proved it again in 1970. We keep the theorem's usual name.

To prove that every Laman graph is generically rigid in the plane, we use the Henneberg moves. The Type I move adds a vertex of degree two, and the Type II move deletes an edge ab and adds a vertex joined to a, b and a third vertex. Every Laman graph is built from K_2 by these moves, and the moves preserve rigidity in the plane. Pollaczek-Geiringer's proof is this induction: it removes a vertex of degree two or three and puts it back by one of the moves (pp. 64–68). Laman's own proof uses the moves implicitly, as Tay and Whiteley [TW85] observe (p. 39). Their §3 makes the argument explicit, and we follow the presentation of Jordán [Jor16] (§2.2). For the converse, a graph on n vertices that is generically rigid in the plane has $2n-3$ edges whose rows in the rigidity map are linearly independent at a suitable framework. A bound on the rank of the rigidity map by the trivial motions shows that these edges form a Laman graph. For background on combinatorial rigidity, see [GSS93, Jor16].

Beyond Laman's theorem, the blueprint proves results on the rigidity matroid in the plane, Lee and Streinu's pebble game for sparse graphs, the union of matroids, and the theorems of Tay on body-bar frameworks and of Tay and Whiteley on body-hinge frameworks. A body-hinge framework in $\mathbb{R}^d$ joins rigid bodies in pairs by hinges, which are affine subspaces of dimension $d - 2$. It is a *panel-hinge* framework if the hinges at each body lie in a common hyperplane, the *panel* of the body. The *molecular conjecture* of Tay and Whiteley [TW84] states that a graph has an infinitesimally rigid body-hinge realization if and only if it has an infinitesimally rigid panel-hinge realization. Katoh and Tanigawa proved it [KT11]. They derive it from their Theorem 5.6: every graph has a panel-hinge realization whose

rigidity matrix has the largest rank of any body-hinge realization of the graph. Following their proof, we prove Theorem 5.6 and the molecular conjecture in every dimension $d \geq 2$ and over every infinite field, for simple graphs and for multigraphs (Theorem 23.56, Theorem 23.70). A formal panel-hinge framework records, for each edge, a pair of bodies whose panels meet in its hinge, and this pair is the ends of the edge (Formalization note 23.6). Theorems 5.5 and 5.6 give each simple graph they cover a realization with nonparallel panels (Definition 23.10), as in Katoh and Tanigawa's Theorem 5.5, so the hinge of every edge is nondegenerate. As formalized, Theorem 5.6 covers every simple graph with at least one vertex, and both sides of Theorem 23.56 ask for a nondegenerate hinge at every edge, its panel side also for nonparallel panels. The statements for multigraphs place each hinge in the panels of the ends of its edge. Phase 41 made the condition on the pair part of the framework and restated these theorems.

From Theorem 5.6, Katoh and Tanigawa derive the *molecule rank formula* (Corollary 26.8), which Jackson and Jordán conjectured. It gives the rank of the square G^2 of a graph G of minimum degree at least two in the generic rigidity matroid in $\mathbb{R}^3$. When G is the graph of the atoms and bonds of a molecule, G^2 is the graph of its bar-joint model. Jackson and Jordán derive from the formula Jacobs' conjecture on the independence of squares of graphs. A later chapter proves this conjecture and their degree-one rank formula (Section 27). Two later chapters (Section 29, Section 30) prove a strengthening of Katoh–Tanigawa's Theorem 5.6 in dimension three that appears to be new, the pencil realization theorem, which this project posed and proved. Its realizations ask in addition that the hinges at each body pass through a common point of the body's panel.

Organization of the blueprint. The Lean development is organized into phases, tracked in `ROADMAP.md`, which also lists the Lean files of each phase. Phases 1–26 form four arcs:

- **Laman's theorem** (phases 1–6): sparse graphs, Laman graphs, Henneberg moves, frameworks, and the two directions of the theorem.
- **The planar rigidity matroid and the pebble game** (phases 7–11): the rigidity matroid in the plane, as a count matroid and as a linear matroid, and the (k, ℓ)-pebble game, a computable test for sparsity that returns a certificate with its answer.
- **Body-bar and body-hinge rigidity** (phases 12–16): matroid union, Tay's theorem on body-bar frameworks, and the theorem of Tay and Whiteley on body-hinge frameworks.
- **The molecular conjecture** (phases 17–26): Katoh and Tanigawa's proof of the molecular conjecture [KT11], cited as KT below, and the rank formula for squares of graphs that they derive from it.

The phases of the four arcs, with the chapters they contribute to:

(1) **Sparsity.** (k, ℓ)-sparse and (k, ℓ)-tight graphs, for all k and ℓ (Section 2, Section 3).
(2) **Laman graphs.** The case $(k, \ell) = (2, 3)$, K_2 as the only Laman graph on two vertices, and the degrees of the vertices of a Laman graph (Section 4).

(3) **Henneberg moves.** The two moves, which preserve the Laman property, and the decomposition of every Laman graph on at least three vertices by a reverse move (Section 5).
(4) **Frameworks.** The rigidity map, and infinitesimal and generic rigidity (Section 6).
(5) **Laman's theorem (⇐).** The Henneberg moves preserve rigidity in the plane, so every Laman graph is generically rigid there (Section 7, Section 9).
(6) **Laman's theorem (⇒).** The trivial motions bound the rank of the rigidity map, and the bound shows that a generically rigid graph in the plane has a spanning Laman subgraph (Section 8, Section 9).
(7) **Row independence and the count matroid.** Every $(2,3)$-sparse edge set is row-independent at some framework in the plane, which completes Pollaczek-Geiringer's and Laman's characterization of the planar rigidity matroid; and the (k,ℓ)-count matroid of Lorea and of White and Whiteley, for $\ell < 2k$ (Section 10, Section 11).
(8) **Linear-matroid framing.** The planar rigidity matroid as the rigidity matroid of one framework of the complete graph (Section 10.4).
(9) **Pebble game.** Lee and Streinu's (k,ℓ)-pebble game for $\ell < 2k$, which generalizes Jacobs and Hendrickson's: on a simple graph it accepts every edge if and only if the graph is (k,ℓ)-sparse (Section 12, Section 13).
(10) **Executable pebble game.** A computable form of the game, procedures that decide (k,ℓ)-sparsity, (k,ℓ)-tightness and the Laman property, and a command-line program that decides whether a graph read from a file is Laman (Section 14).
(11) **Witness extraction.** The game returns a certificate with its answer: the orientation it builds when it accepts, and otherwise a set V' of vertices spanning more than $k|V'| - \ell$ edges. The command-line program prints it (Section 13.7, Section 14.6).
(12) **Matroid union.** The matroid of a submodular function (Edmonds and Rota), Rado's theorem, and the union of matroids with its rank (Nash-Williams and Edmonds) and Edmonds' partition theorem, after Oxley; the formal proofs are ported from Peter Nelson's matroid package for Lean (Section 15).
(13) **Tree packing.** The theorem of Tutte and Nash-Williams: a multigraph is the union of k disjoint forests if and only if it is (k,k)-sparse (Section 16.1, Section 16.3).
(14) **The k-frame.** Whiteley's Theorem 1: the general k-frame of a multigraph, a matrix of indeterminates, represents the union of k copies of its cycle matroid (Section 16.2).
(15) **Tay's theorem.** Let $D = \binom{d+1}{2}$. A multigraph has an independent body-bar framework in $\mathbb{R}^d$ if and only if it is (D,D)-sparse, and an independent and infinitesimally rigid one if and only if it is (D,D)-tight (Section 16.4, Section 16.5).
(16) **Body-hinge frameworks.** The theorem of Tay and Whiteley, proved with each hinge replaced by $D-1$ bars: a multigraph G has an independent and

infinitesimally rigid such framework in $\mathbb{R}^d$ if and only if $(D-1)\cdot G$, with each edge repeated $D-1$ times, is (D,D)-tight (Section 17).

(17) **Extensors.** Homogeneous coordinates, extensors and their join, and KT Lemma 2.1: the extensors of the D sets of $d-1$ of $d+1$ affinely independent points are linearly independent (Section 18).

(18) **The panel-hinge rigidity matrix.** The rigidity matrix $R(G,p)$ of a body-hinge framework, built from the extensors of its hinges (KT §§2.2–2.4), and KT Lemmas 5.1–5.3 on its rank (Section 19).

(19) **Deficiency and k-dof-graphs.** The matroid $M(\tilde{G})$ of the multigraph $\tilde{G} = (D-1)\cdot G$, the deficiency $\mathrm{def}(\tilde{G})$, k-dof-graphs and rigid subgraphs; KT eq. (2.4), which relates the deficiency to the rank of $M(\tilde{G})$; and the lemmas of KT §3 (Section 20).

(20) **Combinatorial induction.** Splitting off, edge-splitting and the contraction of rigid subgraphs, the lemmas of KT §4, and KT Theorem 4.9: every minimal 0-dof-graph with at least two vertices reduces to the two-vertex double edge (Section 21).

(21) **Algebraic induction: the first cases.** The Grassmann–Cayley meet (Section 22), panel-hinge frameworks, and the base and Cases I and II of KT Theorem 5.5, whose proof is an induction over minimal k-dof-graphs of every deficiency k (Section 23).

(22) **Cases I and III.** KT Claim 6.4 of Case I, KT Claim 6.11 of Case III, and KT Lemma 6.10 with KT Claim 6.12, the case $d = 3$ of Case III (Section 23.6, Section 23.10).

(23) **Case III in general dimension.** KT Lemma 6.13, which completes KT Theorem 5.5 for $d \geq 3$. With it come KT Theorem 5.6 and the molecular conjecture for simple graphs (Theorem 23.56). The realizations of KT Theorem 5.6 attain the rank formula of Jackson and Jordán (KT Proposition 2.3).

(24) **The generic bar-joint rigidity matroid.** The generic rigidity matroid $\mathcal{R}_d(V)$ in every dimension d, as the rigidity matroid of the complete graph at a placement that is generic for row independence, and its rank function (Section 24).

(25) **Projective duality and the molecule modelling equivalence.** Projective invariance of infinitesimal rigidity (Crapo and Whiteley), under a polarity that turns molecular frameworks into panel-hinge frameworks, and Whiteley's equivalence between molecular frameworks on a graph G and bar-joint frameworks on its square G^2, both as equalities of dimensions of motion spaces (Section 25).

(26) **The molecule rank formula.** KT Corollary 5.7, which Jackson and Jordán conjectured: for a graph G of minimum degree at least two, $r(G^2) = 3|V| - 6 - \mathrm{def}(\tilde{G})$, where r is the rank function of the generic rigidity matroid in $\mathbb{R}^3$ (Section 26).

Beyond the four arcs, a continuation (phase 32) proves two results of Jackson and Jordán [JJ08a] on squares of graphs (Section 27). The first is Jacobs' conjecture: the square of a graph is independent in the generic rigidity matroid in $\mathbb{R}^3$ if and only if it satisfies the three-dimensional Laman condition of Jackson and Jordán. They

derived it from the molecule rank formula, so the conjecture holds. The second is their degree-one rank formula, for connected graphs with vertices of degree one, to which the molecule rank formula does not apply. It gives the rank of the square of a tree explicitly, a case that Jackson and Jordán credit to Franzblau. For any other such graph it gives the rank of the square through that of the square of the core, the maximal subgraph of minimum degree at least two.

A second continuation (phase 33) proves KT Theorems 5.5 and 5.6 and the molecular conjecture over every infinite field K, of any characteristic. Katoh and Tanigawa's statements over $\mathbb{R}$ are the case $K = \mathbb{R}$. For these theorems the generality appears to be new. For body-bar frameworks there is a precedent: Whiteley states the genericity step of his proof of Tay's theorem over every infinite field [Whi88]. The opening of Section 23 discusses this generality. The molecule rank formula, a statement about the generic rigidity matroid in $\mathbb{R}^3$, stays over $\mathbb{R}$.

A third continuation (phase 34) proves the realization theorems above in generic form, by the coordinate argument of Jackson and Jordán [JJ10] (Section 28). Each theorem holds at every generic realization, and the realizations that are not generic lie in the zero set of a nonzero polynomial. The theorems are the molecule rank formula, at generic placements; Tay's theorem, at generic endpoints of the bars; and, for a simple graph, the rank formula of KT Proposition 2.3, at generic normals of the panels and at generic points spanning the hinges. The last gives the theorem of Tay and Whiteley for frameworks with hinges, which phase 16 proves for bundles of bars. As restated in phase 41, the form at generic normals of the panels holds for every simple graph with at least one vertex, and the form at generic points spanning the hinges for every multigraph with at least one vertex, parallel edges and loops allowed. So they cover the simple minimal k-dof-graphs, such as a single edge, a tree or a triangle, and the form at hinge points also covers the double edge. The form at hinge points holds, too, at every framework that is generic in Katoh and Tanigawa's own sense, whose rigidity matrix and each of its edge-induced submatrices have the largest rank over all realizations of the graph (Definition 19.16); every framework at generic points spanning the hinges is one.

A fourth continuation (phase 35) proves KT Theorem 5.6 and the molecular conjecture for multigraphs, as Katoh and Tanigawa state them (Section 23.5). It works in their own model, the hinge-coplanar model, in which each hinge need only lie in the panels of the two bodies it joins. In the panel-hinge frameworks of the main development, a hinge is the intersection of the two panels. For these frameworks the multigraph form of the molecular conjecture fails.

A fifth continuation (phases 39–40) strengthens the multigraph form of Theorem 5.6 in the hinge-coplanar model, in dimension three. A *pencil realization* gives each body of a hinge-coplanar realization a point of its panel, through which every hinge at the body passes (Definition 29.2). The hinges at a body then form a pencil of lines through a point in a plane. In the molecular reading, the panel is the plane of an atom's bonds and the point is the atom's centre. Over every infinite field, every spanning multigraph G has a pencil realization of rank $6(|V(G)|-1)-\mathrm{def}(\tilde{G})$, the deficiency rank of Theorem 23.66. If G is simple, the points of adjacent bodies

can be chosen projectively distinct. If G also has a nondegenerate pencil realization (Definition 29.23), it has one of that rank. These claims make up the pencil realization theorem (Theorem 29.34).

Katoh and Tanigawa's induction runs over minimal graphs (KT Theorem 4.9). They then add the remaining edges to a realization of a minimal spanning subgraph. In a pencil realization of the subgraph at its deficiency rank, the added edges may have no hinges in both panels through the points of their ends. So the proof runs the induction over every multigraph (Theorem 29.15). Phase 39 reduced the pencil realization theorem to the two *main-component statements* (Definition 29.31). These ask the pencil realization theorem's last two claims of every simple two-edge-connected multigraph on at least three bodies.

Phase 40 proved the two statements (Section 30). The idea is to build pencil realizations by lifting pictures of the graph in the plane. A picture places each body v at a point $q_v = (x_v, y_v)$ of the plane. Lifting that point to a height z_v gives the point $p_v = (x_v, y_v, z_v, 1)$ of K^4. The hinge of an edge uv is the line through p_u and p_v. The configurations at which the points of every closed neighbourhood lie in a plane form the *pencil configuration space*. Over an admissible picture q (Definition 30.1), each of them is a pencil realization. Their heights form a linear space $L(q)$, the lifting space (Definition 30.2). The *main component* is the closure of the part of the pencil configuration space over the main pictures, the admissible pictures at which $\dim L(q)$ is least (Remark 30.19).

For every simple connected graph of minimum degree at least two, a general configuration in the main component has the deficiency rank (Theorem 30.139). The proof is a strong induction on the number of bodies. There are steps at cut vertices, bridges, cycles and ears, and steps by splitting off and by contraction. A case analysis, the structural coverage, shows that some step applies to every such graph (Theorem 30.124). Over an admissible picture adjacent bodies have distinct points, so the first statement follows (Lemma 30.125). The second does not follow in this way. At $K_{2,3}$, no configuration over a general picture is nondegenerate (Lemma 30.127). The second statement is proved instead inside the induction over every multigraph (Theorem 30.137).

The first step of phase 40 extended Theorems 5.5 and 5.6 and the molecular conjecture to dimension two. It also proved a form of Theorem 5.6 for simple graphs that need not span the body set (Theorem 23.54). In the plane a hinge is a pin joining two bodies, and a panel is a line. There the molecular conjecture is a theorem of Jackson and Jordán on pin-collinear body-and-pin frameworks [JJ08b], proved for real frameworks. Here it holds over every infinite field. The main-component argument uses Jackson and Jordán's rank formula for these frameworks, as an equality for the dimension of the lifting space at a general picture. We derive that equality from the planar case of the non-spanning form of Theorem 5.6, so that it too holds over every infinite field (Section 30.3).

Reading this blueprint. In the web version, a statement that is formalized in Lean is marked with a tick, and a button in its heading links to the documentation of its Lean declarations. The *dependency graph*, linked from the navigation bar, shows how the statements depend on each other. A node has a green border when its statement

is formalized, and a green interior when its proof is; the graph's legend explains the other markings. Every node of the graph is green, from Laman's theorem to the pencil realization theorem (Section 29, Section 30): each definition and result the chapters state as a node is formalized. Phase 41 ended a review of the chapters against the Lean, which corrected the statements that claimed more or less than their Lean declarations and gave a Lean declaration to each claim the chapters made without one. From phase 6 on, the graph is written forward: a chapter's nodes and their edges come first, and the statement and proof sketch of a node not yet formalized are the plan for its Lean declaration. So the graph is also the project's list of work to do, and its legend marks the nodes whose prerequisites are all formalized. Outside this introduction and the appendix (Section A), the chapters state the mathematics, not the progress of its formalization.

The proofs give the argument as a paper would, and the Lean proofs behind the links give every detail. The chapters on the molecular conjecture give in full the steps that Katoh and Tanigawa's paper states compactly, so that each of them can be followed without the paper. Among these steps are the cases of KT Theorem 5.5, of which Katoh and Tanigawa call Case III the most difficult (KT p. 650). Where the Lean statement of a node differs from the statement in the text, for instance in how it encodes a construction, a formalization note after the node explains the difference.

An appendix (Section A) records some of the wrong turns that the formalization of KT Theorem 5.5 took: a statement whose formal strength differed from the source's, a decomposition of a proof that did not match the structure the final argument needed, and a decision left with no record of the obligation it created. It stands outside the mathematical development and can be read on its own.

2. Edges in a vertex set

Laman graphs are defined by counting edges: a graph on $n \geq 2$ vertices is a Laman graph if it has $2n-3$ edges and every set of $m \geq 2$ of its vertices spans at most $2m-3$ of them. This section and the next develop this counting with general constants k and ℓ in place of 2 and 3, and Section 4 returns to Laman graphs. Throughout both sections, G is a simple graph on a vertex set V, which we assume to be finite only where a statement says so.

This section studies the set of edges of G with both endpoints in a given vertex set. We show that this set grows with the vertex set and compute it at a few small sets. We then count the edges that remain when a vertex is deleted, and finally compare the counts at two vertex sets with the counts at their union and intersection. Section 3 defines (k,ℓ)-sparse and (k,ℓ)-tight graphs and develops their basic theory.

2.1. **Definition and basic properties.**

Definition 2.1 (Edges induced on a vertex set)**.** For $S \subseteq V$, let

$$E_G[S] \coloneqq \{\, e \in E(G) : e \subseteq S \,\}$$

be the set of edges of G with both endpoints in S, that is, the edge set of the subgraph of G induced on S. We say that S *spans* these edges.

Lemma 2.2 (Monotonicity)**.** *Let $S \subseteq T \subseteq V$. Then $E_G[S] \subseteq E_G[T]$.*

Proof. An edge $e \subseteq S$ also satisfies $e \subseteq T$. □

Lemma 2.3 (Corner cases)**.** *Let $v \in V$ and $S \subseteq V$. Then $E_G[V] = E(G)$, $E_G[\emptyset] = \emptyset$ and $E_G[\{v\}] = \emptyset$. If G has no edges, then $E_G[S] = \emptyset$.*

Proof. All but the third are immediate from the definition. A singleton $\{v\}$ contains no edge, since G has no loops and so the two endpoints of an edge are distinct. □

Formalization note 2.4. The four pins are the four claims, in the order stated.

2.2. **Vertex deletion.** Let $v \in V$, and write $\{v\}^c = V \setminus \{v\}$. The set $E_G[\{v\}^c]$ consists of the edges of G that remain when v is deleted. Let $I_G(v) := \{\, e \in E(G) : v \in e \,\}$ be the set of edges of G incident to v. When V is finite, $|I_G(v)|$ is the degree of v. The next two lemmas show that deleting v removes exactly the edges of $I_G(v)$, and count what remains. The count is used in Lemma 4.5 to show that every vertex of a Laman graph on at least three vertices has degree at least two.

Lemma 2.5 (Edges avoiding a vertex)**.** *Let $v \in V$. Then*

$$E_G[\{v\}^c] \;=\; E(G) \setminus I_G(v);$$

that is, a pair e of vertices lies in $E_G[\{v\}^c]$ if and only if $e \in E(G)$ and $v \notin e$.

Proof. An edge has both endpoints in $\{v\}^c$ exactly when neither endpoint is v, that is, when $v \notin e$. So $E_G[\{v\}^c]$ consists of the edges of G not incident to v, which is $E(G) \setminus I_G(v)$. □

Lemma 2.6 (Vertex-deletion partition of edges)**.** *Let V be finite and let $v \in V$. Then*

$$\big|E_G[\{v\}^c]\big| \;+\; |I_G(v)| \;=\; |E(G)|.$$

Proof. By Lemma 2.5, $E(G)$ is the disjoint union of $E_G[\{v\}^c]$ and $I_G(v)$, so the cardinalities add. □

2.3. **Modularity: intersection, union, and supermodular counts.** The map $S \mapsto E_G[S]$ commutes with intersection but not with union. An edge with one endpoint in $S \setminus T$ and the other in $T \setminus S$ lies in $E_G[S \cup T]$ but in neither $E_G[S]$ nor $E_G[T]$. Counting edges therefore gives an inequality (Lemma 2.8). It becomes an identity, Jordán's Lemma 2.1.1 [Jor16], when the number of these crossing edges is added to its left side. We need only the inequality, which is the counting step in the proof that tight sets are closed under union and intersection (Theorem 3.14).

Lemma 2.7 (Distribution over intersection)**.** *Let $S, T \subseteq V$. Then $E_G[S \cap T] = E_G[S] \cap E_G[T]$.*

Proof. An edge has both endpoints in $S \cap T$ if and only if both endpoints lie in S and both lie in T. □

Lemma 2.8 (Supermodularity of edge counts)**.** *Let V be finite and let $S, T \subseteq V$. Then*

$$\big|E_G[S]\big| + \big|E_G[T]\big| \;\le\; \big|E_G[S \cup T]\big| + \big|E_G[S \cap T]\big|.$$

Proof. Let $A = E_G[S]$ and $B = E_G[T]$, so that $|A| + |B| = |A \cup B| + |A \cap B|$. By Lemma 2.7, $A \cap B = E_G[S \cap T]$. By monotonicity (Lemma 2.2), $A \cup B \subseteq E_G[S \cup T]$, so $|A \cup B| \leq |E_G[S \cup T]|$. Together these give the inequality. □

3. (k, ℓ)-sparsity and (k, ℓ)-tightness

Throughout this section k and ℓ are non-negative integers. We define (k, ℓ)-sparse and (k, ℓ)-tight graphs, as Lee and Streinu [LS08] do for multigraphs, and (k, ℓ)-tight sets of vertices. The Laman graphs of Section 4 are the case $(k, \ell) = (2, 3)$, which Jordán's survey [Jor16] treats in §2.1.

After the definitions we show that sparsity passes to subgraphs and bounds the number of edges. Two facts about $(2, 3)$-sparse graphs follow: such a graph has a vertex of degree at most three, and no vertex has three pairwise adjacent neighbours. Sparsity and tightness are invariant under isomorphism. The main result is that two tight sets in a sparse graph have a tight union and a tight intersection if their intersection is large enough (Theorem 3.14). Two variants enlarge a tight set by counting edges that it does not span. Next comes a counting lemma for reversing the Type II Henneberg move (Definition 5.2), that is, for deleting a vertex of degree three and joining two of its neighbours by an edge. The section ends with the reverse decomposition of a $(2, 3)$-sparse graph with an edge: it has a vertex whose deletion leaves a $(2, 3)$-sparse graph, after such an edge is added if the vertex has degree three (Theorem 3.19).

3.1. **Definitions.**

Definition 3.1 ((k, ℓ)-sparse)**.** G is (k, ℓ)-*sparse* if every finite set $s \subseteq V$ with $\ell \leq k\,|s|$ spans at most $k\,|s| - \ell$ edges, that is,

$$\big|E_G[s]\big| + \ell \;\leq\; k\,|s|.$$

Formalization note 3.2. The formal definition states the bound in the additive form displayed above. Edge counts there are natural numbers, in which $k\,|s| - \ell$ would be truncated at 0. Under the hypothesis $\ell \leq k\,|s|$ the two forms agree.

Definition 3.3 ((k, ℓ)-tight)**.** Let V be finite. The graph G is (k, ℓ)-*tight* if it is (k, ℓ)-sparse and

$$|E(G)| + \ell \;=\; k\,|V|.$$

Definition 3.4 (Tight subset)**.** A finite set $s \subseteq V$ is (k, ℓ)-*tight in* G if it attains the sparsity bound:

$$\big|E_G[s]\big| + \ell \;=\; k\,|s|.$$

Tightness of a set is the local form of tightness of the graph. Since $E_G[V] = E(G)$ (Lemma 2.3), a (k, ℓ)-sparse graph G on a finite set V is (k, ℓ)-tight exactly when V is a (k, ℓ)-tight set in G. At $(k, \ell) = (2, 3)$ the hypothesis $\ell \leq k\,|s|$ of Definition 3.1 says that $|s| \geq 2$. So a finite simple graph is $(2, 3)$-sparse exactly when it is sparse in the sense of Jordán [Jor16] §2.1, and its $(2, 3)$-tight sets are what he calls *critical sets*. We use "tight" throughout.

3.2. **Basic properties and the global edge bound.** Sparsity passes to subgraphs. Applied to the whole vertex set, it bounds the number of edges of G, so that a tight graph has as many edges as a sparse graph on V can have.

Lemma 3.5 (Subgraphs of sparse graphs are sparse)**.** *Let G' be a graph on V with $E(G) \subseteq E(G')$. If G' is (k,ℓ)-sparse, then so is G.*

Proof. For a finite set $s \subseteq V$, every edge of G with both endpoints in s is an edge of G' with both endpoints in s, so $|E_G[s]| \leq |E_{G'}[s]|$. The sparsity bound for G' at s therefore bounds $|E_G[s]|$. □

Lemma 3.6 (Global edge bound)**.** *Let V be finite, and let G be (k,ℓ)-sparse with $\ell \leq k\,|V|$. Then*

$$|E(G)| + \ell \;\leq\; k\,|V|.$$

Proof. Apply the sparsity bound at $s = V$ and use $E_G[V] = E(G)$ (Lemma 2.3). □

Lemma 3.7 (A proper supergraph of a tight graph is not sparse)**.** *Let V be finite, let G be (k,ℓ)-tight, and let H be a graph on V with $E(G) \subsetneq E(H)$. Then H is not (k,ℓ)-sparse.*

Proof. By tightness $|E(G)| + \ell = k\,|V|$; in particular $\ell \leq k\,|V|$. Since $E(G)$ is a proper subset of $E(H)$, we have $|E(G)| < |E(H)|$, and so $|E(H)| + \ell > k\,|V|$. By Lemma 3.6, H is not (k,ℓ)-sparse. □

3.3. **Low-degree vertex and non-adjacency consequences.** Both lemmas below concern $(2,3)$-sparse graphs. The first is applied to Laman graphs in Lemma 4.6. The second is used in the reverse decomposition of a $(2,3)$-sparse graph, at a vertex of degree three (Theorem 3.19).

Lemma 3.8 (A vertex of degree at most 3)**.** *Let V be finite with $|V| \geq 2$, and let G be $(2,3)$-sparse. Then some vertex of G has degree at most 3.*

Proof. Suppose that every vertex has degree at least 4. By the handshake identity, $2|E(G)| = \sum_v \deg_G(v) \geq 4|V|$. The global edge bound (Lemma 3.6) applies at $(k,\ell) = (2,3)$, since $3 \leq 2|V|$ when $|V| \geq 2$. It gives $|E(G)| + 3 \leq 2|V|$, so $2|E(G)| \leq 4|V| - 6 < 4|V|$, a contradiction. □

Lemma 3.9 (Non-adjacent pair among three neighbors)**.** *Let V be finite and let G be $(2,3)$-sparse. Suppose that $v \in V$ has three pairwise distinct neighbors a, b, c. Then at least one of the pairs $\{a,b\}$, $\{a,c\}$, $\{b,c\}$ is non-adjacent in G.*

Proof. Suppose that all three pairs are adjacent, and let $X = \{v, a, b, c\}$. The vertices a, b and c are distinct, and each differs from v because G has no loops, so $|X| = 4$. The set X spans the six distinct edges va, vb, vc, ab, ac and bc, so $|E_G[X]| \geq 6$. But $(2,3)$-sparsity applies at X, since $3 \leq 2 \cdot 4$, and gives $|E_G[X]| \leq 2 \cdot 4 - 3 = 5$, a contradiction. □

3.4. **Invariance under isomorphism.** Invariance under isomorphism is evident for properties defined by counting edges. We record it because a Henneberg move (Section 5) produces a graph on a new vertex set, which is identified with a given graph only up to isomorphism.

Lemma 3.10 (Isomorphisms preserve spanned edges)**.** *Let φ be an isomorphism from G to a graph H, and let $s \subseteq V$. Then*

$$\varphi[E_G[s]] = E_H[\varphi(s)],$$

where φ maps a pair $\{u, v\}$ of vertices to $\{\varphi(u), \varphi(v)\}$.

Proof. Since φ preserves adjacency in both directions, a pair $\{p, q\}$ is an edge of G inside s if and only if $\{\varphi(p), \varphi(q)\}$ is an edge of H inside $\varphi(s)$. Every pair of vertices of H inside $\varphi(s)$ is of this form. □

Lemma 3.11 (Sparsity is preserved by isomorphism)**.** *Let φ be an isomorphism from G to a graph H. If G is (k, ℓ)-sparse, then so is H.*

Proof. Let s be a finite set of vertices of H with $\ell \leq k\,|s|$, and let $s' = \varphi^{-1}(s)$, so that $|s'| = |s|$. The map on pairs induced by φ is injective, and it maps $E_G[s']$ onto $E_H[s]$ (Lemma 3.10). So $|E_H[s]| = |E_G[s']|$, and the sparsity bound for G at s' is the bound for H at s. □

Lemma 3.12 (Tightness is preserved by isomorphism)**.** *Let φ be an isomorphism from G to a graph H. If G is (k, ℓ)-tight, then so is H.*

Proof. The graph H is (k, ℓ)-sparse by Lemma 3.11. The isomorphism is a bijection between the vertex sets of G and H, and it induces a bijection from $E(G)$ to $E(H)$. So G and H have the same numbers of vertices and of edges, and the equation $|E(G)| + \ell = k\,|V|$ gives the same equation for H. □

3.5. **Unions and intersections of tight sets.** Each proof in this subsection shows that a set spans at least as many edges as tightness requires. The sparsity bound gives the reverse inequality, so the bound is attained. The first lemma records this step.

Lemma 3.13 (Tightness from a lower bound)**.** *Let G be (k, ℓ)-sparse, and let $s \subseteq V$ be a finite set with $\ell \leq k\,|s|$ and*

$$k\,|s| \;\leq\; \big|E_G[s]\big| + \ell.$$

Then s is (k, ℓ)-tight in G.

Proof. Sparsity at s gives $|E_G[s]| + \ell \leq k\,|s|$, so equality holds. □

At $(k, \ell) = (2, 3)$ the size hypothesis of the next theorem says that $|s \cap t| \geq 2$, and the theorem is the part of Jordán's Lemma 2.1.2 [Jor16] that concerns critical sets. For general k and $\ell < 2k$, see Lee and Streinu [LS08, Theorem 5(1)], who call the subgraph induced on a tight set a *block*.

Theorem 3.14 (Tight subsets are closed under union and intersection)**.** *Let V be finite and let G be (k, ℓ)-sparse. Let $s, t \subseteq V$ be (k, ℓ)-tight in G, and suppose that $\ell \leq k\,|s \cap t|$. Then $s \cup t$ and $s \cap t$ are (k, ℓ)-tight in G.*

Proof. Since $|s \cup t| \geq |s \cap t|$, the hypothesis $\ell \leq k\,|s \cap t|$ lets us apply sparsity at $s \cap t$ and at $s \cup t$:

$$\big|E_G[s \cap t]\big| + \ell \leq k\,|s \cap t|,$$
$$\big|E_G[s \cup t]\big| + \ell \leq k\,|s \cup t|.$$

Adding these and using $|s \cup t| + |s \cap t| = |s| + |t|$ gives

$$\big|E_G[s \cup t]\big| + \big|E_G[s \cap t]\big| + 2\ell \;\leq\; k\,|s| + k\,|t|.$$

On the other hand, s and t are tight, and supermodularity (Lemma 2.8) gives

$$k\,|s| + k\,|t| \;=\; \big|E_G[s]\big| + \big|E_G[t]\big| + 2\ell \;\leq\; \big|E_G[s \cup t]\big| + \big|E_G[s \cap t]\big| + 2\ell.$$

So the sum of the two sparsity inequalities above is an equality. Hence each of them is an equality, that is, $s \cup t$ and $s \cap t$ are tight. □

The next two results need no size hypothesis. The first combines two tight sets s_1 and s_2 and counts edges that join $s_1 \setminus s_2$ to $s_2 \setminus s_1$, which are the edges that the inequality of Lemma 2.8 leaves out. Without such edges, and under the size hypothesis of Theorem 3.14, it gives the union half of that theorem. The second adds a vertex to a tight set together with k edges joining it to the set. The reverse decomposition of a $(2,3)$-sparse graph at a vertex of degree three uses both (Theorem 3.19).

Theorem 3.15 (Union of tight sets joined by edges)**.** *Let V be finite, let G be (k,ℓ)-sparse, and let $s_1, s_2 \subseteq V$ be (k,ℓ)-tight in G. Let F be a set of edges of G inside $s_1 \cup s_2$ that lie in neither $E_G[s_1]$ nor $E_G[s_2]$, and suppose that*

$$\big|E_G[s_1 \cap s_2]\big| + \ell \;\leq\; k\,|s_1 \cap s_2| + |F|.$$

Then $s_1 \cup s_2$ is (k,ℓ)-tight in G.

Proof. The sets $E_G[s_1]$, $E_G[s_2]$ and F lie in $E_G[s_1 \cup s_2]$ (Lemma 2.2), and F is disjoint from the other two. Their intersection is $E_G[s_1] \cap E_G[s_2] = E_G[s_1 \cap s_2]$ (Lemma 2.7). Counting their union as in the proof of Lemma 2.8 gives

$$\big|E_G[s_1]\big| + \big|E_G[s_2]\big| + |F| \;\leq\; \big|E_G[s_1 \cup s_2]\big| + \big|E_G[s_1 \cap s_2]\big|.$$

Since s_1 and s_2 are tight, the left side equals $k\,|s_1| + k\,|s_2| - 2\ell + |F|$. We substitute $|s_1| + |s_2| = |s_1 \cup s_2| + |s_1 \cap s_2|$ and bound $|E_G[s_1 \cap s_2]|$ by the hypothesis on F. This gives

$$k\,|s_1 \cup s_2| \;\leq\; \big|E_G[s_1 \cup s_2]\big| + \ell.$$

Tightness of s_1 also gives $\ell \leq k\,|s_1| \leq k\,|s_1 \cup s_2|$. So $s_1 \cup s_2$ is tight by Lemma 3.13. □

Theorem 3.16 (Adding a vertex with k edges to a tight set)**.** *Let G be (k,ℓ)-sparse, and let $s \subseteq V$ be (k,ℓ)-tight in G. Let $w \in V \setminus s$, and let F be a set of at least k edges of G inside $s \cup \{w\}$ that are not in $E_G[s]$. Then $s \cup \{w\}$ is (k,ℓ)-tight in G.*

Proof. The sets $E_G[s]$ and F are disjoint, and both lie in $E_G[s \cup \{w\}]$ (Lemma 2.2). Since s is tight and $w \notin s$,

$$\big|E_G[s \cup \{w\}]\big| + \ell \;\geq\; \big|E_G[s]\big| + \ell + |F| \;\geq\; k\,|s| + k \;=\; k\,|s \cup \{w\}|.$$

Tightness of s also gives $\ell \leq k\,|s| \leq k\,|s \cup \{w\}|$. So $s \cup \{w\}$ is tight by Lemma 3.13. □

3.6. **Splittings and tight sets.** Let v be a vertex of degree three with neighbours x, y and z, where x and y are not adjacent. Deleting v and joining x to y by an edge reverses the Type II move on xy and z (Definition 5.2). Jordán [Jor16] calls this operation *splitting* v on the pair vx, vy, and calls the splitting *suitable* if the resulting graph is sparse (§2.1, p. 44). His proof of Lemma 2.1.4(b) begins with an observation about a sparse graph: if a splitting is not suitable, then some critical set, that is, some $(2,3)$-tight set, contains x and y but not v. We prove this for any vertex v and any vertices $x, y \neq v$, since the count does not use the degree of v.

Lemma 3.17 (Tight sets block unsuitable splittings)**.** *Let V be finite, let G be $(2,3)$-sparse, and let v, x and y be vertices of G with $x, y \neq v$. Let $G - v + xy$ be the graph on $V \setminus \{v\}$ obtained from G by deleting v and adding the edge xy. If $G - v + xy$ is not $(2,3)$-sparse, then some set $S \subseteq V \setminus \{v\}$ that contains x and y is $(2,3)$-tight in G.*

Proof. Since $G - v + xy$ is not $(2,3)$-sparse, some finite set $S \subseteq V \setminus \{v\}$ with $|S| \geq 2$ spans more than $2|S| - 3$ edges of $G - v + xy$. As $v \notin S$, these edges are the edges of G inside S, together with xy if both x and y lie in S.

If $x \notin S$ or $y \notin S$, then S spans the same edges in $G - v + xy$ as in G, and this contradicts the sparsity of G. So $x, y \in S$, and S spans at most one more edge in $G - v + xy$ than in G. Hence $|E_G[S]| + 1 > 2|S| - 3$, that is, $2|S| \leq |E_G[S]| + 3$. By Lemma 3.13, S is $(2,3)$-tight in G. □

Formalization note 3.18. The formal statement does not build $G - v + xy$ on the subset $V \setminus \{v\}$ of V. Its vertex set is a separate copy of $V \setminus \{v\}$, which the inclusion map sends into V. Two of its vertices are adjacent if their images are adjacent in G or are x and y. The proof therefore first maps the violating set into V. The inclusion is injective, so the image has as many vertices as the set, and the edges of G inside the image correspond one to one to the edges of $G - v$ inside the set. The reverse decompositions Theorem 3.19 and Theorem 5.12 build $G - v$ and $G - v + xy$ in the same way.

3.7. **Sparse-graph reverse decomposition.** Below is the reverse decomposition of Laman graphs (Section 5.4) in its form for $(2,3)$-sparse graphs, where the vertex may also have degree one. The induction of Section 10.2 removes the vertex that the theorem gives at each step. We write $G - v$ for the graph obtained from G by deleting a vertex v, and $G - v + ab$ for the graph obtained from $G - v$ by adding the edge ab, as in Lemma 3.17.

Jordán [Jor16] finds a vertex of degree one, two or three in his proof of Theorem 2.2.1 (p. 45). He proves the cases of degree two and three as his Lemma 2.1.4 (p. 44): if v has degree two, then $G - v$ is sparse, and if v has degree three, then v has a suitable splitting (Section 3.6). His proof of the second part uses two facts about critical sets, which are our $(2,3)$-tight sets. The first is his Lemma 2.1.2, and the part of it that we use is Theorem 3.14 at $(k, \ell) = (2,3)$. The second is his Lemma 2.1.3, which says in part that if three tight sets meet pairwise in exactly one vertex and have no vertex in common, then their union is tight. We prove the case of Lemma 2.1.3 that we need within the proof of the theorem.

Theorem 3.19 (Sparse-graph reverse decomposition)**.** *Let V be finite, and let G be a $(2,3)$-sparse graph on V with at least one edge. Then G has a vertex v for which one of the following holds.*

1. *The vertex v has degree one, with neighbour a, and $G-v$ is $(2,3)$-sparse.*
2. *The vertex v has degree two, with neighbours a and b, and $G-v$ is $(2,3)$-sparse.*
3. *The vertex v has degree three, with neighbours a, b and c, where a and b are not adjacent, and $G-v+ab$ is $(2,3)$-sparse.*

Here $G-v$ is the graph on $V\setminus\{v\}$ obtained from G by deleting v, and $G-v+ab$ is obtained from $G-v$ by adding the edge ab.

Proof. We find v by the argument of Lemma 3.8, applied to the vertices of positive degree. Let U be the set of these vertices. Every edge of G has both ends in U. So U spans every edge of G, and $|U|\geq 2$, since G has an edge. Sparsity at U gives $|E(G)|+3\leq 2|U|$. If every vertex in U had degree at least four, the handshake identity would give $2|E(G)|=\sum_{u\in U}\deg_G(u)\geq 4|U|$, a contradiction. So some vertex v has degree one, two or three (`IsSparse.exists_one_le_degree_le_three`).

If v has degree one or two, then a set of vertices of $G-v$ spans the same edges in $G-v$ as in G, so $G-v$ is $(2,3)$-sparse. For degree two this is Jordán's Lemma 2.1.4(a).

Let v have degree three, with neighbours a, b and c. We must show that v has a suitable splitting, that is, two neighbours x and y that are not adjacent and for which $G-v+xy$ is $(2,3)$-sparse. Suppose that it has none. We use an observation from Jordán's proof: no $(2,3)$-tight set T with $v\notin T$ contains all three neighbours of v (`IsSparse.no_isTightOn_excluding_three_neighbors`). Otherwise $T\cup\{v\}$ spans the edges inside T and the edges va, vb and vc, so that

$$\big|E_G[T\cup\{v\}]\big| \;\geq\; \big|E_G[T]\big|+3 \;=\; 2|T| \;=\; 2\,|T\cup\{v\}|-2,$$

against sparsity. We derive a contradiction by constructing such a set T.

If two neighbours of v are not adjacent, the splitting of v on them is not suitable, so some tight set that does not contain v contains both of them (Lemma 3.17). At least one pair of neighbours of v is not adjacent (Lemma 3.9). Jordán's proof takes such a tight set for each of the three pairs; for an adjacent pair, the pair itself is such a set. We follow the formal proof instead, which distinguishes how many pairs are not adjacent (`IsSparse.False_of_pairwise_blocker_or_edge`).

Suppose first that exactly one pair is not adjacent, say a and b (`IsSparse.contradiction_one_pair`). Let S be a tight set with $a,b\in S$ and $v\notin S$. If $c\in S$, let $T=S$. Otherwise the edges ca and cb join c to S, and $T=S\cup\{c\}$ is tight by Theorem 3.16 with $k=2$.

Suppose next that exactly two pairs are not adjacent, say a,c and b,c, so that ab is an edge (`IsSparse.contradiction_two_pair`). Let S_1 and S_2 be tight sets that do not contain v, with $a,c\in S_1$ and $b,c\in S_2$, and let $T=S_1\cup S_2$. If $|S_1\cap S_2|\geq 2$, then T is tight by Theorem 3.14. Otherwise $S_1\cap S_2=\{c\}$, so $a\notin S_2$ and $b\notin S_1$. Then the edge ab lies inside T but in neither $E_G[S_1]$ nor $E_G[S_2]$. So Theorem 3.15 applies with $F=\{ab\}$, where its hypothesis reads $0+3\leq 2\cdot 1+1$, and T is tight.

Suppose finally that no two of a, b and c are adjacent (`IsSparse.contradiction_three_pair`). Let S_{ab}, S_{ac} and S_{bc} be tight sets that do not contain v, each containing the two vertices in its index. If two of them meet in at least two vertices, then their union contains a, b and c, and it is tight by Theorem 3.14; let T be this union. Otherwise

$$S_{ab} \cap S_{ac} = \{a\}, \qquad S_{ab} \cap S_{bc} = \{b\}, \qquad S_{ac} \cap S_{bc} = \{c\}.$$

This is the configuration of Jordán's Lemma 2.1.3. We show that $T = S_{ab} \cup S_{ac} \cup S_{bc}$ is tight by applying supermodularity (Lemma 2.8) twice. The set $S_{ab} \cap S_{ac} = \{a\}$ spans no edge, so

$$\big|E_G[S_{ab}]\big| + \big|E_G[S_{ac}]\big| \;\le\; \big|E_G[S_{ab} \cup S_{ac}]\big|.$$

The set $(S_{ab} \cup S_{ac}) \cap S_{bc} = \{b, c\}$ spans no edge either, since b and c are not adjacent, so

$$\big|E_G[S_{ab} \cup S_{ac}]\big| + \big|E_G[S_{bc}]\big| \;\le\; \big|E_G[T]\big|.$$

The three sets are tight, and $|T| = |S_{ab}| + |S_{ac}| + |S_{bc}| - 3$. Hence

$$\big|E_G[T]\big| \;\ge\; (2|S_{ab}| - 3) + (2|S_{ac}| - 3) + (2|S_{bc}| - 3) \;=\; 2|T| - 3,$$

and T is tight by Lemma 3.13.

In each case T is a tight set that contains a, b and c but not v, which is impossible. So v has a suitable splitting. □

4. Laman graphs

This section specializes Section 3 to $(k, \ell) = (2, 3)$, the case of Laman graphs. Pollaczek-Geiringer [PG27] (pp. 60 and 64), and later Laman [Lam70], proved that the Laman graphs are exactly the generically minimally rigid graphs in the plane (see Jordán [Jor16], Theorem 2.2.2). Equivalently, a graph on at least two vertices is generically rigid in the plane if and only if it has a spanning Laman subgraph. We prove this form in Section 9.

The direction from Laman graphs to rigidity is proved by induction on the number of vertices. Its base case is a Laman graph on two vertices, and its inductive step deletes a vertex of degree two or three by reversing a Henneberg move (Section 5). This section proves the facts about Laman graphs that the induction rests on: the Laman property is invariant under isomorphism, the only Laman graph on two vertices is the complete graph K_2, and in a Laman graph on at least three vertices every vertex has degree at least two.

4.1. **Definition and isomorphism invariance.**

Definition 4.1 (Laman graph)**.** Let V be finite. A graph G on V is *Laman* if it is $(2, 3)$-tight, that is, if it is $(2, 3)$-sparse and

$$|E(G)| + 3 \;=\; 2\,|V|.$$

Lemma 4.2 (Laman is preserved by isomorphism)**.** *Let φ be an isomorphism from G to a graph H. If G is Laman, then so is H.*

Proof. Specialize Lemma 3.12 to $(k, \ell) = (2, 3)$. □

4.2. **The base case: K_2.** The first result below is the starting point of the construction of Laman graphs by Henneberg moves, as in the example of Theorem 5.14. The second is the base case of the induction in the proof of Laman's theorem (Theorem 9.2).

Theorem 4.3 (K_2 is Laman)**.** *The complete graph K_2 on two vertices is Laman.*

Proof. The graph K_2 has one edge, and $1 + 3 = 2 \cdot 2$. For sparsity, the only set s of vertices with $3 \leq 2\,|s|$ is the whole vertex set. It spans the one edge (Lemma 2.3), and $1 + 3 \leq 2 \cdot 2$. □

Lemma 4.4 (Laman graphs on two vertices are K_2)**.** *Let G be a Laman graph on a set V of two vertices. Then G is the complete graph on V.*

Proof. Since $|E(G)| + 3 = 2 \cdot 2$, the graph G has one edge. The complete graph on V also has exactly one edge, and it contains every edge of G. So the two edge sets are equal. □

4.3. **Degree bounds.** Both lemmas below concern a Laman graph on at least three vertices. The reverse decomposition uses the first to exclude a vertex of degree one (Theorem 5.12). Lee and Streinu [LS08, Lemma 4] give the same argument for (k, ℓ)-tight graphs. The second lemma is the Laman form of Lemma 3.8.

Lemma 4.5 (Minimum degree)**.** *Let V be finite with $|V| \geq 3$, and let G be a Laman graph on V. Then every vertex of G has degree at least 2.*

Proof. Let $v \in V$. The set $\{v\}^c$ has $|V| - 1 \geq 2$ vertices, so sparsity applies to it: $|E_G[\{v\}^c]| + 3 \leq 2(|V| - 1)$. By Lemma 2.6, $|E_G[\{v\}^c]| = |E(G)| - \deg_G(v)$. Since $|E(G)| + 3 = 2|V|$, this gives $2|V| - \deg_G(v) \leq 2|V| - 2$, that is, $\deg_G(v) \geq 2$. □

Lemma 4.6 (A vertex of degree 2 or 3)**.** *Let V be finite with $|V| \geq 3$, and let G be a Laman graph on V. Then some vertex of G has degree 2 or 3.*

Proof. Since G is $(2, 3)$-sparse, it has a vertex of degree at most 3 (Lemma 3.8). That vertex has degree at least 2 (Lemma 4.5). □

5. Henneberg moves

This section studies two ways of adding a vertex to a graph, the Henneberg moves. The Type I move adds a vertex of degree two. The Type II move deletes an edge ab and adds a vertex of degree three, joined to a, b and a third vertex. The Laman graphs are exactly the graphs that can be built from K_2 by these moves, a characterization due to Henneberg [Hen11], Pollaczek-Geiringer [PG27] (§7, pp. 69–70) and Laman [Lam70]. Jordán [Jor16] proves it as Theorem 2.2.8, for minimally rigid graphs, which are the Laman graphs (Section 4).

Each half of this characterization follows by induction on the number of vertices, and we prove the two inductive steps. After the definitions and a count of the edges that each move adds, we show that the moves preserve the Laman property. The other step is the reverse decomposition (Theorem 5.12): every Laman graph on at least three vertices has a vertex v of degree two or three such that deleting v, and in degree three joining two of its neighbours by an edge, leaves a Laman graph. The

proof that every Laman graph is generically rigid in the plane (Theorem 9.2) is an induction with this step. The section ends with an example.

Throughout, G is a graph on a vertex set V, which is finite where a statement says so. We write ab for the edge with endpoints a and b, and v for the vertex that a move adds, so that the graph the move produces has vertex set $V \cup \{v\}$.

5.1. **Definitions.** In Jordán's survey [Jor16] (§1.2.1, p. 38) the two moves are the 0-extension and the 1-extension in the plane, also called Henneberg moves of type I and type II.

Definition 5.1 (Type I move)**.** Let $a, b \in V$. The *Type I move at a and b* adds to G a new vertex v and the edges va and vb.

Definition 5.2 (Type II move)**.** Let $a, b, c \in V$. The *Type II move on ab and c* deletes the edge ab from G if it is present, and adds a new vertex v and the edges va, vb and vc.

Formalization note 5.3. Both moves are formalized as graphs on Option V, which has an element some u for each $u \in V$ and one further element none. The new vertex v is none, and each vertex u of G becomes some u. This is the standard way in mathlib to adjoin one element, and it makes v new by construction, so no statement needs a hypothesis saying so. The formal statements about a move, here and in later sections, compare a graph on V with a graph on Option V through the map $u \mapsto$ some u, and the text identifies each u with its image.

The definitions allow any vertices a, b and c, but we use the moves only under Jordán's conditions (p. 44). For the Type I move at a and b we assume $a \neq b$, so that v has degree two; if $a = b$, the move adds the single edge va. For the Type II move on ab and c we assume that ab is an edge of G and that c is neither a nor b, so that v has degree three and the number of edges grows by two.

5.2. **Edge sets and edge counts.** We describe the edge set of the graph that each move produces, and count it. We use the counts to verify the equation $|E| + 3 = 2|V|$ of a Laman graph in the preservation theorems below and in the reverse decomposition.

Lemma 5.4 (Edge set of Type I)**.** *Let $a, b \in V$, let G' be obtained from G by the Type I move at a and b, and let v be its new vertex. Then*

$$E(G') \;=\; E(G) \cup \{va, vb\}.$$

Proof. An edge of G' either contains v or joins two vertices of G. By Definition 5.1, the edges of G' at v are va and vb, and two vertices of G are adjacent in G' exactly when they are adjacent in G. □

Lemma 5.5 (Edge count of Type I)**.** *Let V be finite, let $a \neq b$ be vertices of G, and let G' be obtained from G by the Type I move at a and b. Then*

$$|E(G')| \;=\; |E(G)| + 2.$$

Proof. The two sets on the right of Lemma 5.4 are disjoint, since the edges va and vb contain the new vertex v and the edges of G do not. The edges va and vb are distinct because $a \neq b$. □

Lemma 5.6 (Edge set of Type II)**.** *Let $a, b, c \in V$, let G' be obtained from G by the Type II move on ab and c, and let v be its new vertex. Then*

$$E(G') \;=\; (E(G) \setminus \{ab\}) \cup \{va, vb, vc\}.$$

Proof. As in the proof of Lemma 5.4, an edge of G' either contains v or joins two vertices of G. By Definition 5.2, the edges of G' at v are va, vb and vc, and two vertices of G are adjacent in G' exactly when they are adjacent in G and are not a and b. □

Lemma 5.7 (Edge count of Type II)**.** *Let V be finite, let a, b and c be distinct vertices of G, and let G' be obtained from G by the Type II move on ab and c. Then*

$$|E(G')| \;=\; \big|E(G) \setminus \{ab\}\big| + 3.$$

Proof. As in the proof of Lemma 5.5, the two sets on the right of Lemma 5.6 are disjoint. The edges va, vb and vc are distinct because a, b and c are. □

When ab is an edge of G, as in the preservation theorems, the right-hand side is $|E(G)| + 2$.

5.3. **Preservation of the Laman property.** We show that both moves preserve the Laman property, and that the Type I move also reflects it. Jordán [Jor16] proves the sparsity part of these facts as Lemmas 2.1.5 and 2.1.4(a), where the two moves are called the 0-extension and the 1-extension.

Theorem 5.8 (Type I preserves the Laman property)**.** *Let G be a Laman graph and let $a \neq b$ be vertices of G. Let G' be obtained from G by the Type I move at a and b. Then G' is Laman.*

Proof. Let v be the new vertex of G'. The graph G' has one more vertex than G and two more edges (Lemma 5.5), so $|E(G')| + 3 = 2|V(G')|$ follows from $|E(G)| + 3 = 2|V(G)|$.

For $(2,3)$-sparsity (Definition 3.1) we show that every set X of at least two vertices of G' spans at most $2|X| - 3$ edges. Let $X_0 = X \setminus \{v\}$. The edges of G' inside X are the edges of G inside X_0, together with those of the new edges va and vb that lie inside X. If $v \notin X$, then $X = X_0$, and the bound holds because G is sparse. Otherwise $|X| = |X_0| + 1$. If $|X_0| = 1$, then X_0 spans no edge and at most one new edge lies inside X, so X spans at most $1 = 2|X| - 3$ edges. If $|X_0| \geq 2$, then

$$|E_{G'}[X]| \;\leq\; |E_G[X_0]| + 2 \;\leq\; (2|X_0| - 3) + 2 \;=\; 2|X| - 3.$$

□

Theorem 5.9 (Type II preserves the Laman property)**.** *Let G be a Laman graph, let ab be an edge of G, and let $c \neq a, b$ be a vertex of G. Let G' be obtained from G by the Type II move on ab and c. Then G' is Laman.*

Proof. Let v be the new vertex of G'. The move deletes ab and adds va, vb and vc, so G' has one more vertex than G and two more edges (Lemma 5.7). The count follows as in Theorem 5.8.

For sparsity, let X and X_0 be as in the proof of Theorem 5.8. The edges of G' inside X are the edges of G inside X_0 other than ab, together with those of va, vb

and vc that lie inside X. The cases $v \notin X$ and $|X_0| = 1$ go as before. Suppose that $v \in X$ and $|X_0| \geq 2$. If a and b both lie in X_0, then ab is an edge of G inside X_0 that G' does not have, and at most three new edges lie inside X. Otherwise one of va and vb does not lie inside X, so at most two new edges do. In both cases $|E_{G'}[X]| \leq |E_G[X_0]| + 2 \leq 2|X| - 3$. □

The reverse decomposition (Section 5.4) needs the converse of Theorem 5.8, to show that removing a vertex of degree two from a Laman graph leaves a Laman graph.

Theorem 5.10 (Type I reflects the Laman property)**.** *Let G be a finite graph and let $a \neq b$ be vertices of G. Let G' be obtained from G by the Type I move at a and b. If G' is Laman, then G is Laman.*

Proof. Let v be the new vertex of G'. Both new edges contain v, so a set of vertices of G spans the same edges in G' as in G. Hence G is $(2,3)$-sparse because G' is. The count follows from Lemma 5.5, as in Theorem 5.8. □

Theorem 5.11 (Type I preserves and reflects the Laman property)**.** *Let G be a finite graph and let $a \neq b$ be vertices of G. Let G' be obtained from G by the Type I move at a and b. Then G' is Laman if and only if G is Laman.*

Proof. Combine Theorem 5.8 and Theorem 5.10. □

5.4. **Reverse decomposition.** Let v be a vertex of G. We write $G - v$ for the graph on $V \setminus \{v\}$ obtained from G by deleting v, and, for vertices $a, b \neq v$, we write $G - v + ab$ for the graph obtained from $G - v$ by adding the edge ab, as in Section 3.6. Deleting v reverses a move. If v has degree two, with neighbours a and b, then the Type I move at a and b applied to $G - v$ gives a copy of G. If v has degree three, with neighbours a, b and c, where a and b are not adjacent, then the Type II move on ab and c applied to $G - v + ab$ gives a copy of G. The theorem below finds a vertex at which the smaller graph is again Laman.

Most of the work is in the same statement for $(2,3)$-sparse graphs, Theorem 3.19, in which v may also have degree one. A sparse graph with an edge has a vertex v of degree one, two or three, and Jordán's Lemma 2.1.4 [Jor16] treats the last two cases. If v has degree two, then $G - v$ is sparse. If v has degree three, then v has a suitable splitting (Section 3.6): two neighbours a and b of v that are not adjacent and make $G - v + ab$ sparse. We proved the sparse statement in Section 3.7. It is also the inductive step in the proof that the edge set of every $(2,3)$-sparse graph is independent in the rigidity matroid (Section 10).

Theorem 5.12 (Laman reverse decomposition)**.** *Let V be finite with $|V| \geq 3$, and let G be a Laman graph on V. Then G has a vertex v for which one of the following holds.*

(1) The vertex v has degree two, with neighbours a and b, and $G - v$ is Laman.
(2) The vertex v has degree three, with neighbours a, b and c, where a and b are not adjacent, and $G - v + ab$ is Laman.

Here $G - v$ is the graph on $V \setminus \{v\}$ obtained from G by deleting v, and $G - v + ab$ is obtained from $G - v$ by adding the edge ab.

Proof. Since G is Laman and $|V| \geq 3$, it is $(2,3)$-sparse and has $2|V| - 3 \geq 3$ edges. So Theorem 3.19 gives a vertex v of degree one, two or three. Every vertex of G has degree at least two (Lemma 4.5), so v has degree two or three.

Suppose that v has degree two, with neighbours a and b. Then G is isomorphic to the graph obtained from $G - v$ by the Type I move at a and b, so that graph is Laman (Lemma 4.2). Hence $G - v$ is Laman by Theorem 5.11.

Suppose that v has degree three. Then the sparse statement gives neighbours a, b and c of v, where a and b are not adjacent, such that $H = G - v + ab$ is $(2,3)$-sparse. The Type II move on ab and c deletes from H the edge ab that H adds to $G - v$, and it adds v back. So G is isomorphic to the graph H' obtained from H by this move, and H' is Laman (Lemma 4.2). As H' has $|V|$ vertices, $|E(H')| + 3 = 2|V|$. Since ab is an edge of H, Lemma 5.7 gives $|E(H')| = |E(H)| + 2$. Hence $|E(H)| + 3 = 2(|V| - 1)$, and H is Laman. □

Formalization note 5.13. The formal statement builds $G - v$ and $G - v + ab$ on a separate copy of $V \setminus \{v\}$, as in Formalization note 3.18. The isomorphisms between G and the moved graphs, in the proof above and in the induction of Laman's theorem (Theorem 9.2), are built from the neighbours of v. Formalization note 10.1 explains why a reverse decomposition describes the smaller graph directly, and not as a graph to which a move is applied.

5.5. **An example: K_4 minus an edge.** We build K_4 minus an edge from K_2 by two Type I moves.

Theorem 5.14 (K_4 minus an edge is Laman)**.** *The graph on* $\{0,1,2,3\}$ *obtained from the complete graph* K_4 *by deleting the edge* 23 *is Laman.*

Proof. The complete graph on $\{0,1\}$ is Laman (Theorem 4.3). The Type I move at 0 and 1 adds a new vertex 2 and the edges 20 and 21. The result is a triangle, which is Laman by Theorem 5.8. A second Type I move at 0 and 1 adds a new vertex 3 and the edges 30 and 31. The result is Laman by Theorem 5.8, and its edges are the pairs from $\{0,1,2,3\}$ other than 23.

In the formal proof the two new vertices are not the numbers 2 and 3 (Formalization note 5.3). The graph built is therefore isomorphic to the graph of the statement, by the bijection that sends the first new vertex to 2 and the second to 3. The Laman property passes along this isomorphism (Lemma 4.2). □

6. Frameworks and infinitesimal rigidity

This section defines the rigidity of a graph placed in Euclidean space, and proves the facts about it that Laman's theorem (Section 9) uses. We follow the presentation in Jordán's survey [Jor16] (§1.2). A framework places each vertex of a graph at a point of $\mathbb{R}^d$. Its infinitesimal motions assign a velocity to each vertex so that no edge changes length to first order. They form the kernel of a linear map, the rigidity map. The translations and rotations of $\mathbb{R}^d$ give the trivial infinitesimal motions, and a framework is infinitesimally rigid when it has no others. A graph is generically rigid when some framework of it is infinitesimally rigid.

After the definitions we bound the rank of the rigidity map by the number of edges. We show that rigidity is preserved by adding edges and by isomorphisms, and that

infinitesimal rigidity survives a small change of the framework. We then prove that a graph on n vertices that is generically rigid in $\mathbb{R}^d$ has at least $dn - d(d+1)/2$ edges. The section ends with the complete graph K_2, which is generically rigid in every dimension and is the base case of the induction in Section 9.

Throughout, G is a graph on a set V, which is finite where a statement says so, and $d \in \mathbb{N}$. We equip $\mathbb{R}^d$ with its standard inner product $\langle \cdot, \cdot \rangle$. As in Section 5, we write uv for the edge with endpoints u and v.

6.1. **Definitions.**

Definition 6.1 (Framework)**.** A *framework* on V in $\mathbb{R}^d$ is a map $p : V \to \mathbb{R}^d$, which places each vertex v at the point p_v. The frameworks on V in $\mathbb{R}^d$ form the real vector space $(\mathbb{R}^d)^V$.

Jordán's framework is the pair (G, p) of a graph and such a map. Here a framework is the map alone, and the graph enters each notion below as a separate argument. We also call a framework a *placement* of V.

Lemma 6.2 (Dimension of the space of frameworks)**.** *Let V be finite. Then* $\dim_{\mathbb{R}} (\mathbb{R}^d)^V = d\,|V|$.

Proof. The space $(\mathbb{R}^d)^V$ is a product of $|V|$ copies of $\mathbb{R}^d$, each of dimension d. □

Let p be a framework, and let each vertex v move with velocity p'_v, along the line $t \mapsto p_v + t\,p'_v$. The squared length of an edge uv then has derivative $2\langle p_u - p_v,\, p'_u - p'_v \rangle$ at $t = 0$. The rigidity map records these derivatives, without the factor 2.

Definition 6.3 (Rigidity map)**.** Let p be a framework on V in $\mathbb{R}^d$. The *rigidity map* of G at p is the linear map $R(G,p) : (\mathbb{R}^d)^V \to \mathbb{R}^{E(G)}$ given by

$$R(G,p)(p')(uv) \;=\; \langle p_u - p_v,\, p'_u - p'_v \rangle$$

for $p' \in (\mathbb{R}^d)^V$ and each edge uv of G.

The right-hand side is unchanged when u and v are exchanged, since both of its factors change sign, so it defines a value at the edge uv. It is linear in p'. In the standard bases, the matrix of $R(G,p)$ is Jordán's rigidity matrix (p. 36). The kernel of $R(G,p)$ is the space of *infinitesimal motions* of G at p.

We record the defining formula as a lemma, which later proofs cite when they evaluate the rigidity map at an edge.

Lemma 6.4 (The rigidity map at an edge)**.** *Let uv be an edge of G, and let p and p' be frameworks on V in $\mathbb{R}^d$. Then*

$$R(G,p)(p')(uv) \;=\; \langle p_u - p_v,\, p'_u - p'_v \rangle.$$

Proof. Immediate from the definition. □

Each translation and each infinitesimal rotation of $\mathbb{R}^d$ gives an infinitesimal motion of G at p, whatever the graph G is. Together they give the motions $p'_v = Ap_v + b$, for a skew-symmetric $d \times d$ matrix A and a vector $b \in \mathbb{R}^d$. These are the *trivial* infinitesimal motions (Jordán, p. 36; Section 8). The pairs (A, b) form a space of dimension $d(d+1)/2$, so the trivial motions form a space of dimension at most

$d(d+1)/2$. Equality holds when the points p_v affinely span $\mathbb{R}^d$ (Lemma 8.12). A framework is infinitesimally rigid when all of its infinitesimal motions are trivial. We state this as a bound on the dimension of the kernel, the form that rank–nullity arguments use.

Definition 6.5 (Infinitesimally rigid framework)**.** Let V be finite, and let p be a framework on V in $\mathbb{R}^d$. We say that p is *infinitesimally rigid* for G if

$$\dim_{\mathbb{R}} \ker R(G,p) \;\leq\; \frac{d(d+1)}{2}.$$

Formalization note 6.6. When the points p_v affinely span $\mathbb{R}^d$, the kernel contains the trivial motions, a space of dimension $d(d+1)/2$, so the definition says that every infinitesimal motion is trivial. In general, by rank–nullity and Lemma 6.2, it says that $R(G,p)$ has rank at least $d|V| - d(d+1)/2$. Jordán (pp. 36–37) asks instead that the rank attain the bound $S(|V|,d)$ of his Lemma 1.2.1. This bound is $d|V| - d(d+1)/2$ when $|V| \geq d+2$ and $|V|(|V|-1)/2$ when $|V| \leq d+1$; the two expressions agree at $|V| = d$ and at $|V| = d+1$. The rank never exceeds $S(|V|,d)$, so the two definitions agree when $|V| \geq d$. They also agree when $|V| \leq 1$, where both hold for every p. In particular they agree for every V in the plane, the case of Laman's theorem. When $2 \leq |V| \leq d-1$, which needs $d \geq 3$, the definition above is weaker. For instance, two vertices with no edge between them, placed anywhere in $\mathbb{R}^3$, satisfy it, since the kernel has dimension 6; yet the two vertices can move apart.

Definition 6.7 (Generically rigid graph)**.** Let V be finite. The graph G is *generically rigid* in dimension d if some framework on V in $\mathbb{R}^d$ is infinitesimally rigid for G.

Defining generic rigidity through a single framework avoids generic frameworks, those whose coordinates are algebraically independent over $\mathbb{Q}$. Jordán (p. 38) calls G rigid in $\mathbb{R}^d$ when its generic frameworks are rigid: every continuous motion of the vertices that keeps each edge length fixed comes from an isometry of $\mathbb{R}^d$. For $|V| \geq d$, where infinitesimal rigidity has Jordán's meaning (Formalization note 6.6), the two notions agree, for two reasons. At a generic framework, rigidity and infinitesimal rigidity are equivalent, by a theorem of Asimow and Roth [AR78]. And the rigidity map has its largest rank at a generic framework, since its entries are polynomials in the coordinates with integer coefficients (Jordán, pp. 38–39).

Definition 6.8 (Generically rigid graph, injective form)**.** Let V be finite. The graph G is *injectively generically rigid* in dimension d if some injective framework on V in $\mathbb{R}^d$ is infinitesimally rigid for G.

An injectively generically rigid graph is generically rigid. The induction that proves Laman's theorem (Theorem 9.2) keeps an injective framework at each step, and weakens to generic rigidity at the end. Its step for each Henneberg move places the new vertex using that the move's vertices a and b lie at distinct points (Section 7). In positive dimension the two notions agree, since an infinitesimally rigid framework can be moved slightly to an injective one (Section 6.5).

6.2. **Rank and kernel bounds.** We compare the kernels of the rigidity maps of two graphs, one of which contains the edges of the other, and we bound the rank of the rigidity map by the number of edges. The first gives the monotonicity results below, and the second gives the edge-count bound.

Lemma 6.9 (Kernel monotonicity)**.** *Let G' be a graph on V with $E(G) \subseteq E(G')$, and let p be a framework on V in $\mathbb{R}^d$. Then*

$$\ker R(G', p) \subseteq \ker R(G, p).$$

Proof. Let $p' \in \ker R(G', p)$. At each edge of G, the value of $R(G, p)(p')$ equals the value of $R(G', p)(p')$ at the same edge of G', which is zero. □

Lemma 6.10 (Rank is bounded by edge count)**.** *Let V be finite, and let p be a framework on V in $\mathbb{R}^d$. Then the rank of $R(G, p)$ is at most $|E(G)|$.*

Proof. The range of $R(G, p)$ lies in $\mathbb{R}^{E(G)}$, which has dimension $|E(G)|$. □

6.3. **Graph monotonicity.** Adding edges to a graph preserves each form of rigidity, at the same framework.

Lemma 6.11 (Infinitesimal rigidity is monotone)**.** *Let V be finite, let G' be a graph on V with $E(G) \subseteq E(G')$, and let p be a framework on V in $\mathbb{R}^d$. If p is infinitesimally rigid for G, then p is infinitesimally rigid for G'.*

Proof. The kernel of $R(G', p)$ lies in that of $R(G, p)$ (Lemma 6.9), so its dimension is at most $d(d+1)/2$. □

Lemma 6.12 (Generic rigidity is monotone)**.** *Let V be finite, and let G' be a graph on V with $E(G) \subseteq E(G')$. If G is generically rigid in dimension d, then so is G'.*

Proof. A framework that is infinitesimally rigid for G is infinitesimally rigid for G' (Lemma 6.11). □

Lemma 6.13 (Injective generic rigidity is monotone)**.** *Let V be finite, and let G' be a graph on V with $E(G) \subseteq E(G')$. If G is injectively generically rigid in dimension d, then so is G'.*

Proof. An injective framework that is infinitesimally rigid for G is infinitesimally rigid for G' (Lemma 6.11), and it is still injective. □

6.4. **Transport along graph isomorphism.** Rigidity is invariant under isomorphism, as sparsity is (Lemma 3.11). The induction in Section 9 uses this for graphs that are identified only up to isomorphism, such as a Laman graph on two vertices and K_2.

Lemma 6.14 (Infinitesimal rigidity transports along graph isomorphism)**.** *Let V and W be finite, let φ be an isomorphism from G to a graph H on W, and let p be a framework on V in $\mathbb{R}^d$. If p is infinitesimally rigid for G, then $p \circ \varphi^{-1}$ is infinitesimally rigid for H.*

Proof. Composition with φ is a linear isomorphism from $(\mathbb{R}^d)^W$ to $(\mathbb{R}^d)^V$, sending q' to $q' \circ \varphi$. Let uv be an edge of G, so that $\varphi(u)\varphi(v)$ is an edge of H. The value of $R(H, p \circ \varphi^{-1})(q')$ at $\varphi(u)\varphi(v)$ is $\langle p_u - p_v,\, q'_{\varphi(u)} - q'_{\varphi(v)} \rangle$ (Lemma 6.4), which is the value of $R(G,p)(q' \circ \varphi)$ at uv. Since φ maps the edges of G onto those of H, the vector q' is an infinitesimal motion of H at $p \circ \varphi^{-1}$ if and only if $q' \circ \varphi$ is an infinitesimal motion of G at p. So composition with φ maps the one kernel isomorphically onto the other, and the two kernels have the same dimension. □

Lemma 6.15 (Generic rigidity transports along graph isomorphism)**.** *Let V and W be finite, and let φ be an isomorphism from G to a graph H on W. If G is generically rigid in dimension d, then so is H.*

Proof. If p is infinitesimally rigid for G, then $p \circ \varphi^{-1}$ is infinitesimally rigid for H (Lemma 6.14). □

Lemma 6.16 (Injective generic rigidity transports along graph isomorphism)**.** *Let V and W be finite, and let φ be an isomorphism from G to a graph H on W. If G is injectively generically rigid in dimension d, then so is H.*

Proof. If p is injective and infinitesimally rigid for G, then $p \circ \varphi^{-1}$ is infinitesimally rigid for H (Lemma 6.14). It is injective, since φ^{-1} is a bijection. □

6.5. **Openness of infinitesimal rigidity.** Infinitesimal rigidity is an open condition on the framework: every framework close enough to an infinitesimally rigid one is again infinitesimally rigid. So an infinitesimally rigid framework can be moved slightly, while staying infinitesimally rigid, to satisfy a further condition that holds on a dense open set of frameworks. The Type II step of the Henneberg induction (Theorem 7.5) moves a point in this way, off the line through two others.

Lemma 6.17 (Infinitesimal rigidity is an open condition on the placement)**.** *Let V be finite, and let p_0 be a framework on V in $\mathbb{R}^d$ that is infinitesimally rigid for G. Then every framework in some neighbourhood of p_0 in $(\mathbb{R}^d)^V$ is infinitesimally rigid for G.*

Proof. Let r be the rank of $R(G, p_0)$, and choose $\xi_1, \dots, \xi_r \in (\mathbb{R}^d)^V$ whose images under $R(G, p_0)$ form a basis of its range. The entry of $R(G,p)(\xi_i)$ at an edge uv is the inner product of $p_u - p_v$ with $(\xi_i)_u - (\xi_i)_v$ (Lemma 6.4), so $R(G,p)(\xi_i)$ depends continuously on p. Linear independence of r vectors in the finite-dimensional space $\mathbb{R}^{E(G)}$ is an open condition. So for every p in some neighbourhood of p_0 the vectors $R(G,p)(\xi_1), \dots, R(G,p)(\xi_r)$ are linearly independent, and $R(G,p)$ has rank at least r.

For such p, rank–nullity in $(\mathbb{R}^d)^V$ shows that the kernel of $R(G,p)$ has dimension at most that of the kernel of $R(G, p_0)$, which is at most $d(d+1)/2$. □

6.6. **The edge-count bound.** Rank–nullity gives a lower bound on the number of edges of a generically rigid graph. In the plane it says that a graph on V that is generically rigid has at least $2|V| - 3$ edges, as many as a Laman graph on V (Lemma 9.1).

Theorem 6.18 (Edge-count bound for generically rigid graphs)**.** *Let V be finite, and let G be generically rigid in dimension d. Then*

$$d\,|V| \;\le\; |E(G)| + \frac{d(d+1)}{2}.$$

Proof. Let p be a framework that is infinitesimally rigid for G. By rank–nullity, $\dim(\mathbb{R}^d)^V$ is the rank of $R(G,p)$ plus the dimension of its kernel. Here $\dim(\mathbb{R}^d)^V = d\,|V|$ (Lemma 6.2), the rank is at most $|E(G)|$ (Lemma 6.10), and the kernel has dimension at most $d(d+1)/2$, since p is infinitesimally rigid. Together these give the bound. □

Formalization note 6.19. The bound is stated in additive form because the formal statement counts in the natural numbers, in which $d\,|V| - d(d+1)/2$ would be truncated at 0.

6.7. **Worked example: K_2 is generically rigid in every dimension.** The complete graph K_2 on two vertices is the base case of the induction in Section 9, which needs it in the injective form.

Theorem 6.20 (K_2 is injectively generically rigid in every positive dimension)**.** *Let $d \ge 1$. Then the complete graph K_2 on two vertices is injectively generically rigid in dimension d.*

Proof. Let u and v be the two vertices, and let e be the first standard basis vector of $\mathbb{R}^d$. The framework p with $p_u = 0$ and $p_v = e$ is injective. The velocities $p'_u = e$ and $p'_v = 0$ give $R(K_2,p)(p')(uv) = \langle 0-e,\, e-0\rangle = -1$ (Lemma 6.4), so $R(K_2,p)$ has rank at least 1. The space of frameworks has dimension $2d$ (Lemma 6.2), so by rank–nullity the kernel has dimension at most $2d-1$. Finally $2d-1 \le d(d+1)/2$, since the difference is $(d-1)(d-2)/2 \ge 0$. □

Theorem 6.21 (K_2 is generically rigid in every dimension)**.** *Let $d \in \mathbb{N}$. Then the complete graph K_2 on two vertices is generically rigid in dimension d.*

Proof. For $d \ge 1$ this follows from Theorem 6.20. For $d = 0$ the space of frameworks is $\{0\}$, so the kernel of the rigidity map has dimension $0 = d(d+1)/2$. □

7. Rigidity preservation under Henneberg moves (dimension 2)

We show that both Henneberg moves (Section 5) preserve injective generic rigidity in the plane (Definition 6.8). This is the inductive step in the proof that every Laman graph is generically rigid in the plane (Theorem 9.2). That induction is on the number of vertices and starts from K_2 (Theorem 6.20). By the reverse decomposition (Theorem 5.12), every Laman graph on at least three vertices is obtained, up to isomorphism, from a smaller Laman graph by a move.

Jordán [Jor16] calls the two moves the 0-extension and the 1-extension. His Lemmas 1.2.2 and 1.2.3 (p. 38) state that in $\mathbb{R}^d$, if the points involved are in general position, each of these moves raises the rank of the rigidity matrix of a framework by d. He cites Whiteley's Vertex d-Addition Lemma 11.1.1 and Edge d-Split Theorem 11.1.7 [Whi96]. In the plane our theorems assume the same general-position conditions. We prove the half of the conclusion that rigidity needs: the kernel of the

rigidity map does not grow. By rank–nullity, this means that the rank grows by at least 2.

Let p be an infinitesimally rigid framework of G, and let G' be obtained from G by a move with new vertex v. We extend p to G' by placing v at a point q, and we restrict each infinitesimal motion of G' to the vertices of G. We show that the restriction is an infinitesimal motion of G, and that it determines the velocity of v. It does so when two new edges at v have linearly independent directions. Then the kernel of the rigidity map of G' has dimension at most that of G, so the extended framework is again infinitesimally rigid.

For the Type I move at a and b, every edge of G is an edge of G', and we place v off the line through p_a and p_b. The Type II move on ab and c deletes the edge ab, so the restriction must also satisfy the constraint of ab. As in Whiteley's proof of his Theorem 11.1.7, we place v on the line through p_a and p_b, distinct from both. The new edges va and vb then point along this line, and together they imply the constraint of ab. The new edges va and vc determine the velocity of v if p_a, p_b and p_c are not collinear. For each move we first prove that the extended framework is infinitesimally rigid for a given point q, and then choose q.

Throughout, frameworks are in $\mathbb{R}^2$. A framework p on a finite set V is infinitesimally rigid for a graph G on V when $\dim\ker R(G,p) \le 3$ (Definition 6.5). An infinitesimal motion x, an element of this kernel, gives each vertex u a velocity $x_u \in \mathbb{R}^2$. As in Section 5, v is the new vertex of a move, and the graph that the move produces has vertex set $V \cup \{v\}$. Formalization note 5.3 describes how the new vertex is encoded formally.

7.1. **The Type I move.** To choose a point q off the line through p_a and p_b, we need $p_a \neq p_b$. An injective framework has $p_a \neq p_b$ when $a \neq b$, and this is why we work with injective frameworks.

Theorem 7.1 (Infinitesimal rigidity under the Type I move)**.** *Let G be a graph on a finite set V, let p be a framework on V in $\mathbb{R}^2$ that is infinitesimally rigid for G, and let G' be obtained from G by the Type I move at vertices a and b, with new vertex v. Let $q \in \mathbb{R}^2$ be a point such that $q - p_a$ and $q - p_b$ are linearly independent, that is, q, p_a and p_b are not collinear. Let $\hat{p}$ be the framework on $V \cup \{v\}$ that agrees with p on V and places v at q. Then $\hat{p}$ is infinitesimally rigid for G'.*

Proof. Let x be an infinitesimal motion of G' at $\hat{p}$, and let $\rho(x)$ be its restriction to V. Every edge uw of G is an edge of G'. The entry of the rigidity map at uw depends only on the points and the velocities of u and w (Lemma 6.4). So $\rho(x)$ is an infinitesimal motion of G at p, and ρ is a linear map from $\ker R(G', \hat{p})$ to $\ker R(G,p)$.

We show that ρ is injective. Suppose that $\rho(x) = 0$, so that $x_u = 0$ for every $u \in V$. The entries of $R(G', \hat{p})(x)$ at the new edges va and vb vanish:

$$\langle q - p_a,\, x_v - x_a \rangle \;=\; 0 \quad \text{and} \quad \langle q - p_b,\, x_v - x_b \rangle \;=\; 0.$$

Since $x_a = x_b = 0$, the velocity x_v is orthogonal to $q - p_a$ and to $q - p_b$. These two vectors are linearly independent, so they span $\mathbb{R}^2$, and hence $x_v = 0$. So $x = 0$.

Since ρ is injective and p is infinitesimally rigid for G,

$$\dim\ker R(G', \hat{p}) \;\le\; \dim\ker R(G,p) \;\le\; 3.$$

□

Theorem 7.2 (Type I preserves injective generic rigidity in dimension 2)**.** *Let G be a graph on a finite set V that is injectively generically rigid in dimension 2, and let $a \neq b$ be vertices of G. Let G' be obtained from G by the Type I move at a and b. Then G' is injectively generically rigid in dimension 2.*

Proof. Let p be an injective framework on V in $\mathbb{R}^2$ that is infinitesimally rigid for G. Since p is injective and $a \neq b$, the points p_a and p_b are distinct. The line through them and the finitely many points p_u, for $u \in V$, do not cover $\mathbb{R}^2$. So we can choose a point q off this line that is none of the points p_u. Since q is off the line through p_a and p_b, the vectors $q - p_a$ and $q - p_b$ are linearly independent.

Let $\hat{p}$ be the framework that extends p by placing the new vertex at q. It is infinitesimally rigid for G' (Theorem 7.1). It is injective, since p is injective and q is none of the points p_u. □

7.2. **The Type II move.** Here the point q lies on the line through p_a and p_b. We choose it first when p_a, p_b and p_c are not collinear. We then remove this assumption by moving p_c slightly, which keeps the framework infinitesimally rigid (Section 6.5).

Theorem 7.3 (Infinitesimal rigidity under the Type II move)**.** *Let G be a graph on a finite set V, let p be a framework on V in $\mathbb{R}^2$ that is infinitesimally rigid for G, and let $a, b, c \in V$. Let G' be obtained from G by the Type II move on ab and c, with new vertex v. Let $q = p_a + \alpha(p_b - p_a)$ for a real number $\alpha \neq 0, 1$, suppose that $q - p_a$ and $q - p_c$ are linearly independent, and let $\hat{p}$ be the framework on $V \cup \{v\}$ that agrees with p on V and places v at q. Then $\hat{p}$ is infinitesimally rigid for G'.*

Proof. As in the proof of Theorem 7.1, let $\rho(x)$ be the restriction to V of an infinitesimal motion x of G' at $\hat{p}$. We first show that $\rho(x)$ is an infinitesimal motion of G at p. Every edge of G other than ab is an edge of G', so $\rho(x)$ satisfies the constraint of each such edge, as before.

If ab is an edge of G, the move deletes it, and we recover its constraint from the new edges va and vb. We have $q - p_a = \alpha(p_b - p_a)$ and $q - p_b = (\alpha - 1)(p_b - p_a)$. The entries of $R(G', \hat{p})(x)$ at va and vb vanish (Lemma 6.4). Dividing them by α and by $\alpha - 1$, which are not zero, gives

$$\langle p_b - p_a,\, x_v - x_a \rangle \;=\; 0 \quad \text{and} \quad \langle p_b - p_a,\, x_v - x_b \rangle \;=\; 0.$$

Subtracting the first equation from the second gives $\langle p_b - p_a,\, x_a - x_b \rangle = 0$. This is the constraint $\langle p_a - p_b,\, x_a - x_b \rangle = 0$ of the edge ab, up to sign.

It remains to show that ρ is injective. If $\rho(x) = 0$, the entries at the new edges va and vc show that x_v is orthogonal to $q - p_a$ and to $q - p_c$. These vectors are linearly independent, so $x_v = 0$ and $x = 0$. As in the proof of Theorem 7.1, it follows that $\dim \ker R(G', \hat{p}) \leq \dim \ker R(G, p) \leq 3$. □

Theorem 7.4 (Type II preserves injective generic rigidity, non-collinear case)**.** *Let G be a graph on a finite set V, and let p be an injective framework on V in $\mathbb{R}^2$ that is infinitesimally rigid for G. Let $a, b, c \in V$ be such that $p_b - p_a$ and $p_c - p_a$ are linearly independent, that is, p_a, p_b and p_c are not collinear. Let G' be obtained*

from G by the Type II move on ab and c. Then G' is injectively generically rigid in dimension 2.

Proof. For $\alpha \in \mathbb{R}$, let $q_\alpha = p_a + \alpha(p_b - p_a)$. Since $p_b - p_a \neq 0$, different values of α give different points q_α. So all but finitely many α satisfy $\alpha \neq 0, 1$ and $q_\alpha \neq p_u$ for every $u \in V$. Choose such an α, and let $q = q_\alpha$. The vectors $q - p_a = \alpha(p_b - p_a)$ and $q - p_c = \alpha(p_b - p_a) - (p_c - p_a)$ are linearly independent, since $p_b - p_a$ and $p_c - p_a$ are and $\alpha \neq 0$.

Let $\hat{p}$ be the framework that extends p by placing the new vertex at q. It is infinitesimally rigid for G' (Theorem 7.3). It is injective, since p is injective and q is none of the points p_u. □

The placement of the new vertex v on the line through p_a and p_b needs p_c off this line. If p_c lies on the line, then all three new edges at v point along it. Moving v perpendicular to the line, and no other vertex, is then an infinitesimal motion of G'. This motion is not trivial, since a trivial motion that vanishes at two distinct points vanishes everywhere. Jordán also uses openness, in his proof of Laman's theorem (Theorem 2.2.1, p. 45), to make the three neighbours of the new vertex of a 1-extension non-collinear.

Theorem 7.5 (Type II preserves injective generic rigidity in dimension 2)**.** *Let G be a graph on a finite set V that is injectively generically rigid in dimension* 2*, and let a, b and c be distinct vertices of G. Let G' be obtained from G by the Type II move on ab and c. Then G' is injectively generically rigid in dimension* 2.

Proof. Let p be an injective framework on V in $\mathbb{R}^2$ that is infinitesimally rigid for G. If $p_b - p_a$ and $p_c - p_a$ are linearly independent, then Theorem 7.4 applies to p.

Otherwise, since $p_b \neq p_a$ (as p is injective and $a \neq b$), we have $p_c - p_a = \delta(p_b - p_a)$ for some $\delta \in \mathbb{R}$. Choose a vector $w \in \mathbb{R}^2$ such that w and $p_b - p_a$ are linearly independent. For $t \in \mathbb{R}$, let p^t be the framework that agrees with p except at c, where $p^t_c = p_c + tw$. The map $t \mapsto p^t$ is continuous, and $p^0 = p$. Infinitesimal rigidity is an open condition on the framework (Lemma 6.17). So is injectivity, since V is finite. Hence p^t is injective and infinitesimally rigid for G for every t close enough to 0.

Since c is neither a nor b, the points of a and b do not move, and

$$p^t_b - p^t_a \;=\; p_b - p_a, \qquad p^t_c - p^t_a \;=\; \delta(p_b - p_a) + tw.$$

For $t \neq 0$ these two vectors are linearly independent, because w and $p_b - p_a$ are. Choose such a $t \neq 0$ close enough to 0, and apply Theorem 7.4 to p^t. □

8. Trivial infinitesimal motions

The translations and rotations of $\mathbb{R}^d$ give infinitesimal motions of every framework in $\mathbb{R}^d$, whatever the graph (Section 6). In this section we define the space $T(p)$ of these trivial motions at a framework p, and we show that it lies in the kernel of the rigidity map $R(G, p)$ for every graph G. We then show that $T(p)$ has dimension at least $d(d+1)/2$ when the points p_v affinely span $\mathbb{R}^d$, so that the kernel of $R(G, p)$ has at least this dimension too. By rank–nullity, the rank of $R(G, p)$ is then at most

$d|V|-d(d+1)/2$. The proof of Laman's theorem uses this bound in the plane, where $d(d+1)/2 = 3$ (Lemma 9.10).

We define the translations first, and then the infinitesimal rotations, which apply a skew-symmetric linear map to every point. The trivial motions are the linear combinations of the two. For the dimension bound we exhibit $d(d+1)/2$ trivial motions that are linearly independent: the translations along the d coordinate axes, and one infinitesimal rotation in each of the $d(d-1)/2$ coordinate planes. Whiteley [Whi96] (§11.1, before Lemma 11.1.2) generates the trivial motions in the same way, and notes that these generators are linearly independent once the points affinely span a hyperplane of $\mathbb{R}^d$. Jordán [Jor16] (p. 36) states the resulting dimension, $d(d+1)/2$. Our proof uses the stronger assumption that the points affinely span $\mathbb{R}^d$. We prove only the lower bound, since the rank bound needs no more; the upper bound is the count in Section 6.

Throughout, G is a graph on a set V, and $d \in \mathbb{N}$. An element x of $(\mathbb{R}^d)^V$ gives each vertex v a velocity $x_v \in \mathbb{R}^d$. It is an infinitesimal motion of G at a framework p when $R(G,p)(x) = 0$ (Definition 6.3). We write $e_1, \dots, e_d$ for the standard basis of $\mathbb{R}^d$, and y_i for the i-th coordinate of a vector $y \in \mathbb{R}^d$. We identify a linear map of $\mathbb{R}^d$ with its matrix in this basis, and we write E_{ij} for the matrix with entry 1 in row i and column j and all other entries 0. A linear map A of $\mathbb{R}^d$ is *skew-symmetric* if $\langle y, Ay\rangle = 0$ for every $y \in \mathbb{R}^d$. Applying this to $y+z$ shows that it holds exactly when $\langle y, Az\rangle = -\langle Ay, z\rangle$ for all $y, z \in \mathbb{R}^d$, that is, when $A^\top = -A$.

8.1. **Translations.** A translation of $\mathbb{R}^d$ moves every point with the same velocity.

Definition 8.1 (Translation motion)**.** Let $b \in \mathbb{R}^d$. The *translation* by b is the element τ_b of $(\mathbb{R}^d)^V$ with $(\tau_b)_v = b$ for every vertex v.

Lemma 8.2 (Translations are infinitesimal motions)**.** *Let p be a framework on V in $\mathbb{R}^d$, and let $b \in \mathbb{R}^d$. Then $\tau_b \in \ker R(G,p)$.*

Proof. At each edge uv of G, the entry of $R(G,p)(\tau_b)$ is $\langle p_u - p_v, b - b\rangle = 0$ (Lemma 6.4). □

8.2. **Infinitesimal rotations.** Let A be a skew-symmetric linear map of $\mathbb{R}^d$. The maps e^{tA}, for $t \in \mathbb{R}$, are rotations of $\mathbb{R}^d$, and the point $e^{tA}p_v$ has velocity Ap_v at $t = 0$. So the velocities Ap_v describe a rotation of the framework to first order.

Definition 8.3 (Infinitesimal rotation)**.** Let A be a linear map of $\mathbb{R}^d$, and let p be a framework on V in $\mathbb{R}^d$. We write $A \circ p$ for the element of $(\mathbb{R}^d)^V$ with $(A \circ p)_v = Ap_v$ for every vertex v. When A is skew-symmetric, $A \circ p$ is an *infinitesimal rotation* at p.

Lemma 8.4 (Infinitesimal rotations are infinitesimal motions)**.** *Let A be a skew-symmetric linear map of $\mathbb{R}^d$, and let p be a framework on V in $\mathbb{R}^d$. Then $A \circ p \in \ker R(G,p)$.*

Proof. At each edge uv of G, the entry of $R(G,p)(A \circ p)$ is $\langle p_u - p_v, Ap_u - Ap_v\rangle$ (Lemma 6.4). By linearity it equals $\langle p_u - p_v, A(p_u - p_v)\rangle$, which is zero since A is skew-symmetric. □

8.3. **The space of trivial motions.** In Section 6 the trivial motions at p are the motions $v \mapsto Ap_v + b$, with A skew-symmetric and $b \in \mathbb{R}^d$. We define them as a span.

Definition 8.5 (Trivial motions)**.** Let p be a framework on V in $\mathbb{R}^d$. The space $T(p)$ of *trivial infinitesimal motions* at p is the linear span in $(\mathbb{R}^d)^V$ of the translations τ_b, for $b \in \mathbb{R}^d$, and the infinitesimal rotations $A \circ p$, for skew-symmetric linear maps A of $\mathbb{R}^d$.

The translations form a linear subspace of $(\mathbb{R}^d)^V$, and so do the infinitesimal rotations at p. So $T(p)$ is their sum, which consists of the motions $v \mapsto Ap_v + b$ above.

Lemma 8.6 (Trivial motions are infinitesimal motions)**.** *Let p be a framework on V in $\mathbb{R}^d$. Then $T(p) \subseteq \ker R(G,p)$.*

Proof. The translations lie in $\ker R(G,p)$ (Lemma 8.2), and so do the infinitesimal rotations at p (Lemma 8.4). The kernel is a linear subspace, so it contains their span $T(p)$. □

8.4. **A lower bound on the dimension.** The points p_v affinely span $\mathbb{R}^d$ when their affine hull is all of $\mathbb{R}^d$, or equivalently when V is nonempty and the differences $p_v - p_w$, for $v, w \in V$, span $\mathbb{R}^d$. Under this condition we exhibit $d(d+1)/2$ linearly independent trivial motions. The rotations among them come from the following linear maps.

Definition 8.7 (Elementary skew-symmetric map)**.** Let $1 \le i, j \le d$. The *elementary skew-symmetric map* $E_{ij} - E_{ji}$ is the linear map of $\mathbb{R}^d$ that sends y to $y_j e_i - y_i e_j$.

For $i \neq j$, the map $E_{ij} - E_{ji}$ sends e_j to e_i and e_i to $-e_j$, and it vanishes on the other basis vectors. So the maps $e^{t(E_{ij}-E_{ji})}$ rotate the plane spanned by e_i and e_j, and they fix the other basis vectors. The map is zero when $i = j$, and exchanging i and j changes its sign.

Lemma 8.8 ($E_{ij} - E_{ji}$ is skew-symmetric)**.** *Let $1 \le i, j \le d$. Then $\langle y, (E_{ij} - E_{ji})y \rangle = 0$ for every $y \in \mathbb{R}^d$, that is, $E_{ij} - E_{ji}$ is skew-symmetric.*

Proof. For $y \in \mathbb{R}^d$, we have $\langle y,\, y_j e_i - y_i e_j \rangle = y_j y_i - y_i y_j = 0$. □

Since exchanging i and j only changes the sign of $E_{ij} - E_{ji}$, we use the pairs with $j < i$.

Definition 8.9 (Elementary trivial motions)**.** Let p be a framework on V in $\mathbb{R}^d$. The *elementary trivial motions* at p are the translations τ_{e_i}, for $1 \le i \le d$, and the motions $(E_{ij} - E_{ji}) \circ p$, for $1 \le j < i \le d$.

Lemma 8.10 (The elementary trivial motions lie in $T(p)$)**.** *Let p be a framework on V in $\mathbb{R}^d$. Then every elementary trivial motion at p lies in $T(p)$.*

Proof. The translations τ_{e_i} are among the generators of $T(p)$. So are the motions $(E_{ij} - E_{ji}) \circ p$, since $E_{ij} - E_{ji}$ is skew-symmetric (Lemma 8.8). □

Lemma 8.11 (Linear independence of the elementary trivial motions)**.** *Let p be a framework on V in $\mathbb{R}^d$ whose points affinely span $\mathbb{R}^d$. Then the elementary trivial motions at p are linearly independent in $(\mathbb{R}^d)^V$.*

Proof. Suppose that a linear combination of the elementary trivial motions vanishes, with coefficient c_i at τ_{e_i} and coefficient c_{ij} at $(E_{ij} - E_{ji}) \circ p$. Let $b = \sum_i c_i e_i$ and $S = \sum_{j<i} c_{ij}(E_{ij} - E_{ji})$. The combination is the motion $v \mapsto b + Sp_v$, so $b + Sp_v = 0$ for every vertex v.

Subtracting this equation at two vertices v and w gives $S(p_v - p_w) = 0$. Since the points affinely span $\mathbb{R}^d$, these differences span $\mathbb{R}^d$, so $S = 0$. For the same reason V is nonempty, and $b = -Sp_v = 0$ for any vertex v. The coordinates of b are the c_i, so every c_i is zero.

Let $j < i$. Among the matrices $E_{kl} - E_{lk}$ with $l < k$, only $E_{ij} - E_{ji}$ has a nonzero entry in row i and column j. So this entry of S is c_{ij}, and it is zero since $S = 0$. □

Lemma 8.12 (Dimension lower bound for trivial motions)**.** *Let V be finite, and let p be a framework on V in $\mathbb{R}^d$ whose points affinely span $\mathbb{R}^d$. Then*

$$\dim T(p) \;\geq\; \frac{d(d+1)}{2}.$$

Proof. The elementary trivial motions at p lie in $T(p)$ (Lemma 8.10). They are linearly independent (Lemma 8.11), so $\dim T(p)$ is at least their number. There are d translations, and for each i there are $i - 1$ pairs (i, j) with $j < i$. So there are $d + \sum_{i=1}^{d}(i-1) = d + d(d-1)/2 = d(d+1)/2$ elementary trivial motions. □

Lemma 8.13 (Dimension lower bound for the rigidity-map kernel)**.** *Let V be finite, and let p be a framework on V in $\mathbb{R}^d$ whose points affinely span $\mathbb{R}^d$. Then*

$$\dim \ker R(G,p) \;\geq\; \frac{d(d+1)}{2}.$$

Proof. The kernel of $R(G,p)$ contains $T(p)$ (Lemma 8.6). Since the points affinely span $\mathbb{R}^d$, the dimension of $T(p)$ is at least $d(d+1)/2$ (Lemma 8.12). □

9. Laman's theorem

In this section we prove Laman's theorem [PG27, Lam70] in the form stated in Section 4: a graph on at least two vertices is generically rigid in the plane if and only if it has a spanning subgraph that is Laman (Theorem 9.14). Jordán [Jor16] states it in an equivalent form as his Theorem 2.2.2: a graph is generically minimally rigid in the plane if and only if it is Laman.

We first specialize the edge count of Section 6 to the plane, and then prove the two directions in turn. That every Laman graph is generically rigid in the plane is proved by induction on the number of vertices, with the Henneberg moves of Section 5 and the fact that they preserve rigidity (Section 7). For the converse we study the rows of the rigidity map of a generically rigid graph G at a suitable framework. Some $2|V| - 3$ edges of G have linearly independent rows. A bound on the rank of the rigidity map, which comes from the trivial motions of Section 8, shows that these edges form a Laman graph. The last subsection combines the two directions.

Throughout, G is a graph on a set V, which is finite where a statement says so. Frameworks, the rigidity map $R(G,p)$, and infinitesimal and generic rigidity are as in Section 6. As there, we write uv for the edge with endpoints u and v.

9.1. **The edge count in the plane.** In the plane, the edge-count bound (Theorem 6.18) says that a generically rigid graph on V has at least $2|V|-3$ edges, as many as a Laman graph on V.

Lemma 9.1 (Edge-count bound in dimension 2)**.** *Let V be finite, and let G be generically rigid in dimension 2. Then*

$$2|V| \;\leq\; |E(G)|+3.$$

Proof. Specialize Theorem 6.18 to $d=2$, where $d(d+1)/2=3$. □

9.2. **The $\Leftarrow$ direction.** The sufficiency half of Laman's theorem [Lam70] says that every Laman graph is generically rigid in the plane. We prove it by induction on the number of vertices, along the reverse decomposition of Section 5.4. The step for each Henneberg move needs an injective framework (Section 7), so the induction proves the stronger statement for injective generic rigidity.

Theorem 9.2 (Every Laman graph is generically rigid in dimension 2)**.** *Let V be finite, and let G be a Laman graph on V. Then G is generically rigid in dimension 2.*

Proof. We show by strong induction on $|V|$ that G is injectively generically rigid in dimension 2 (Definition 6.8). An injectively generically rigid graph is generically rigid.

Since $|E(G)|+3=2|V|$, the set V has at least two elements. If $|V|=2$, then G is the complete graph on V (Lemma 4.4), which is isomorphic to K_2. The graph K_2 is injectively generically rigid in dimension 2 (Theorem 6.20). This property passes along isomorphisms (Lemma 6.16), so G has it too.

Let $|V|\geq 3$. The reverse decomposition (Theorem 5.12) gives a vertex v of G of degree two or three. Suppose first that v has degree two, with neighbours a and b, and that $G-v$ is Laman. The graph $G-v$ has $|V|-1$ vertices, so by induction it is injectively generically rigid. Hence so is the graph obtained from $G-v$ by the Type I move at a and b (Theorem 7.2). This graph is isomorphic to G, by the map that sends its new vertex to v and fixes every other vertex (Section 5.4). So G is injectively generically rigid (Lemma 6.16).

Otherwise v has degree three, with neighbours a, b and c, where a and b are not adjacent, and $G-v+ab$ is Laman. By induction, $G-v+ab$ is injectively generically rigid. Hence so is the graph obtained from it by the Type II move on ab and c (Theorem 7.5). This move deletes the edge ab again, and the graph it gives is isomorphic to G in the same way. So G is injectively generically rigid, as before. □

9.3. **The $\Rightarrow$ direction.** We now prove that every graph that is generically rigid in the plane contains a Laman spanning subgraph. This is the necessity half of Laman's theorem [Lam70]. We find the edges of the subgraph among the rows of the rigidity map. Let p be a framework on V in $\mathbb{R}^2$. A set I of edges is row-independent at p when the rows of $R(G,p)$ at the edges of I are linearly independent (Definition 9.3). We prove two facts about such sets.

First, if no three of the points p_v are collinear, then the edges of every row-independent set form a $(2,3)$-sparse graph (Lemma 9.12). The necessity of this count for minimal rigidity goes back to Maxwell [Max64]. Whiteley [Whi96] proves the count in every dimension as his d-Counting Lemma 11.1.3, and Jordán [Jor16] as his Lemma 1.3.1.

Second, if p is infinitesimally rigid for G, then $R(G,p)$ has rank at least $2|V|-3$. So some $2|V|-3$ edges are row-independent at p (see the proof of Lemma 9.9). Infinitesimal rigidity is an open condition (Lemma 6.17). So an infinitesimally rigid framework can be moved slightly to one that is still infinitesimally rigid and has no three points collinear (Lemma 9.11). There the $2|V|-3$ row-independent edges form a $(2,3)$-sparse graph with $2|V|-3$ edges, which is a Laman graph (Theorem 9.13).

Jordán (p. 39) calls the row-independent edge sets of a framework the independent sets of its rigidity matroid. For a generic framework in the plane, Pollaczek-Geiringer and Laman characterized them: an edge set is row-independent if and only if it forms a $(2,3)$-sparse graph (Jordán, Theorem 2.2.1, which he credits to Laman; Pollaczek-Geiringer proves the harder half in §8, p. 70). The first fact above is one half of this characterization, and Laman's theorem needs only this half. We prove the converse in Section 10, where the two halves identify the rigidity matroid in the plane with the $(2,3)$-count matroid.

9.3.1. *Row-independent edge sets.* Let p be a framework on V in $\mathbb{R}^d$. The row of $R(G,p)$ at an edge $e=uv$ is the linear functional $p' \mapsto R(G,p)(p')(e) = \langle p_u - p_v,\, p'_u - p'_v\rangle$ on $(\mathbb{R}^d)^V$. In the standard bases it is the row of Jordán's rigidity matrix indexed by e.

Definition 9.3 (Row-independent edge set at a placement)**.** Let p be a framework on V in $\mathbb{R}^d$, and let $I \subseteq E(G)$. We say that I is *row-independent at p* if the linear functionals $p' \mapsto R(G,p)(p')(e)$ on $(\mathbb{R}^d)^V$, for $e \in I$, are linearly independent.

9.3.2. *The rows of the rigidity map.* To find a large row-independent edge set, we use that the row rank of a matrix equals its column rank. For this we regard the rows of $R(G,p)$ as elements of the dual space $((\mathbb{R}^d)^V)^*$ of linear functionals on $(\mathbb{R}^d)^V$. The *transpose* of $R(G,p)$ is the linear map $R(G,p)^\top$ from the dual space of $\mathbb{R}^{E(G)}$ to $((\mathbb{R}^d)^V)^*$ that sends a functional φ to $\varphi \circ R(G,p)$. In the standard bases its matrix is the transpose of the rigidity matrix. When V is finite, $R(G,p)^\top$ has the same rank as $R(G,p)$.

Definition 9.4 (Rigidity row at a placement)**.** Let p be a framework on V in $\mathbb{R}^d$, and let e be an edge of G. The *rigidity row* of G at p indexed by e is the linear functional $r_e \in ((\mathbb{R}^d)^V)^*$ given by $r_e(p') = R(G,p)(p')(e)$.

Lemma 9.5 (The rigidity rows span the image of the transpose)**.** *Let G have finitely many edges, and let p be a framework on V in $\mathbb{R}^d$. Then the rigidity rows r_e, for $e \in E(G)$, span the image of $R(G,p)^\top$.*

Proof. For an edge e, let e^* be the coordinate functional of $\mathbb{R}^{E(G)}$ at e. Then $r_e = e^* \circ R(G,p) = R(G,p)^\top(e^*)$. Since $E(G)$ is finite, the functionals e^* form a basis of the dual space of $\mathbb{R}^{E(G)}$. So their images under $R(G,p)^\top$ span its image. □

Lemma 9.6 (Row-independence as linear independence of the rigidity rows)**.** *Let p be a framework on V in $\mathbb{R}^d$, and let $I \subseteq E(G)$. Then I is row-independent at p if and only if the rigidity rows r_e, for $e \in I$, are linearly independent in $((\mathbb{R}^d)^V)^*$.*

Proof. The rigidity row r_e is the functional of Definition 9.3 at the edge e. So both sides say that the same family of linear functionals is linearly independent. □

Formalization note 9.7. The formal definition of row-independence regards the rows as elements of the space of all real functions on $(\mathbb{R}^d)^V$. The rigidity rows are elements of its linear subspace $((\mathbb{R}^d)^V)^*$, in which the transpose of $R(G,p)$ takes its values. The lemma passes between the two: a family of vectors of a subspace is linearly independent in the subspace exactly when it is linearly independent in the whole space.

9.3.3. *A lower bound on the rank.* The proof of the edge-count bound (Theorem 6.18) first bounds the rank of the rigidity map from below, and then bounds the rank by the number of edges. We record the first step.

Lemma 9.8 (Rank lower bound for generically rigid graphs)**.** *Let V be finite, and let G be generically rigid in dimension d. Then there is a framework p on V in $\mathbb{R}^d$ with*

$$d\,|V| \;\le\; \operatorname{rank} R(G,p) + \frac{d(d+1)}{2}.$$

Proof. Let p be a framework that is infinitesimally rigid for G. By rank–nullity, $\dim(\mathbb{R}^d)^V$ is the rank of $R(G,p)$ plus the dimension of its kernel. Here $\dim(\mathbb{R}^d)^V = d\,|V|$ (Lemma 6.2). The kernel has dimension at most $d(d+1)/2$, since p is infinitesimally rigid (Definition 6.5). □

Lemma 9.9 (A row-independent set of $2|V|-3$ edges)**.** *Let V be finite, and let G be generically rigid in dimension 2. Then there are a framework p on V in $\mathbb{R}^2$ and an edge set $I \subseteq E(G)$ with $|I| = \max(2|V|-3,\,0)$ that is row-independent at p.*

Proof. Let p be a framework given by Lemma 9.8 at $d=2$, so that $R(G,p)$ has rank at least $2|V|-3$. The rigidity rows span the image of $R(G,p)^\top$ (Lemma 9.5). This image has the same dimension as the image of $R(G,p)$, since the transpose has the same rank. So there is a set B of edges whose rigidity rows are linearly independent and span the image of $R(G,p)^\top$, and $|B| \ge 2|V|-3$.

Let I be a subset of B with $\max(2|V|-3,\,0)$ elements. The rigidity rows r_e, for $e \in I$, are linearly independent. So I is row-independent at p (Lemma 9.6). □

9.3.4. *An upper bound on the rank.* When the points p_v affinely span $\mathbb{R}^d$, the kernel of $R(G,p)$ contains the space $T(p)$ of trivial motions, of dimension at least $d(d+1)/2$ (Lemma 8.13). By rank–nullity this bounds the rank from above. Jordán proves the same bound for every framework on at least $d+2$ vertices, as his Lemma 1.2.1 [Jor16]. In the plane the bound gives the sparsity of row-independent edge sets (Lemma 9.12).

Lemma 9.10 (Rank upper bound for affinely spanning frameworks)**.** *Let V be finite, and let p be a framework on V in $\mathbb{R}^d$ whose points affinely span $\mathbb{R}^d$. Then*

$$\operatorname{rank} R(G,p) + \frac{d(d+1)}{2} \;\le\; d\,|V|.$$

Proof. By rank–nullity, the rank of $R(G,p)$ plus the dimension of its kernel is $\dim(\mathbb{R}^d)^V$, which is $d\,|V|$ (Lemma 6.2). The kernel has dimension at least $d(d+1)/2$ (Lemma 8.13). □

9.3.5. *A perturbation lemma.* The following lemma gives the framework that the beginning of this subsection asks for, one that is still infinitesimally rigid and has no three points collinear. We state it in any dimension d and for any property that holds at all frameworks near a given one, since Section 10 applies it also to row-independence (Lemma 10.6).

Lemma 9.11 (A perturbation making every $d+1$ points affinely span $\mathbb{R}^d$)**.** *Let V be finite, let $d \in \mathbb{N}$, and let P be a property of frameworks on V in $\mathbb{R}^d$ that holds at every framework in some neighbourhood of a framework p_0. Then there is a framework p on V in $\mathbb{R}^d$ with property P such that, for every set S of at least $d+1$ vertices, the points p_v, $v \in S$, affinely span $\mathbb{R}^d$.*

Proof. Choose an injective map $\varphi : V \to \mathbb{R}$, for instance a numbering of the vertices by $0, 1, \dots, |V|-1$. Let $w \in (\mathbb{R}^d)^V$ place each vertex v on the moment curve, at $w_v = (\varphi(v), \varphi(v)^2, \dots, \varphi(v)^d)$. For $t \in \mathbb{R}$, let $p(t) = p_0 + t\,w$ be the framework that places v at $p(t)_v = (p_0)_v + t\,w_v$. The map $t \mapsto p(t)$ is continuous and $p(0) = p_0$, so there is an $\varepsilon > 0$ such that $p(t)$ has property P whenever $|t| < \varepsilon$.

Let $v_0, \dots, v_d$ be distinct vertices. The points $p(t)_{v_0}, \dots, p(t)_{v_d}$ are affinely independent if and only if the $d \times d$ matrix with rows $p(t)_{v_i} - p(t)_{v_0}$, for $1 \le i \le d$, has nonzero determinant. This determinant is a polynomial $f(t)$ in t of degree at most d. Its coefficient of t^d is the determinant of the matrix with rows $w_{v_i} - w_{v_0}$, whose entry in row i and column j is $\varphi(v_i)^j - \varphi(v_0)^j$. Subtract the first row of the Vandermonde matrix of $\varphi(v_0), \dots, \varphi(v_d)$ from the other rows, and expand along the first column. This shows that the coefficient of t^d is the Vandermonde determinant

$$\prod_{0 \le i < j \le d} (\varphi(v_j) - \varphi(v_i)),$$

which is nonzero because φ is injective. So f is a nonzero polynomial, and it has finitely many roots.

There are finitely many tuples $(v_0, \dots, v_d)$ of distinct vertices, so only finitely many t are roots of one of their polynomials. The interval $(0, \varepsilon)$ is infinite, so we can choose t in it that is a root of none of them. Let $p = p(t)$. Then p has property P. Let S be a set of at least $d+1$ vertices, and let $v_0, \dots, v_d$ be distinct elements of S. The points $p_{v_0}, \dots, p_{v_d}$ are affinely independent, so they affinely span $\mathbb{R}^d$, and hence so do the points p_v, $v \in S$. □

9.3.6. *Sparsity from row-independence.* We now prove the first of the two facts stated at the beginning of this subsection. Jordán derives his Lemma 1.3.1 from his rank bound, Lemma 1.2.1, and we argue in the same way: we apply the rank upper bound (Lemma 9.10) to the subgraph induced on each set of vertices. That bound needs the points of the framework to affinely span the plane, which is why we assume that no three of them are collinear.

Lemma 9.12 ($(2,3)$-sparsity from row-independence, dimension 2)**.** *Let p be a framework on V in $\mathbb{R}^2$ such that no three of the points p_v are collinear, that is, for every set S of at least three vertices the points p_v, $v \in S$, affinely span $\mathbb{R}^2$. Let $I \subseteq E(G)$ be row-independent at p. Then the spanning subgraph of G with edge set I is $(2,3)$-sparse.*

Proof. Let H be the spanning subgraph of G with edge set I. By Definition 3.1, we must show that $|E_H[S]| + 3 \leq 2|S|$ for every set S of at least two vertices. If $|S| = 2$, then S spans at most one edge, and $1 + 3 \leq 4$.

Let $|S| \geq 3$. Let $H[S]$ be the subgraph of H induced on S, whose edges are the edges in $E_H[S]$, and let $p|_S$ be the framework on S that places each $v \in S$ at p_v. For an edge e of $H[S]$, let r_e be the rigidity row of G at p indexed by e, and let r'_e be the rigidity row of $H[S]$ at $p|_S$ indexed by e (Definition 9.4). Let $\rho : (\mathbb{R}^2)^V \to (\mathbb{R}^2)^S$ be the restriction $x \mapsto x|_S$. If $e = uv$, then $u, v \in S$, and both $r_e(x)$ and $r'_e(\rho(x))$ equal $\langle p_u - p_v,\, x_u - x_v \rangle$. So $r_e = r'_e \circ \rho$.

The rows r_e, for $e \in E_H[S]$, are linearly independent, since $E_H[S] \subseteq I$ and I is row-independent at p (Lemma 9.6). Suppose that $\sum_e c_e r'_e = 0$. Composing with ρ gives $\sum_e c_e r_e = 0$, so every c_e is zero. Hence the rows r'_e are linearly independent. They span the image of $R(H[S], p|_S)^\top$ (Lemma 9.5), which has the same dimension as the image of $R(H[S], p|_S)$. So their number is the rank of $R(H[S], p|_S)$:

$$|E_H[S]| \;=\; \operatorname{rank} R(H[S], p|_S).$$

By hypothesis the points p_v, $v \in S$, affinely span $\mathbb{R}^2$. So this rank is at most $2|S| - 3$ (Lemma 9.10 at $d = 2$), and $|E_H[S]| + 3 \leq 2|S|$. □

9.3.7. *The Laman subgraph.* We now combine the two facts with the perturbation lemma.

Theorem 9.13 (Every generically rigid graph in the plane has a spanning Laman subgraph)**.** *Let V be finite with $|V| \geq 2$, and let G be generically rigid in dimension 2. Then G has a spanning subgraph that is Laman.*

Proof. Since G is generically rigid, some framework p_0 on V in $\mathbb{R}^2$ is infinitesimally rigid for G (Definition 6.7). Every framework in some neighbourhood of p_0 is infinitesimally rigid for G too (Lemma 6.17). So there is a framework p that is infinitesimally rigid for G and has no three of its points collinear (Lemma 9.11 at $d = 2$).

By rank–nullity, as in the proof of Lemma 9.8, the rank of $R(G, p)$ is at least $2|V| - 3$. As in the proof of Lemma 9.9, it follows that some set I of $2|V| - 3$ edges of G is row-independent at p; this number is positive, since $|V| \geq 2$. Let H be the spanning subgraph of G with edge set I. Then H is $(2,3)$-sparse (Lemma 9.12), and $|E(H)| + 3 = 2|V|$. So H is Laman (Definition 4.1). □

9.4. **Laman's theorem.** The two directions combine into Laman's theorem.

Theorem 9.14 (Laman's theorem)**.** *Let V be finite with $|V| \geq 2$, and let G be a graph on V. Then G is generically rigid in dimension 2 if and only if it has a spanning subgraph that is Laman.*

Proof. If G is generically rigid in dimension 2, then it has a spanning subgraph that is Laman (Theorem 9.13). Conversely, let H be a spanning subgraph of G that is Laman. Then H is generically rigid in dimension 2 (Theorem 9.2). Since $E(H) \subseteq E(G)$, so is G (Lemma 6.12). □

10. The planar rigidity matroid

In Section 9 we proved one half of the characterization of the independent sets of the rigidity matroid in the plane by Pollaczek-Geiringer [PG27] and Laman [Lam70]: an edge set that is row-independent at a framework in the plane with no three points collinear forms a $(2,3)$-sparse graph (Lemma 9.12). In this section we prove the other half: the edge set of every $(2,3)$-sparse graph is row-independent at some framework in the plane, which Pollaczek-Geiringer proves in her §8 (p. 70). The two halves together are Jordán's [Jor16] Theorem 2.2.1, which he credits to Laman, and we follow his proof (p. 45). We then describe the rigidity matroid in the plane as a matroid on the edges of a complete graph.

The proof is an induction that removes a vertex and puts it back by a Henneberg move (Section 5). A vertex whose removal leaves a smaller $(2,3)$-sparse graph exists by the reverse decomposition at the end of Section 3 (Theorem 3.19). We show in Section 10.1 that putting it back preserves row-independence at a suitable framework, and we complete the induction in Section 10.2.

For a finite set V, let K_V be the complete graph on V. In Section 10.3 we define the planar rigidity matroid $\mathcal{R}(V)$ by counting: its independent sets are the sets of edges of K_V that form $(2,3)$-sparse graphs. By the characterization, these are the edge sets that are row-independent at some framework in the plane. In Section 10.4 we show that $\mathcal{R}(V)$ is the rigidity matroid of a suitable framework of K_V.

Throughout, frameworks and the rigidity map $R(G,p)$ are as in Section 6, and we write uv for the edge with endpoints u and v. The rows r_e of $R(G,p)$ and row-independent edge sets are defined in Section 9.

Formalization note 10.1. The reverse decomposition (Theorem 3.19) describes the smaller graphs $G - v$ and $G - v + ab$ directly, and not as graphs from which a move rebuilds G. In the formalization the difference matters. A move produces a graph on a new vertex set (Formalization note 5.3), so the move applied to $G - v$ gives a graph isomorphic to G, not G itself. A reverse decomposition starts from G, and it is stated in terms of G. The lifts of Section 10.1, like the results of Section 7, are statements about what a move does, and they are stated for the graph that the move produces. Where a proof uses both kinds of result, as in Theorem 10.9, we pass between them by an isomorphism from G to the graph obtained by the move, built from the neighbours of v.

10.1. **Lifting row-independence through Henneberg moves.** In this subsection we prove the inductive step of the induction in Section 10.2: if $E(G)$ is row-independent at a framework p, and G' is obtained from G by a move, then $E(G')$ is row-independent at some framework. For the Type I move at a and b we assume that $p_a \neq p_b$. In Jordán's proof the step is his Lemmas 1.2.2 and 1.2.3 (p. 38): if the points involved are in general position, a 0-extension or a 1-extension in the plane

raises the rank of the rigidity matrix by two. As in Section 7, we first prove the statement for the framework $\hat{p}$ that extends a given framework p by placing the new vertex v at a given point q, and then choose q.

Lemma 10.2 (Row-independence under the Type I move)**.** *Let G be a graph on a set V, and let p be a framework on V in $\mathbb{R}^2$ at which $E(G)$ is row-independent. Let $a, b \in V$, and let G' be obtained from G by the Type I move at a and b, with new vertex v. Let $q \in \mathbb{R}^2$ be a point such that $q - p_a$ and $q - p_b$ are linearly independent, and let $\hat{p}$ be the framework on $V \cup \{v\}$ that agrees with p on V and places v at q. Then $E(G')$ is row-independent at $\hat{p}$.*

Proof. For an edge e of G, let r_e be the rigidity row of G at p indexed by e, and for an edge e of G', let $\hat{r}_e$ be the rigidity row of G' at $\hat{p}$ indexed by e (Definition 9.4). Since $q - p_a$ and $q - p_b$ are linearly independent, $a \neq b$. So the edges of G' are the edges of G and the two new edges va and vb (Lemma 5.4). Suppose that

$$\sum_{e \in E(G)} c_e \, \hat{r}_e + c_a \, \hat{r}_{va} + c_b \, \hat{r}_{vb} \;=\; 0.$$

We show that every coefficient is zero.

For $\alpha \in \mathbb{R}^2$, let $x^\alpha \in (\mathbb{R}^2)^{V \cup \{v\}}$ give v the velocity α and every vertex of G the velocity zero. If e is an edge of G, then $\hat{r}_e(x^\alpha) = 0$, since both ends of e have velocity zero (Lemma 6.4). At a new edge, $\hat{r}_{va}(x^\alpha) = \langle q - p_a, \, \alpha \rangle$. Evaluating the relation at x^α therefore gives $\langle c_a(q - p_a) + c_b(q - p_b), \, \alpha \rangle = 0$ for every α. Hence $c_a(q - p_a) + c_b(q - p_b) = 0$, and $c_a = c_b = 0$, since $q - p_a$ and $q - p_b$ are linearly independent.

It remains to show that $\sum_e c_e \, \hat{r}_e = 0$ forces every c_e to be zero. Let $\rho : (\mathbb{R}^2)^{V \cup \{v\}} \to (\mathbb{R}^2)^V$ be the restriction $x \mapsto x|_V$. If $e = uw$ is an edge of G, then both $\hat{r}_e(x)$ and $r_e(\rho(x))$ equal $\langle p_u - p_w, \, x_u - x_w \rangle$, so $\hat{r}_e = r_e \circ \rho$. Hence $(\sum_e c_e \, r_e) \circ \rho = 0$. Since ρ is surjective, $\sum_e c_e \, r_e = 0$. So every c_e is zero, because $E(G)$ is row-independent at p (Lemma 9.6). □

Lemma 10.3 (Type I lift of row-independence)**.** *Let G be a graph on a set V, and let p be a framework on V in $\mathbb{R}^2$ at which $E(G)$ is row-independent. Let $a, b \in V$ with $p_a \neq p_b$, and let G' be obtained from G by the Type I move at a and b, with new vertex v. Then there is a framework $\hat{p}$ on $V \cup \{v\}$ that agrees with p on V and at which $E(G')$ is row-independent.*

Proof. Since $p_a \neq p_b$, there is a unique line through p_a and p_b. Choose a point q off this line. Then $q - p_a$ and $q - p_b$ are linearly independent. Let $\hat{p}$ extend p by placing v at q. Then $E(G')$ is row-independent at $\hat{p}$ (Lemma 10.2). □

A vertex of degree one, a pendant vertex, is put back by the Type I move at a and a, which adds a new vertex v and the single edge va (Definition 5.1). Jordán's proof treats this case together with the Type I move.

Lemma 10.4 (Pendant lift of row-independence)**.** *Let G be a graph on a set V, let p be a framework on V in $\mathbb{R}^2$ at which $E(G)$ is row-independent, and let $a \in V$. Let G' be obtained from G by the Type I move at a and a, with new vertex v. Then*

there is a framework $\hat{p}$ on $V \cup \{v\}$ that agrees with p on V and at which $E(G')$ is row-independent.

Proof. Choose a point $q \neq p_a$, and let $\hat{p}$ extend p by placing v at q. Let r_e, $\hat{r}_e$ and x^α be as in the proof of Lemma 10.2. The edges of G' are the edges of G and the new edge va. Suppose that $\sum_{e \in E(G)} c_e \hat{r}_e + c_a \hat{r}_{va} = 0$. Evaluating this relation at x^α with $\alpha = q - p_a$ gives $c_a |q - p_a|^2 = 0$, so $c_a = 0$. Then $\sum_e c_e \hat{r}_e = 0$, and every c_e is zero, as in the proof of Lemma 10.2. □

Lemma 10.5 (Row-independence under the Type II move)**.** *Let G be a graph on a set V, let ab be an edge of G, and let p be a framework on V in $\mathbb{R}^2$ at which $E(G)$ is row-independent. Let $c \in V$, and let G' be obtained from G by the Type II move on ab and c, with new vertex v. Let $q = p_a + s\,(p_b - p_a)$ for a real number $s \neq 0, 1$, suppose that $q - p_a$ and $q - p_c$ are linearly independent, and let $\hat{p}$ be the framework on $V \cup \{v\}$ that agrees with p on V and places v at q. Then $E(G')$ is row-independent at $\hat{p}$.*

Proof. Let r_e, $\hat{r}_e$, x^α and ρ be as in the proof of Lemma 10.2. The vertices a, b and c are distinct: $a \neq b$ since ab is an edge, and $c \neq a, b$ since $q - p_a$ and $q - p_c$ are linearly independent. So the edges of G' are the edges of G other than ab and the three new edges va, vb and vc (Lemma 5.6). Suppose that

$$\sum_e c_e \hat{r}_e + c_a \hat{r}_{va} + c_b \hat{r}_{vb} + c_c \hat{r}_{vc} \;=\; 0,$$

where e runs over the edges of G other than ab.

Let $\delta = p_b - p_a$, so that $q - p_a = s\,\delta$ and $q - p_b = (s-1)\,\delta$. Evaluating the relation at x^α gives

$$\big\langle (sc_a + (s-1)c_b)\,\delta + c_c\,(q - p_c),\ \alpha \big\rangle \;=\; 0$$

for every α, so $(sc_a + (s-1)c_b)\,\delta + c_c\,(q - p_c) = 0$. Since $s \neq 0$, the vector δ is a multiple of $q - p_a$, and $q - p_a$ and $q - p_c$ are linearly independent. Hence $sc_a + (s-1)c_b = 0$ and $c_c = 0$.

We use the first equation to recover the row of the deleted edge ab, as in the proof of Theorem 7.3. For every x we have, by Lemma 6.4 and the first equation,

$$c_a \hat{r}_{va}(x) + c_b \hat{r}_{vb}(x) \;=\; sc_a \langle \delta,\, x_v - x_a \rangle + (s-1)c_b \langle \delta,\, x_v - x_b \rangle \;=\; sc_a \langle \delta,\, x_b - x_a \rangle,$$

and $\langle \delta,\, x_b - x_a \rangle = \langle p_a - p_b,\, x_a - x_b \rangle = r_{ab}(\rho(x))$. So $c_a \hat{r}_{va} + c_b \hat{r}_{vb} = sc_a\,(r_{ab} \circ \rho)$. The other rows satisfy $\hat{r}_e = r_e \circ \rho$, so the relation becomes

$$\Big(\sum_e c_e\, r_e + sc_a\, r_{ab} \Big) \circ \rho \;=\; 0.$$

Since ρ is surjective, $\sum_e c_e\, r_e + sc_a\, r_{ab} = 0$. As $E(G)$ is row-independent at p, every c_e is zero and $sc_a = 0$. So $c_a = 0$, since $s \neq 0$, and then $c_b = 0$, since $s \neq 1$. □

For the Type II move we move one point of the framework slightly, as in the proof of Theorem 7.5. This needs row-independence to be an open condition on the framework, as infinitesimal rigidity is (Lemma 6.17). In the induction we also transport row-independence along graph isomorphisms, as we did for infinitesimal rigidity (Lemma 6.14).

Lemma 10.6 (Row-independence is an open condition on the framework)**.** *Let G be a graph on a finite set V, let $I \subseteq E(G)$, and let p_0 be a framework on V in $\mathbb{R}^d$ at which I is row-independent. Then I is row-independent at every framework in some neighbourhood of p_0 in $(\mathbb{R}^d)^V$.*

Proof. Fix a basis $\xi_1, \dots, \xi_n$ of $(\mathbb{R}^d)^V$. A linear functional ℓ on $(\mathbb{R}^d)^V$ is determined by the vector $(\ell(\xi_1), \dots, \ell(\xi_n)) \in \mathbb{R}^n$, and the map sending ℓ to this vector is a linear isomorphism. For a framework p, let r_e be the rigidity row of G at p indexed by e (Definition 9.4). By Lemma 9.6 and this isomorphism, I is row-independent at p if and only if the vectors $(r_e(\xi_1), \dots, r_e(\xi_n))$, for $e \in I$, are linearly independent in $\mathbb{R}^n$.

For $e = uw$, the entry $r_e(\xi_k)$ is the inner product of $p_u - p_w$ with $(\xi_k)_u - (\xi_k)_w$ (Lemma 6.4), so it depends continuously on p. Linear independence of finitely many vectors in $\mathbb{R}^n$ is an open condition. Hence it holds for every p in some neighbourhood of p_0. □

Lemma 10.7 (Row-independence transports along a graph isomorphism)**.** *Let G and H be graphs on sets V and W, let φ be an isomorphism from G to H, and let q be a framework on W in $\mathbb{R}^d$. If $E(H)$ is row-independent at q, then $E(G)$ is row-independent at $q \circ \varphi$.*

Proof. Let $L : (\mathbb{R}^d)^V \to (\mathbb{R}^d)^W$ be the linear isomorphism $x \mapsto x \circ \varphi^{-1}$. For an edge $e = uv$ of G, let r_e be the rigidity row of G at $q \circ \varphi$ indexed by e, and let $r^H_{\varphi(e)}$ be the rigidity row of H at q indexed by the edge $\varphi(u)\varphi(v)$ (Definition 9.4). Both $r_e(x)$ and $r^H_{\varphi(e)}(L(x))$ equal $\langle q_{\varphi(u)} - q_{\varphi(v)},\, x_u - x_v \rangle$, so $r_e = r^H_{\varphi(e)} \circ L$.

Suppose that $\sum_e c_e\, r_e = 0$. Then $(\sum_e c_e\, r^H_{\varphi(e)}) \circ L = 0$, and so $\sum_e c_e\, r^H_{\varphi(e)} = 0$, since L is surjective. The map $e \mapsto \varphi(e)$ is a bijection from the edges of G to those of H, and $E(H)$ is row-independent at q. So every c_e is zero. □

Lemma 10.8 (Type II lift of row-independence)**.** *Let G be a graph on a finite set V, let ab be an edge of G, and let $c \in V$ be distinct from a and b. Let p be a framework on V in $\mathbb{R}^2$ at which $E(G)$ is row-independent, and let G' be obtained from G by the Type II move on ab and c, with new vertex v. Then there is a framework on $V \cup \{v\}$ at which $E(G')$ is row-independent.*

Proof. We shall place v on the line through p_a and p_b, which needs p_c off this line (Lemma 10.5). So we first move p, keeping $E(G)$ row-independent, until p_a, p_b and p_c are not collinear. The row r_{ab} of $R(G,p)$ is not zero, since $E(G)$ is row-independent at p. Hence $p_a \neq p_b$ (Lemma 6.4). If $p_b - p_a$ and $p_c - p_a$ are linearly independent, we keep p.

Otherwise $p_c - p_a = \gamma\,(p_b - p_a)$ for some $\gamma \in \mathbb{R}$. Choose a vector $w \in \mathbb{R}^2$ such that w and $p_b - p_a$ are linearly independent. For $t \in \mathbb{R}$, let p^t be the framework that agrees with p except at c, where $p^t_c = p_c + t\,w$. The map $t \mapsto p^t$ is continuous, and $p^0 = p$. So $E(G)$ is row-independent at p^t for every t close enough to 0 (Lemma 10.6). Since c is neither a nor b, we have $p^t_b - p^t_a = p_b - p_a$ and $p^t_c - p^t_a = \gamma\,(p_b - p_a) + t\,w$. For $t \neq 0$ these two vectors are linearly independent, because w and $p_b - p_a$ are. Choose $t \neq 0$ close enough to 0, and replace p by p^t.

Now let $q = p_a + \frac{1}{2}(p_b - p_a)$, the midpoint of p_a and p_b. Then $q - p_a = \frac{1}{2}(p_b - p_a)$ and $q - p_c = \frac{1}{2}(p_b - p_a) - (p_c - p_a)$. These two vectors are linearly independent,

because $p_b - p_a$ and $p_c - p_a$ are. So Lemma 10.5 applies with $s = \frac{1}{2}$, and $E(G')$ is row-independent at the framework that extends p by placing v at q. □

10.2. **Row-independent edge sets in the plane.** We now complete the induction. Together with Lemma 9.12, its result characterizes the edge sets that are row-independent at some framework in the plane.

Theorem 10.9 (Every $(2,3)$-sparse graph has a row-independent framework in the plane)**.** *Let V be finite, and let G be a $(2,3)$-sparse graph on V. Then there is a framework p on V in $\mathbb{R}^2$ at which $E(G)$ is row-independent.*

Proof. We use induction on $|V|$; Jordán's proof inducts on the number of edges instead. If G has no edges, for instance if $|V| \leq 1$, then $E(G)$ is row-independent at every framework, since the empty family of rows is linearly independent.

Suppose that G has an edge. By Theorem 3.19, G has a vertex v of degree one, two or three for which the following graph H on $V \setminus \{v\}$ is $(2,3)$-sparse. If v has degree one or two, with neighbour a or neighbours a and b, then $H = G - v$. If v has degree three, with neighbours a, b and c, where a and b are not adjacent, then $H = G - v + ab$. By induction, $E(H)$ is row-independent at some framework p on $V \setminus \{v\}$ in $\mathbb{R}^2$.

Let H' be obtained from H by the move that puts v back: the Type I move at a and a if v has degree one, the Type I move at a and b if v has degree two, and the Type II move on ab and c if v has degree three. The map φ that sends v to the new vertex of H' and fixes every other vertex is an isomorphism from G to H' (Formalization note 10.1). If $E(H')$ is row-independent at a framework $\hat{p}$, then $E(G)$ is row-independent at $\hat{p} \circ \varphi$ (Lemma 10.7). So it remains to find such a framework $\hat{p}$.

If v has degree one, then Lemma 10.4 gives $\hat{p}$. If v has degree three, then ab is an edge of H and c is distinct from a and b, so Lemma 10.8 gives $\hat{p}$.

Let v have degree two. If $p_a \neq p_b$, then Lemma 10.3 gives $\hat{p}$. Otherwise we first move the point p_a. Choose a nonzero vector $w \in \mathbb{R}^2$, and for $t \in \mathbb{R}$ let p^t be the framework that agrees with p except at a, where $p^t_a = p_a + t\,w$. The map $t \mapsto p^t$ is continuous, and $p^0 = p$. So $E(H)$ is row-independent at p^t for every t close enough to 0 (Lemma 10.6). Since $a \neq b$, we have $p^t_a - p^t_b = t\,w$, which is not zero for $t \neq 0$. Choose $t \neq 0$ close enough to 0. Then Lemma 10.3 applies to p^t and gives $\hat{p}$. □

Theorem 10.10 (Planar row-independence is $(2,3)$-sparsity (Pollaczek-Geiringer, Laman))**.** *Let V be finite, let G be a graph on V, and let $I \subseteq E(G)$. Then I is row-independent at some framework on V in $\mathbb{R}^2$ if and only if the spanning subgraph of G with edge set I is $(2,3)$-sparse.*

Proof. Suppose first that I is row-independent at a framework p_0. Then I is row-independent at every framework in some neighbourhood of p_0 (Lemma 10.6). So there is a framework p at which I is row-independent and no three of the points p_v are collinear (Lemma 9.11 at $d = 2$). Hence the spanning subgraph of G with edge set I is $(2,3)$-sparse (Lemma 9.12).

Conversely, let H be the spanning subgraph of G with edge set I, and suppose that H is $(2,3)$-sparse. Then $E(H) = I$ is row-independent at some framework p

(Theorem 10.9). For an edge $e = uv$ in I, the rigidity rows of H and of G at p indexed by e are the same functional $x \mapsto \langle p_u - p_v,\, x_u - x_v\rangle$ (Lemma 6.4). So I, as a set of edges of G, is row-independent at p. □

Corollary 10.11 (Every Laman graph has a row-independent framework in the plane)**.** *Let V be finite, and let G be a Laman graph on V. Then there is a framework p on V in $\mathbb{R}^2$ at which $E(G)$ is row-independent.*

Proof. A Laman graph is $(2,3)$-sparse (Definition 4.1), so Theorem 10.9 applies. □

10.3. **The rigidity matroid as a count matroid.** Let V be finite. For $\ell < 2k$, the sets of edges of K_V that form (k,ℓ)-sparse graphs are the independent sets of a matroid on $E(K_V)$, the (k,ℓ)-count matroid. We prove this in the next section (Section 11), and use it here for $(k,\ell) = (2,3)$, where $3 < 4$. We take the $(2,3)$-count matroid as the definition of the rigidity matroid in the plane, and then identify its independent sets with the edge sets that are row-independent at some framework.

Definition 10.12 (Planar rigidity matroid)**.** Let V be finite. The *planar rigidity matroid* $\mathcal{R}(V)$ is the $(2,3)$-count matroid on V (Definition 11.7): the matroid on $E(K_V)$ whose independent sets are the sets $I \subseteq E(K_V)$ for which the graph on V with edge set I is $(2,3)$-sparse.

Theorem 10.13 (Independent sets of the planar rigidity matroid)**.** *Let V be finite, and let $I \subseteq E(K_V)$. Then I is independent in $\mathcal{R}(V)$ if and only if I is row-independent at some framework on V in $\mathbb{R}^2$.*

Proof. Since $I \subseteq E(K_V)$, the set I is independent in $\mathcal{R}(V)$ if and only if the graph on V with edge set I is $(2,3)$-sparse (Theorem 11.9). This graph is the spanning subgraph of K_V with edge set I. So the claim is Theorem 10.10 at $G = K_V$. □

10.4. **The rigidity matroid of a framework.** Jordán (p. 39) defines the rigidity matroid of a framework (G,p) as the matroid on $E(G)$ whose independent sets are the edge sets that are row-independent at p, and notes that all generic frameworks of G have the same rigidity matroid. By Theorem 10.13, a set of edges of K_V is independent in $\mathcal{R}(V)$ exactly when it is row-independent at some framework, which may depend on the set. Rather than define genericity, we find a single framework p at which every independent set of $\mathcal{R}(V)$ is row-independent. This identifies $\mathcal{R}(V)$ with the rigidity matroid of the framework (K_V, p).

We now define the rigidity matroid of a framework of K_V, in any dimension. Its formal construction starts from the rigidity rows of K_V, extended to all unordered pairs of vertices.

Definition 10.14 (The rigidity rows extended to all pairs of vertices)**.** Let p be a framework on V in $\mathbb{R}^d$, and let r_e, for $e \in E(K_V)$, be the rigidity rows of K_V at p (Definition 9.4). The *extended rigidity rows* at p are the functionals $\tilde{r}_e \in ((\mathbb{R}^d)^V)^*$, for unordered pairs e of elements of V, given by $\tilde{r}_e = r_e$ if $e \in E(K_V)$ and $\tilde{r}_e = 0$ if $e = \{v, v\}$ for some $v \in V$.

Formalization note 10.15. The extension is needed only for the formal construction of the matroid in the next definition. That construction, `Matroid.ofFun` in Nelson's

matroid library, takes a function defined on all unordered pairs of vertices, together with a ground set. The pairs $\{v, v\}$ lie outside the ground set $E(K_V)$, so the values $\tilde{r}_{\{v,v\}} = 0$ do not affect the matroid.

Definition 10.16 (Linear rigidity matroid at a framework)**.** Let p be a framework on V in $\mathbb{R}^d$. The *linear rigidity matroid* $\mathcal{L}_p(V)$ at p is the matroid on $E(K_V)$ represented by the extended rigidity rows at p: a set $I \subseteq E(K_V)$ is independent if the functionals $\tilde{r}_e$, for $e \in I$, are linearly independent.

Lemma 10.17 (Independent sets of the linear rigidity matroid)**.** *Let p be a framework on V in $\mathbb{R}^d$, and let $I \subseteq E(K_V)$. Then I is independent in $\mathcal{L}_p(V)$ if and only if I is row-independent at p.*

Proof. For $e \in E(K_V)$, the extended row $\tilde{r}_e$ is the rigidity row r_e of K_V at p. So I is independent in $\mathcal{L}_p(V)$ if and only if the rows r_e, for $e \in I$, are linearly independent, that is, if and only if I is row-independent at p (Lemma 9.6). □

So $\mathcal{L}_p(V)$ is the rigidity matroid of the framework (K_V, p), and it remains to choose p.

Lemma 10.18 (A framework at which every $(2,3)$-sparse edge set is row-independent)**.** *Let V be finite. Then there is a framework p on V in $\mathbb{R}^2$ such that every $I \subseteq E(K_V)$ is row-independent at p whenever the graph on V with edge set I is $(2,3)$-sparse.*

Proof. Let $\mathcal{S}$ be the family of sets $I \subseteq E(K_V)$ for which the graph on V with edge set I is $(2,3)$-sparse. It is finite, since $E(K_V)$ is. We show, by induction on n, that for any n sets in $\mathcal{S}$ there is a framework at which all of them are row-independent; for $n = |\mathcal{S}|$ this is the lemma. For $n = 0$ any framework will do.

Let $\mathcal{F} \subseteq \mathcal{S}$ have n elements, let p_0 be a framework at which every set in $\mathcal{F}$ is row-independent, and let $I_0 \in \mathcal{S} \setminus \mathcal{F}$. Then I_0 is row-independent at some framework q (Theorem 10.10 at $G = K_V$). For $t \in \mathbb{R}$, let $p^t = p_0 + t\,(q - p_0)$, so that $p^0 = p_0$ and $p^1 = q$. The rigidity row of an edge uv at a framework p' is $x \mapsto \langle p'_u - p'_v,\, x_u - x_v \rangle$ (Lemma 6.4), which depends linearly on p'. So the row r_e^t of an edge e at p^t is $r_e^0 + t\, s_e$, where s_e is the row of e at $q - p_0$.

Let I be a set in $\mathcal{F}$, or I_0, and let t_0 be 0 or 1 accordingly, so that the rows $r_e^{t_0}$, for $e \in I$, are linearly independent. Fix a basis of $((\mathbb{R}^2)^V)^*$. The coordinate vectors of the rows r_e^t, for $e \in I$, are the rows of a matrix $A + t\,B$, where A and B do not depend on t. Since the rows of $A + t_0 B$ are linearly independent, some $|I| \times |I|$ submatrix of $A + t_0 B$ has nonzero determinant. The determinant of the same submatrix of $A + t\,B$ is a polynomial in t that is nonzero at t_0, so it has finitely many roots. At every other t, the rows r_e^t, for $e \in I$, are linearly independent.

Hence there are only finitely many $t \in \mathbb{R}$ at which some set in $\mathcal{F}$, or I_0, fails to be row-independent at p^t. Choose t outside these finitely many values. Then every set in $\mathcal{F}$, and I_0, is row-independent at p^t. □

Theorem 10.19 (The planar rigidity matroid as a linear matroid)**.** *Let V be finite. Then there is a framework p on V in $\mathbb{R}^2$ such that $\mathcal{L}_p(V) = \mathcal{R}(V)$.*

Proof. Let p be a framework as in Lemma 10.18. Both matroids have ground set $E(K_V)$, so it suffices to show that they have the same independent sets. Let $I \subseteq E(K_V)$. If I is independent in $\mathcal{L}_p(V)$, then I is row-independent at p (Lemma 10.17), and so I is independent in $\mathcal{R}(V)$ (Theorem 10.13). Conversely, if I is independent in $\mathcal{R}(V)$, then the graph on V with edge set I is $(2,3)$-sparse (Definition 10.12). So I is row-independent at p, by the choice of p, and it is independent in $\mathcal{L}_p(V)$. □

11. The (k,ℓ)-count matroid

Let V be a finite set, let K_V be the complete graph on V, and let k and ℓ be non-negative integers with $\ell < 2k$. We call a set $I \subseteq E(K_V)$ (k,ℓ)-*sparse* if the graph on V with edge set I is (k,ℓ)-sparse (Definition 3.1). In this section we show that the (k,ℓ)-sparse subsets of $E(K_V)$ are the independent sets of a matroid on $E(K_V)$, the (k,ℓ)-count matroid. In Section 10.3 we took its case $(k,\ell) = (2,3)$ as the definition of the planar rigidity matroid $\mathcal{R}(V)$. Theorem 10.19 identifies $\mathcal{R}(V)$ with the rigidity matroid of a framework of K_V.

The count matroid goes back to Lorea and, independently, to White and Whiteley. Whiteley [Whi96, Proposition A.1.1] proves it in the generality of hypergraphs and credits both. Lee and Streinu [LS08, Theorem 2] prove a version for multigraphs. Their proof rests on the structure of tight sets, which we also use.

Tight sets are as in Section 3, where the paragraph after Definition 3.4 relates them to Jordán's critical sets. Let G be a (k,ℓ)-sparse graph on V. Following Lee and Streinu, who speak of the induced subgraphs, we call a (k,ℓ)-tight set of G a *component* of G if no other (k,ℓ)-tight set of G contains it.

A family of subsets of a finite set is the family of independent sets of a matroid if it contains the empty set, is closed under taking subsets, and has the augmentation property: if I and J are members with $|I| < |J|$, then $I \cup \{e\}$ is a member for some $e \in J \setminus I$. For the (k,ℓ)-sparse subsets of $E(K_V)$ the first two properties are immediate. We prove the third in Section 11.1 (Lemma 11.6), and define the matroid in Section 11.2.

We sketch the proof of the augmentation property. Let I and J be (k,ℓ)-sparse with $|I| < |J|$, and let G be the graph on V with edge set I. Since $\ell < 2k$, two distinct vertices that lie in a tight set of G lie in exactly one component of G. So two distinct components share at most one vertex, and no edge lies inside both. Suppose that $I \cup \{e\}$ is (k,ℓ)-sparse for no $e \in J \setminus I$. Then the endpoints of each edge of $J \setminus I$ lie in a tight set of G, and hence in a component. Inside each of these components, J has at most as many edges as I, since the component is tight in G and J is sparse. Every other edge of J is an edge of I. Hence $|J| \leq |I|$, a contradiction.

Lee and Streinu also decide whether a graph is (k,ℓ)-sparse by an algorithm, the (k,ℓ)-pebble game, which we formalize in Section 13.

11.1. **Components and augmentation.** The first lemma is the case $\ell < 2k$ of Theorem 3.14, and it follows from Lee and Streinu's Theorem 5(1) [LS08]. From it we obtain the component that contains a given set of at least two vertices. We then show that two distinct components share at most one vertex, which is their Theorem 5(2). These lemmas and Lemma 11.2 give the augmentation property.

Lemma 11.1 (Tight sets that share two vertices)**.** *Let V be finite, let $\ell < 2k$, and let G be a (k,ℓ)-sparse graph on V. Let $s, t \subseteq V$ be (k,ℓ)-tight in G, and suppose that two distinct vertices u and v lie in both s and t. Then $s \cup t$ and $s \cap t$ are (k,ℓ)-tight in G.*

Proof. Since u and v lie in $s \cap t$, we have $|s \cap t| \geq 2$, and so $\ell < 2k \leq k\,|s \cap t|$. Hence Theorem 3.14 applies. □

The next lemma is one direction of Lee and Streinu's Corollary 6 [LS08].

Lemma 11.2 (An edge that cannot be added lies in a tight set)**.** *Let G be a (k,ℓ)-sparse graph on a set V, which need not be finite, and let u and v be distinct vertices that are not adjacent in G. Let $G + uv$ be obtained from G by adding the edge uv. If $G + uv$ is not (k,ℓ)-sparse, then some set containing u and v is (k,ℓ)-tight in G.*

Proof. Since $G+uv$ is not sparse, there is a set $S \subseteq V$ with $\ell \leq k\,|S|$ and $|E_{G+uv}[S]| + \ell > k\,|S|$ (Definition 3.1). If u or v lies outside S, then $G + uv$ and G have the same edges inside S, and this contradicts the sparsity of G at S.

So u and v lie in S. Since uv is not an edge of G, the set S spans exactly one more edge of $G + uv$ than of G. Hence $k\,|S| < |E_G[S]| + 1 + \ell$, while the sparsity of G gives $|E_G[S]| + \ell \leq k\,|S|$. So $|E_G[S]| + \ell = k\,|S|$, that is, S is tight in G. □

By Lemma 11.1, the tight sets of a sparse graph that contain two given vertices are closed under union. We therefore consider the union of all tight sets that contain a given set of vertices.

Definition 11.3 (The union of the tight sets containing a set)**.** Let V be finite, let G be a graph on V, and let $X \subseteq V$. We write $C_G(X)$ for the union of all sets that contain X and are (k,ℓ)-tight in G. If there is no such set, then $C_G(X)$ is empty.

Lemma 11.4 (The union of the tight sets containing a set is tight)**.** *Let V be finite, let $\ell < 2k$, and let G be a (k,ℓ)-sparse graph on V. Let $X \subseteq V$ have at least two elements, and suppose that some (k,ℓ)-tight set of G contains X. Then $C_G(X)$ is (k,ℓ)-tight in G.*

Proof. Let $\mathcal{F}$ be the family of tight sets of G that contain X. It is finite, since V is, and it is non-empty by hypothesis. Choose distinct vertices u and v in X. Every member of $\mathcal{F}$ contains u and v, so the union of two members of $\mathcal{F}$ is tight (Lemma 11.1) and is again a member of $\mathcal{F}$. By induction on the number of members, the union of all members of $\mathcal{F}$ is a member of $\mathcal{F}$. This union is $C_G(X)$, which is therefore tight. □

Under these hypotheses, $C_G(X)$ is therefore the largest tight set of G that contains X. It is a component of G: a tight set that contains $C_G(X)$ also contains X, and so it is contained in $C_G(X)$.

Lemma 11.5 (Distinct components share at most one vertex)**.** *Let V be finite, let $\ell < 2k$, and let G be a (k,ℓ)-sparse graph on V. Let $X, Y \subseteq V$ each have at least two elements. Suppose that some (k,ℓ)-tight set of G contains X, and that $Y \subseteq C_G(X)$. Then $C_G(Y) = C_G(X)$. In particular, two distinct components of G share at most one vertex, and so no edge.*

Proof. The set $C_G(X)$ is tight (Lemma 11.4). It contains Y, so it is one of the sets whose union is $C_G(Y)$. Hence $C_G(X) \subseteq C_G(Y)$. Since some tight set contains Y, the set $C_G(Y)$ is tight as well (Lemma 11.4). Both $C_G(X)$ and $C_G(Y)$ contain Y, which has at least two elements, so their union is tight (Lemma 11.1). This union contains X, so it is contained in $C_G(X)$. Hence $C_G(Y) \subseteq C_G(X)$.

For the last claim, let C be a component of G with at least two vertices. Then $C \subseteq C_G(C)$, and $C_G(C)$ is tight (Lemma 11.4). Since no other tight set contains C, we have $C = C_G(C)$. If distinct vertices u and v lie in C, then the first claim, with $X = C$ and $Y = \{u, v\}$, gives $C = C_G(\{u, v\})$. So two components that share u and v are both equal to $C_G(\{u, v\})$. □

We can now prove the augmentation property.

Lemma 11.6 (Augmentation)**.** *Let V be finite, let $\ell < 2k$, and let $I, J \subseteq E(K_V)$ be (k, ℓ)-sparse with $|I| < |J|$. Then there is an edge $e \in J \setminus I$ such that $I \cup \{e\}$ is (k, ℓ)-sparse.*

Proof. Let G and H be the graphs on V with edge sets I and J, so that G and H are (k, ℓ)-sparse. Suppose, to the contrary, that $I \cup \{e\}$ is (k, ℓ)-sparse for no $e \in J \setminus I$. Let $e = uv$ be an edge in $J \setminus I$. Since $e \notin I$, the vertices u and v are not adjacent in G, and $G + uv$ is the graph on V with edge set $I \cup \{e\}$. It is not sparse, so some tight set of G contains u and v (Lemma 11.2). Let $C_e = C_G(\{u, v\})$. This set contains u and v, and it is tight in G (Lemma 11.4).

Let $\mathcal{C}$ be the family of the sets C_e, for $e \in J \setminus I$. An edge xy of K_V lies inside at most one member of $\mathcal{C}$. Indeed, if x and y lie in C_e, then $C_G(\{x, y\}) = C_e$ (Lemma 11.5). Let I_0 and J_0 be the sets of edges of I and of J that lie inside no member of $\mathcal{C}$. Then I is the disjoint union of I_0 and the sets $E_G[C]$, for $C \in \mathcal{C}$, and likewise for J and H. So

$$|I| = \sum_{C \in \mathcal{C}} |E_G[C]| + |I_0| \quad \text{and} \quad |J| = \sum_{C \in \mathcal{C}} |E_H[C]| + |J_0|.$$

We compare the two sums term by term. Every edge e of $J \setminus I$ lies inside C_e. So every edge of J_0 is an edge of I, and $J_0 \subseteq I_0$. Let $C \in \mathcal{C}$. Since C is tight in G, we have $|E_G[C]| + \ell = k\,|C|$, and in particular $\ell \leq k\,|C|$. So the sparsity of H at C gives

$$|E_H[C]| + \ell \;\leq\; k\,|C| \;=\; |E_G[C]| + \ell.$$

Summing, we get $|J| \leq |I|$, which contradicts $|I| < |J|$. □

11.2. **The matroid.** The (k, ℓ)-sparse subsets of $E(K_V)$ satisfy the independence axioms of a matroid. The empty set is (k, ℓ)-sparse, since no set of vertices spans an edge of the graph on V with no edges (Lemma 2.3). A subset of a (k, ℓ)-sparse set is (k, ℓ)-sparse (Lemma 3.5). The augmentation property is Lemma 11.6.

Definition 11.7 ((k, ℓ)-count matroid)**.** Let V be finite, and let k and ℓ be non-negative integers with $\ell < 2k$. The *(k, ℓ)-count matroid* on V is the matroid on $E(K_V)$ whose independent sets are the sets $I \subseteq E(K_V)$ for which the graph on V with edge set I is (k, ℓ)-sparse.

Formalization note 11.8. Formally, the count matroid is a matroid on the set of all unordered pairs of elements of V, the pairs $\{v, v\}$ included, with ground set $E(K_V)$. So a set of pairs is independent only if it is contained in $E(K_V)$.

Theorem 11.9 (Independent sets of the count matroid)**.** *Let V be finite, let k and ℓ be non-negative integers with $\ell < 2k$, and let I be a set of unordered pairs of elements of V. Then I is independent in the (k, ℓ)-count matroid on V if and only if $I \subseteq E(K_V)$ and the graph on V with edge set I is (k, ℓ)-sparse.*

Proof. The independent sets of the count matroid are defined to be these sets (Definition 11.7). □

12. Verified depth-first search

The pebble game of Section 13 keeps an orientation of the edges it has accepted, with pebbles on the vertices. To bring a pebble to a vertex u, it looks for a directed path from u to a vertex that has a free pebble, and it reverses the arcs of that path (Definition 13.9). Lee and Streinu find such a path by depth-first search [LS08, §3]. In this section we define a search of this kind and prove it correct.

The search works with any relation R on a finite set V, given by a list of out-neighbours at each vertex, and with any predicate P on V. Started at a vertex v, it follows walks from v along R depth first, trying the out-neighbours of a vertex in the order of its list. It marks each vertex it enters and never enters a marked vertex again, and it stops at the first vertex that satisfies P. It then returns that vertex together with the walk from v to it. If it finds no such vertex, it fails. The section uses nothing from the earlier sections.

We first define directed walks (Definition 12.1) and the search (Definition 12.2). We then prove that the search is sound and complete (Theorem 12.4): a returned walk is simple and ends at a vertex satisfying P, and the search fails only if no vertex reachable from v satisfies P. In Section 12.1 we compute the set of vertices reachable from v, by running the search once for each vertex. The pebble game uses this set when it rejects an edge (Definition 13.12).

Definition 12.1 (Directed walk)**.** Let R be a relation on a set V, whose elements we call vertices. A *directed R-walk* from u to w is a sequence $u = v_0, v_1, \dots, v_m = w$ of vertices, with $m \geq 0$, such that $(v_{i-1}, v_i) \in R$ for $1 \leq i \leq m$. Its *length* is m. The walk is *simple* if $v_0, \dots, v_m$ are distinct.

We say that w is *reachable* from u along R if (u, w) lies in the reflexive-transitive closure of R, that is, if there is a directed R-walk from u to w.

We define the search with an arbitrary set of vertices marked at the start, since we prove its correctness by induction over all such sets.

Definition 12.2 (Depth-first reachability search)**.** Let V be a finite set, and let each vertex $a \in V$ have a list $N^+(a)$ of vertices, its out-neighbours. Let R be the relation with $(a, b) \in R$ when b appears in $N^+(a)$, and let P be a predicate on V. For a set $S \subseteq V$ of *marked* vertices and a vertex x, the *search for P from x with S marked* returns a set $S' \supseteq S$ of marked vertices, together with either a vertex w and a walk from x to w, or a failure. It runs as follows.

(1) If $x \in S$, the search fails, with $S' = S$.
(2) Otherwise, if $P(x)$ holds, it returns x and the walk of length 0 at x, with $S' = S$.
(3) Otherwise, let $y_1, \dots, y_k$ be the out-neighbours of x in the order of $N^+(x)$, and let $S_0 = S \cup \{x\}$. For $i = 1, \dots, k$ in turn, it runs the search for P from y_i with S_{i-1} marked. If this search returns a vertex w and a walk p from y_i to w, it stops, and returns w, the walk from x that steps to y_i and follows p, and the set that this search returned. If it fails, with the set S_i, the search goes on to y_{i+1}. If all k of these searches fail, the search fails, with $S' = S_k$.

The *search for P from v* is the case $S = \emptyset$ and $x = v$, without its set S'.

Formalization note 12.3. The out-neighbours are given as lists, not as finite sets, so that the search can be computed. A finite set has no preferred order of its elements, and mathlib lists them by a choice that cannot be computed; the search would inherit that choice. The marked set can be a finite set, since the search only tests whether a vertex lies in it and never lists its elements. For these tests the formal search assumes that equality of vertices can be decided, and it takes P as a function from V to $\{\text{true}, \text{false}\}$.

The search terminates, since each search started in step 3 starts with a set that contains $S \cup \{x\}$, so with fewer unmarked vertices than the search that starts it, and V is finite. A vertex that the search enters is marked, it stays marked for the rest of the search, and the search does not enter a marked vertex again. So the search from v enters each vertex at most once, and starts at most one search for each entry of each list $N^+(a)$.

Theorem 12.4 (Correctness of the search)**.** *Let V, R and P be as in Definition 12.2, and let $v \in V$.*

(1) (Soundness) If the search for P from v returns a vertex w and a walk p, then p is a simple directed R-walk from v to w, and $P(w)$ holds.
(2) (Completeness) If some vertex reachable from v along R satisfies P, then the search for P from v does not fail.

Proof. For each part we prove a statement about the search for P from x with S marked, for every set S and vertex x, by induction on the number of unmarked vertices. The part follows from the case $S = \emptyset$ and $x = v$.

For soundness we show more: if the search for P from x with S marked returns w and p, then $P(w)$ holds, p is simple, and no vertex of p lies in S. If the search returns in step 2, then $w = x$, $P(x)$ holds, $x \notin S$, and p has the single vertex x. Otherwise it returns in step 3, and p steps from x to an out-neighbour y_i and then follows a walk p', which the search from y_i with S_{i-1} marked returned. By induction, $P(w)$ holds, p' is simple, and no vertex of p' lies in S_{i-1}, which contains $S \cup \{x\}$. Since $x \notin S$, the walk p is simple and avoids S.

For completeness we show: if the search for P from x with S marked fails, with the set S', then $x \in S'$, and every vertex of $S' \setminus S$ fails P and has all its out-neighbours in S'. If $x \in S$, then $S' = S$, which contains x. If $P(x)$ holds, the search does not fail. So let $x \notin S$, and let $P(x)$ fail. Then the searches from $y_1, \dots, y_k$ in step 3 all

fail, with sets $S_0 \subseteq S_1 \subseteq \cdots \subseteq S_k = S'$. By induction, $y_i \in S_i \subseteq S'$, and every vertex of $S_i \setminus S_{i-1}$ fails P and has all its out-neighbours in $S_i \subseteq S'$. Since $S_0 = S \cup \{x\}$, a vertex of $S' \setminus S$ is either x, which fails P and whose out-neighbours are $y_1, \ldots, y_k$, or a vertex of $S_i \setminus S_{i-1}$ for some i. And $x \in S_0 \subseteq S'$.

This gives completeness. If the search for P from v fails, with the set S', then $v \in S'$, and every vertex of S' fails P and has all its out-neighbours in S', since nothing was marked at the start. By induction on the length of a directed R-walk from v, every vertex reachable from v lies in S', so none satisfies P. □

12.1. **Reachability regions.** Lee and Streinu call the set of vertices reachable from a vertex v its reachability region, and write it $\mathrm{Reach}(v)$ [LS08, §4]. When the pebble game rejects an edge uv, its witness is the union of the reachability regions of u and v (Definition 13.12). We compute the reachability region of v by running the search from v once for each vertex w, with the predicate that holds at w alone. Since the search can be computed, so can the region.

Definition 12.5 (Reachability region)**.** Let V and R be as in Definition 12.2, and let $v \in V$. For $w \in V$, let P_w be the predicate that holds at w and nowhere else. The *reachability region* $\mathrm{Reach}_R(v)$ is the set of vertices $w \in V$ for which the search for P_w from v does not fail.

Lemma 12.6 (Membership in the reachability region)**.** *Let V and R be as in Definition 12.2, and let $v, w \in V$. Then $w \in \mathrm{Reach}_R(v)$ if and only if w is reachable from v along R.*

Proof. Suppose that $w \in \mathrm{Reach}_R(v)$. Then the search for P_w from v returns a vertex w' and a walk. By part (1) of Theorem 12.4, $P_w(w')$ holds, so $w' = w$, and the walk is a directed R-walk from v to w. Hence w is reachable from v.

Conversely, suppose that w is reachable from v. Then w is a vertex reachable from v that satisfies P_w. By part (2) of Theorem 12.4, the search for P_w from v does not fail, so $w \in \mathrm{Reach}_R(v)$. □

13. The (k, ℓ)-pebble game

Lee and Streinu [LS08] decide whether a graph is (k, ℓ)-sparse by an algorithm, the *(k, ℓ)-pebble game*. It generalizes the pebble game of Jacobs and Hendrickson [JH97] for rigidity in the plane, which in Lee and Streinu's terms is the $(2, 3)$-pebble game [LS08, p. 1426]. The game considers the edges of the input graph one at a time. It keeps the accepted edges oriented, with pebbles on the vertices, and it accepts an edge when at least $\ell+1$ free pebbles can be gathered on its two endpoints. We formalize Lee and Streinu's basic pebble game [LS08, §3] on simple graphs, for $\ell < 2k$, the range they consider. We prove that it accepts every edge of a graph G if and only if G is (k, ℓ)-sparse. When it rejects an edge, it returns a set of vertices that shows that G is not (k, ℓ)-sparse.

In Section 13.1 we describe the state of the game, a partial orientation from which the free pebbles are counted, and its two moves, which reverse a directed path and insert an arc. In Section 13.2 we prove Lee and Streinu's four invariants for the orientations that these moves reach. In Section 13.3 we define the algorithm; it

finds pebbles with the depth-first search of Section 12. We prove the two directions of correctness in Section 13.4 and Section 13.5, and combine them in Section 13.6. The first direction follows from the fourth invariant. For the second, we show that when the game rejects an edge, the set V' of vertices reachable from its endpoints spans more than $k|V'| - \ell$ edges of G. In the formalization, the game returns its answer together with these proofs, as a verdict (Section 13.7). In Section 13.8 we return to the (k, ℓ)-count matroid of Section 11: the game decides whether the edge set of a graph is independent in it, and so, at $(k, \ell) = (2, 3)$, in the planar rigidity matroid.

Lee and Streinu play the game on multigraphs, which may have loops and parallel edges. On simple graphs their invariants simplify (Formalization note 13.8). We do not formalize their component pebble game (§5). It maintains the components of the accepted graph (Section 11), rejects an edge at once when both its endpoints lie in one component, and runs in time $O(n^2)$ on n vertices [LS08, p. 1436]. For the details of its implementation, Lee and Streinu refer to Lee, Streinu and Theran [LST05], who introduce for them a data structure problem, union pair-find. Nor do we formalize the Henneberg sequences of Lee and Streinu's §6, which extend the Henneberg construction (Section 5) from Laman graphs to tight graphs.

Lee and Streinu bound the running time of their basic game by $O(m_a n)$, where m_a is the number of accepted edges, counting at most $\ell + 1$ depth-first searches for each edge [LS08, p. 1430]. Our pebble search enters each vertex at most once (Section 12), but we prove no bound on the running time of our game.

13.1. **State and moves.** Lee and Streinu play the pebble game on a directed graph D on V, which starts with no edges and with k pebbles on each vertex [LS08, §3]. An edge is inserted into D oriented away from an endpoint that holds a pebble, and that pebble is removed. On a simple graph, D has no loops and at most one arc between two vertices, so it is a partial orientation (Definition 13.1). By Lee and Streinu's Invariant (1) [LS08, Lemma 10], the number of free pebbles at a vertex is then k minus its out-degree. So we do not keep the pebbles separately, and define them from D (Definition 13.2). The game changes D by two moves. It reverses a directed path, which moves a pebble from the end of the path to its start (Definition 13.4). It also inserts arcs (Definition 13.5).

Definition 13.1 (Partial orientation)**.** Let V be a set. A *partial orientation* on V is a finite set $D \subseteq V \times V$ such that:

(1) (no loops) $(v, v) \notin D$ for every $v \in V$;
(2) (no antiparallel pairs) if $(u, v) \in D$, then $(v, u) \notin D$.

The elements of D are its *arcs*, and an arc (u, v) has *tail* u and *head* v. An edge uv of the complete graph K_V is *oriented* in D if $(u, v) \in D$ or $(v, u) \in D$. The *underlying edge set* $\underline{D} \subseteq E(K_V)$ is the set of edges oriented in D. The *empty orientation* has no arcs.

Definition 13.2 (Out-degree, free pebbles and span)**.** Let D be a partial orientation on V, and let k be a non-negative integer. For a vertex v, the *out-degree* $\mathrm{out}_D(v)$ is

the number of arcs of D with tail v, and the number of *free pebbles* at v is

$$\mathrm{peb}_{D,k}(v) = \max(k - \mathrm{out}_D(v), 0).$$

For a finite set $V' \subseteq V$, let

- $\mathrm{peb}_{D,k}(V') = \sum_{v \in V'} \mathrm{peb}_{D,k}(v)$;
- $\mathrm{span}_D(V')$ be the number of arcs of D with both ends in V';
- $\mathrm{out}_D(V')$ be the number of arcs of D with tail in V' and head outside V'.

These are Lee and Streinu's functions peb, span and out [LS08, §4], for a graph without loops. The maximum with 0 matters only when $\mathrm{out}_D(v) > k$. This does not happen at the orientations the game reaches, the (k, ℓ)-reachable ones of Definition 13.6, as Invariant (1) of Lemma 13.7 shows.

Formalization note 13.3. In order, the five pins define $\mathrm{out}_D(v)$, $\mathrm{peb}_{D,k}(v)$, $\mathrm{span}_D(V')$, $\mathrm{out}_D(V')$ and $\mathrm{peb}_{D,k}(V')$.

Definition 13.4 (Path reversal)**.** Let D be a partial orientation on V. A *directed path* in D from u_0 to u_m is a simple directed walk $u_0, u_1, \ldots, u_m$ along the arcs of D (Definition 12.1): the vertices u_i are distinct, and $(u_{i-1}, u_i) \in D$ for $1 \le i \le m$. *Reversing* the path replaces D by the partial orientation

$$D' = (D \setminus \{(u_{i-1}, u_i) : 1 \le i \le m\}) \cup \{(u_i, u_{i-1}) : 1 \le i \le m\}.$$

The underlying edge sets of D' and D are equal.

The set D' is a partial orientation. It has no loops, since the vertices of the path are distinct. Each arc of D' is either an arc of D off the path or the reverse (u_i, u_{i-1}) of an arc of the path. Two arcs of D off the path are not antiparallel, since D has no antiparallel pairs. The arc (u_i, u_{i-1}) is not antiparallel to an arc of D off the path, since (u_{i-1}, u_i) is on the path. Nor is it antiparallel to the reverse of another arc of the path, since a path with distinct vertices does not use the edge $u_{i-1}u_i$ twice.

Let $m \ge 1$. Reversing the path lowers $\mathrm{out}_D(u_0)$ by 1, raises $\mathrm{out}_D(u_m)$ by 1, and leaves every other out-degree unchanged: an inner vertex u_i loses the arc (u_i, u_{i+1}) and gains the arc (u_i, u_{i-1}). Suppose that $\mathrm{out}_D(u_0) \le k$ and $\mathrm{out}_D(u_m) < k$. Then u_0 gains a free pebble and u_m loses one, so the move takes a pebble from u_m to u_0. This is the move by which Lee and Streinu collect a pebble [LS08, §3, Fig. 4].

Definition 13.5 (Arc insertion)**.** Let D be a partial orientation on V, and let $u \ne v$ be vertices with $(v, u) \notin D$. *Inserting* the arc (u, v) replaces D by the partial orientation $D \cup \{(u, v)\}$. Its underlying edge set is $\underline{D} \cup \{uv\}$.

The set $D \cup \{(u, v)\}$ is a partial orientation, since $u \ne v$ rules out a loop and $(v, u) \notin D$ rules out an antiparallel pair. If the arc is new, that is, if $(u, v) \notin D$, the insertion raises $\mathrm{out}_D(u)$ by 1 and leaves every other out-degree unchanged. If moreover $\mathrm{out}_D(u) < k$, then u loses a free pebble. This is Lee and Streinu's edge insertion [LS08, §3, Fig. 4].

13.2. **Invariants.** Lee and Streinu [LS08, Lemma 10] prove four invariants of their game. We prove them for the orientations obtained from the empty one by the two moves, each made under the conditions the game imposes on it (Definition 13.6). Invariant (4) bounds the number of underlying edges inside a set of vertices by the sparsity count, and the soundness of the game (Section 13.4) follows from it.

Definition 13.6 ((k,ℓ)-reachable orientation)**.** Let k and ℓ be non-negative integers. The (k,ℓ)-*reachable* partial orientations on V form the smallest set of partial orientations that contains the empty orientation and is closed under the following two moves, applied to a member D:

(1) reversing a directed path in D from u_0 to u_m with $\mathrm{out}_D(u_m) < k$ (Definition 13.4);
(2) inserting an arc (u,v) with $u \neq v$, $(u,v) \notin D$, $(v,u) \notin D$, $\mathrm{out}_D(u) < k$ and $\mathrm{peb}_{D,k}(u) + \mathrm{peb}_{D,k}(v) \geq \ell + 1$ (Definition 13.5).

Lemma 13.7 (Invariants of the pebble game)**.** *Let V be a set, let k and ℓ be non-negative integers, and let D be a (k,ℓ)-reachable partial orientation on V. Let $v \in V$, and let $V' \subseteq V$ be a finite set. Then:*

(1) $\mathrm{peb}_{D,k}(v) + \mathrm{out}_D(v) = k$*;*
(2) $\mathrm{peb}_{D,k}(V') + \mathrm{span}_D(V') + \mathrm{out}_D(V') = k|V'|$*;*
(3) if $\ell \leq k|V'|$*, then* $\mathrm{peb}_{D,k}(V') + \mathrm{out}_D(V') \geq \ell$*;*
(4) if $\ell \leq k|V'|$*, then* $\mathrm{span}_D(V') \leq k|V'| - \ell$*.*

Proof. We argue by induction on the moves that produce D from the empty orientation (Definition 13.6).

For (1) it suffices to show that $\mathrm{out}_D(v) \leq k$, by the definition of $\mathrm{peb}_{D,k}(v)$. All out-degrees are 0 at the empty orientation. Reversing a path raises no out-degree other than that of its last vertex u_m, which it raises by at most 1, and the move requires $\mathrm{out}_D(u_m) < k$. Inserting an arc raises no out-degree other than that of its tail u, which it raises by 1, and the move requires $\mathrm{out}_D(u) < k$.

For (2) we need no induction. Summing (1) over V' gives

$$\mathrm{peb}_{D,k}(V') + \sum_{v \in V'} \mathrm{out}_D(v) = k|V'|.$$

An arc with tail in V' has its head either in V' or outside it. So the sum of the out-degrees over V' is $\mathrm{span}_D(V') + \mathrm{out}_D(V')$.

For (3), let $\ell \leq k|V'|$. At the empty orientation the left-hand side is $k|V'| \geq \ell$. Reversing a path does not change $\mathrm{span}_D(V')$, since it replaces each arc of the path by an arc with the same ends. By (2), before and after the move, it does not change $\mathrm{peb}_{D,k}(V') + \mathrm{out}_D(V')$ either. Inserting a new arc (u,v) with $\mathrm{out}_D(u) < k$ lowers $\mathrm{peb}_{D,k}(V')$ by 1 if $u \in V'$, and leaves it unchanged otherwise. It raises $\mathrm{out}_D(V')$ by 1 if $u \in V'$ and $v \notin V'$, and leaves it unchanged otherwise. So the insertion lowers the left-hand side of (3) by 1 if $u, v \in V'$, and leaves it unchanged otherwise. If $u, v \in V'$, then before the move

$$\mathrm{peb}_{D,k}(V') \;\geq\; \mathrm{peb}_{D,k}(u) + \mathrm{peb}_{D,k}(v) \;\geq\; \ell + 1,$$

so the left-hand side is at least ℓ after it.

Part (4) follows from (2) and (3). □

Formalization note 13.8. Lee and Streinu state their Lemma 10 for multigraphs, and the setting here simplifies it in two ways. First, a simple graph has no loops. Their invariant $\mathrm{peb}(v) + \mathrm{span}(v) + \mathrm{out}(v) = k$, in which $\mathrm{span}(v)$ counts the loops at v, therefore becomes (1). Second, they assume $|V'| \geq 1$ when $\ell \leq k$ and $|V'| \geq 2$ when $k < \ell < 2k$. Both give $\ell \leq k|V'|$, which is the hypothesis of (3) and (4) here; no part needs $\ell < 2k$. The formal statement of (4) is $\mathrm{span}_D(V') + \ell \leq k|V'|$, which avoids the truncated subtraction of natural numbers. Under $\ell \leq k|V'|$ the two forms agree. The first two pins are (1), in the equivalent form $\mathrm{out}_D(v) \leq k$ and as stated, and the other three are (2) to (4).

13.3. **The basic algorithm.** Lee and Streinu's basic pebble game [LS08, §3, Fig. 4] considers the edges of the input graph one at a time. An edge uv is accepted, and inserted as an arc, when u and v together hold at least $\ell + 1$ free pebbles. While they hold fewer, the game searches D for a free pebble on another vertex, reachable from u or from v, and brings it to u or v by reversing a directed path. If there is no such pebble, the edge is rejected. Lee and Streinu go on after a rejected edge, in order to classify the input graph. Our game stops at the first rejected edge, and returns a set of vertices which shows that the graph is not (k, ℓ)-sparse.

We define the algorithm in three levels. The pebble search (Definition 13.9) looks for a vertex with a given property, here a free pebble, and reverses the path to it. Processing an edge (Definition 13.12) repeats pebble searches until it can insert the edge or must reject it. The pebble game (Definition 13.14) processes a list of edges in turn.

The search of Section 12 visits the out-neighbours of a vertex in the order of a list. In this subsection, each partial orientation D on V comes with a list $N_D^+(a)$ for each vertex a, whose entries are the out-neighbours of a in D, that is, the vertices b with $(a, b) \in D$. Every result below holds for every choice of these lists.

Definition 13.9 (Pebble search)**.** Let D be a partial orientation on a finite set V, and let each vertex a have a list $N_D^+(a)$ whose entries are the out-neighbours of a in D. Let P be a predicate on V, and let $v \in V$. The *pebble search for P from v in D* runs the search for P from v (Definition 12.2) with these lists, so that the search follows the arcs of D.

(1) If the search returns a vertex w and a walk p, then p is a directed path in D from v to w, and $P(w)$ holds (Theorem 12.4). The pebble search returns w, p, and the partial orientation obtained from D by reversing p (Definition 13.4).
(2) If the search fails, the pebble search fails.

Formalization note 13.10. The first two pins are the pebble search, with the lists $N_D^+(a)$ as an argument and with mathlib's lists (Formalization note 13.15). The third computes the partial orientation obtained from D by reversing p, from the data that the search returns.

When the processing of an edge uv cannot gather $\ell + 1$ free pebbles on u and v, it rejects uv and returns the following data.

Definition 13.11 (Failure witness)**.** Let V be a set, and let k and ℓ be non-negative integers. A *failure witness* consists of a partial orientation D on V, a finite set $V' \subseteq V$, and a pair (u, v) of distinct vertices in V', such that:

(1) D is (k, ℓ)-reachable (Definition 13.6);
(2) the edge uv is not oriented in D;
(3) no arc of D has its tail in V' and its head outside V', that is, $\mathrm{out}_D(V') = 0$;
(4) $\mathrm{peb}_{D,k}(V') \leq \ell$.

A failure witness does not mention a graph. Let $\ell < 2k$, and let G be a simple graph on V with finitely many edges, whose edges include uv and the edges in $\underline{D}$. Then $\ell \leq k|V'|$, and V' spans more than $k|V'| - \ell$ edges of G (Lemma 13.22).

Definition 13.12 (Processing an edge)**.** Let V be a finite set, and let k and ℓ be non-negative integers. Let each partial orientation on V come with lists $N_D^+(a)$ as in Definition 13.9. Let D be a (k, ℓ)-reachable partial orientation on V, and let $u \neq v$ be vertices such that the edge uv is not oriented in D. Let P_D be the predicate that holds at a vertex w when $w \neq u$, $w \neq v$ and $\mathrm{peb}_{D,k}(w) > 0$. *Processing uv at D* runs as follows.

(1) If $\mathrm{peb}_{D,k}(u) + \mathrm{peb}_{D,k}(v) \geq \ell + 1$, it accepts uv. It inserts the arc (u, v) if $\mathrm{peb}_{D,k}(u) > 0$, and the arc (v, u) otherwise (Definition 13.5), and returns the result.
(2) Otherwise, it runs the pebble search for P_D from u in D. If this search does not fail, it processes uv at the partial orientation D' that the search returns.
(3) Otherwise, it runs the pebble search for P_D from v in D. If this search does not fail, it processes uv at the partial orientation D' that the search returns.
(4) Otherwise it rejects uv. It returns the failure witness made of D, the set $\mathrm{Reach}_D(u) \cup \mathrm{Reach}_D(v)$ and the pair (u, v). Here $\mathrm{Reach}_D(x)$ is the reachability region of x in D, the set of vertices reachable from x along the arcs of D (Definition 12.5).

We check that each step is defined and that the processing stops. In step 1 the inserted arc is new, and its tail has a free pebble, since $\mathrm{peb}_{D,k}(v) \geq \ell + 1 \geq 1$ when $\mathrm{peb}_{D,k}(u) = 0$. So the insertion is a move of Definition 13.6, and the result is (k, ℓ)-reachable.

In step 2, the pebble search returns a directed path from u to a vertex w with $w \neq u, v$ and $\mathrm{peb}_{D,k}(w) > 0$. So the path has at least one arc, and $\mathrm{out}_D(w) < k$. Hence D' is (k, ℓ)-reachable, and uv is not oriented in D', since the reversal keeps the underlying edge set. Since D is (k, ℓ)-reachable, $\mathrm{out}_D(u) \leq k$ by Invariant (1) of Lemma 13.7. So the reversal raises $\mathrm{peb}_{D,k}(u)$ by 1. It leaves $\mathrm{peb}_{D,k}(v)$ unchanged, since v is not an end of the path. Step 3 is the same with u and v exchanged. Hence $\ell + 1 - \mathrm{peb}_{D,k}(u) - \mathrm{peb}_{D,k}(v)$ drops by 1 each time the processing starts again, so step 1 or step 4 is reached.

Lee and Streinu mark u and v before the search, so that it neither passes through them nor moves their pebbles [LS08, §3, Fig. 4]. Here the predicate P_D only excludes u and v as ends of the path. A search may pass through u or v, but reversing a path does not change the out-degrees of its inner vertices.

The data returned in step 4 form a failure witness. By hypothesis, D is (k,ℓ)-reachable and uv is not oriented in D. The set $V' = \mathrm{Reach}_D(u) \cup \mathrm{Reach}_D(v)$ contains u and v. No arc of D leaves V', since the head of an arc whose tail is reachable from u is reachable from u, and likewise for v. Both pebble searches failed, so no vertex reachable from u or from v satisfies P_D (Theorem 12.4, part (2)). Hence every vertex of V' other than u and v has no free pebble, and

$$\mathrm{peb}_{D,k}(V') \;=\; \mathrm{peb}_{D,k}(u) + \mathrm{peb}_{D,k}(v) \;\leq\; \ell.$$

The set V', and the argument that no arc leaves it, are from Lee and Streinu's proof of their Lemma 13 [LS08].

When the game rejects a graph, it returns a set of vertices of the following kind.

Definition 13.13 (Blocking witness)**.** Let G be a simple graph on V, and let k and ℓ be non-negative integers. A *blocking witness* for G is a finite set $V' \subseteq V$ with $\ell \leq k|V'|$ that spans more than $k|V'| - \ell$ edges of G, that is, $|E_G[V']| + \ell > k|V'|$.

A blocking witness shows that G is not (k,ℓ)-sparse (Definition 3.1).

Definition 13.14 (The pebble game)**.** Let V, k, ℓ and the lists $N^+_D(a)$ be as in Definition 13.12. Let D_0 be a (k,ℓ)-reachable partial orientation on V, and let $(u_1,v_1),\dots,(u_m,v_m)$ be a list of pairs of vertices. The *pebble game on this list from* D_0 starts with $D = D_0$, and for $i = 1,\dots,m$ in turn runs as follows.

(1) If $u_i = v_i$, or $(u_i,v_i) \in D$, or $(v_i,u_i) \in D$, it skips the pair.
(2) Otherwise it processes u_iv_i at D. If this rejects u_iv_i, the game stops and returns the failure witness. If it accepts u_iv_i and returns D', the game sets $D = D'$.

If no pair is rejected, the game returns the final orientation D.

Now let G be a simple graph on V, and let $\ell < 2k$. The *pebble game on* G is the pebble game from the empty orientation on a list of the edges of G in which each edge uv appears once, as (u,v) or as (v,u). It *accepts* G if this game returns a partial orientation D. It *rejects* G, with the set V', if the game returns a failure witness whose set is V'.

The game is defined at every step, since processing an edge returns a (k,ℓ)-reachable orientation when it accepts. It stops, since the list is finite. In Section 13.4 we show that if the game accepts G with D, then $\underline{D} = E(G)$, and G is (k,ℓ)-sparse. In Section 13.6 we show that if it rejects G with V', then V' is a blocking witness for G. The formal pebble game on G (Definition 13.26) returns its answer together with proofs. With D it returns proofs that $\underline{D} = E(G)$ and that D is (k,ℓ)-reachable. With V' it returns a proof that V' is a blocking witness for G, for which it uses $\ell < 2k$.

Formalization note 13.15. The pebble search, the processing of an edge and the pebble game on a list are formalized with the lists $N^+_D(a)$ as an argument, one for each partial orientation D and vertex a, together with a proof that the entries of $N^+_D(a)$ are the out-neighbours of a in D. These forms (the declarations whose names end in "With") can be computed whenever the lists can, and the results below are proved for them. A second form of each procedure takes for $N^+_D(a)$ mathlib's list of

the elements of the finite set of out-neighbours, so it is the first form with particular lists. For the pebble game, the second form is the formal pebble game on G. It also lists the edges of G in this way, and writes each edge as an ordered pair by a choice. As in Section 12, mathlib lists the elements of a finite set by a choice that cannot be computed, so the second forms cannot be computed. In Section 14 we use lists sorted by a linear order on V instead, which can be computed.

13.4. **Soundness.** We show that if the pebble game on G accepts G, then G is (k,ℓ)-sparse. Lee and Streinu prove this direction of their Theorem 8 first, as part of their Corollary 12 [LS08]. Like theirs, our proof rests on Invariant (4) of Lemma 13.7. Suppose that the game accepts G with D. It runs on a list in which each edge of G appears once, so it skips no edge, and since it returns D, it accepts every edge. Hence $\underline{D} = E(G)$ (Lemma 13.17). For a set V' of vertices, $\mathrm{span}_D(V')$ is then the number of edges of G inside V' (Lemma 13.18), and Invariant (4) becomes the count that defines sparsity.

Lemma 13.16 (The edges oriented by an accepted run)**.** *Let V, k, ℓ and the lists $N_D^+(a)$ be as in Definition 13.12, and let D be a (k,ℓ)-reachable partial orientation on V.*

1. *Let $u \neq v$ be vertices such that the edge uv is not oriented in D. If processing uv at D accepts uv and returns D', then $\underline{D'} = \underline{D} \cup \{uv\}$.*
2. *Let $(u_1, v_1), \ldots, (u_m, v_m)$ be a list of pairs of vertices. If the pebble game on this list from D returns a partial orientation D', then $\underline{D} \subseteq \underline{D'}$, and every edge in $\underline{D'} \setminus \underline{D}$ is $u_i v_i$ for some i.*

Proof. For (1), processing uv at D stops in step 1 or step 4 of Definition 13.12. Otherwise, in step 2 or step 3, it processes uv again at a partial orientation D'' obtained from D by reversing a directed path. We argue by induction along this recursion, which stops, as shown after Definition 13.12. Step 4 rejects uv, so it is not reached. Step 1 inserts the arc (u, v) or the arc (v, u), and either way the underlying edge set becomes $\underline{D} \cup \{uv\}$ (Definition 13.5). In steps 2 and 3, $\underline{D''} = \underline{D}$ (Definition 13.4), and processing uv at D'' accepts uv and returns D'. By induction, $\underline{D'} = \underline{D''} \cup \{uv\} = \underline{D} \cup \{uv\}$.

For (2) we argue by induction on m. If $m = 0$, then $D' = D$. Let $m \geq 1$. If the game skips (u_1, v_1), it plays the rest of the list from D, and the claim follows by induction. Otherwise it processes $u_1 v_1$ at D. Since the game returns a partial orientation, the processing accepts $u_1 v_1$ and returns a (k,ℓ)-reachable partial orientation D_1, with $\underline{D_1} = \underline{D} \cup \{u_1 v_1\}$ by (1). The game then plays $(u_2, v_2), \ldots, (u_m, v_m)$ from D_1 and returns D'. The induction hypothesis at D_1 gives both claims. □

Lemma 13.17 (An accepted run orients every listed edge)**.** *Let V, k, ℓ and the lists $N_D^+(a)$ be as in Definition 13.12, and let D be a (k,ℓ)-reachable partial orientation on V. Let $(u_1, v_1), \ldots, (u_m, v_m)$ be a list of pairs of vertices with $u_i \neq v_i$ and $u_i v_i \notin \underline{D}$ for every i, such that the edges $u_1 v_1, \ldots, u_m v_m$ are distinct. If the pebble game on this list from D returns a partial orientation D', then $u_i v_i \in \underline{D'}$ for every i.*

Moreover, let G be a simple graph on V, and let the pairs (u_i, v_i) list the edges of G, each once, as (u, v) or as (v, u). If the pebble game on this list from the empty orientation returns D', then $\underline{D'} = E(G)$.

Proof. We argue by induction on m; for $m = 0$ there is nothing to prove. Let $m \geq 1$. Since $u_1 \neq v_1$ and $u_1v_1 \notin \underline{D}$, neither (u_1, v_1) nor (v_1, u_1) is an arc of D, so the game does not skip (u_1, v_1). As in the proof of Lemma 13.16 (2), processing u_1v_1 at D accepts it and returns a (k, ℓ)-reachable partial orientation D_1 with $\underline{D_1} = \underline{D} \cup \{u_1v_1\}$. For $i \geq 2$, the edge u_iv_i is neither in $\underline{D}$ nor equal to u_1v_1, so the rest of the list satisfies the hypotheses at D_1. By induction, $u_iv_i \in \underline{D'}$ for $i \geq 2$. Also $u_1v_1 \in \underline{D_1}$, and $\underline{D_1} \subseteq \underline{D'}$ by Lemma 13.16 (2).

For the second part, the edges of G are not loops, none of them is oriented in the empty orientation, and each is listed once. So the first part gives $E(G) \subseteq \underline{D'}$. Conversely, by Lemma 13.16 (2), every edge in $\underline{D'}$ is one of the u_iv_i, and so an edge of G.

The pebble game on G (Definition 13.14) runs on a list of this kind, since a list of the elements of a finite set has no repetitions (Formalization note 13.15). So if it accepts G with D, then $\underline{D} = E(G)$. □

Lemma 13.18 (Arcs and edges inside a set)**.** *Let D be a partial orientation on V, and let G be a simple graph on V with $\underline{D} = E(G)$. Then every finite set $V' \subseteq V$ satisfies*

$$\operatorname{span}_D(V') = |E_G[V']|.$$

Proof. Map each arc (a, b) of D with both ends in V' to the edge ab. This edge lies in $\underline{D} = E(G)$ and inside V', so it is in $E_G[V']$. The map is injective: two arcs with the same underlying edge are equal or antiparallel, and D has no antiparallel pairs. It is surjective: an edge ab of G inside V' lies in $\underline{D}$, so (a, b) or (b, a) is an arc of D, and both its ends are in V'. □

Theorem 13.19 (Soundness of the pebble game)**.** *Let V be a finite set, let G be a simple graph on V, and let k and ℓ be non-negative integers. Let the lists $N_D^+(a)$ be as in Definition 13.12, and let the pairs $(u_1, v_1), \dots, (u_m, v_m)$ list the edges of G, each once, as (u, v) or as (v, u). If the pebble game on this list from the empty orientation returns a partial orientation, then G is (k, ℓ)-sparse. In particular, if $\ell < 2k$ and the pebble game on G accepts G, then G is (k, ℓ)-sparse.*

Proof. Let D be the partial orientation returned. It is (k, ℓ)-reachable, since processing an edge returns a (k, ℓ)-reachable partial orientation when it accepts (Definition 13.12). By Lemma 13.17, $\underline{D} = E(G)$. Let $V' \subseteq V$ with $\ell \leq k|V'|$. Invariant (4) of Lemma 13.7 gives $\operatorname{span}_D(V') \leq k|V'| - \ell$. By Lemma 13.18, the left-hand side is $|E_G[V']|$. Hence G is (k, ℓ)-sparse. The pebble game on G runs on a list of the kind above (Lemma 13.17), which gives the last claim.

Only the last claim assumes $\ell < 2k$, the range in which the pebble game on G is defined (Definition 13.14). □

13.5. **Completeness.** We now prepare the converse, which Section 13.6 completes: if the pebble game on G rejects G with V', then V' is a blocking witness for G, so G is not (k, ℓ)-sparse.

Following Lee and Streinu [LS08, §2], we call an edge uv that is not oriented in a partial orientation D *independent* in D if the graph on V with edge set $\underline{D} \cup \{uv\}$ is (k, ℓ)-sparse. Their Lemma 14 states that the game inserts uv if and only if uv

is independent in D. One direction follows from Invariant (4): if processing uv at D accepts uv, then uv is independent in D (Lemma 13.23). For the other, Lee and Streinu apply their Lemma 13 repeatedly. While u and v hold at most ℓ free pebbles, their Lemma 13 gives a free pebble on another vertex reachable from u or from v, and reversing a directed path brings it to u or to v. They remark that the vertices reachable from u and from v may change after such a move, and that each application of their Lemma 13 needs uv to be independent [LS08, p. 1433]. It is, since reversing a path does not change $\underline{D}$.

We prove Lee and Streinu's Lemma 13 as Lemma 13.21, from a form that does not mention a graph (Lemma 13.20). For the correctness of the game, however, we argue at the rejection. Processing uv at D rejects uv only in step 4 of Definition 13.12. The failure witness that it returns there shows that uv is not independent in D (Lemma 13.22). The inequalities are those of Lee and Streinu's Lemma 13, read in the contrapositive.

Lemma 13.20 (A free pebble for an independent edge, counting form)**.** *Let V be a set, let k and ℓ be non-negative integers, and let D be a (k,ℓ)-reachable partial orientation on V. Let $u \neq v$ be vertices, and let $V' \subseteq V$ be a finite set that contains u and v and that no arc of D leaves, that is, $\mathrm{out}_D(V') = 0$. Suppose that*

$$\mathrm{span}_D(V') + 1 + \ell \;\leq\; k|V'| \qquad \text{and} \qquad \mathrm{peb}_{D,k}(u) + \mathrm{peb}_{D,k}(v) \;\leq\; \ell.$$

Then $\mathrm{peb}_{D,k}(w) \geq 1$ for some vertex $w \in V'$ other than u and v.

Proof. By Invariant (2) of Lemma 13.7,

$$\mathrm{peb}_{D,k}(V') + \mathrm{span}_D(V') + \mathrm{out}_D(V') = k|V'|.$$

Since $\mathrm{out}_D(V') = 0$, the first inequality gives $\mathrm{peb}_{D,k}(V') \geq \ell + 1$. By the second, u and v hold at most ℓ of these pebbles, so another vertex of V' holds one.

In Lee and Streinu's proof of their Lemma 13 [LS08], V' is the set of vertices reachable from u or from v, and the first inequality follows from the independence of uv, as in the proof of Lemma 13.21. □

Lemma 13.21 (A free pebble for an independent edge)**.** *Let V be a finite set, let k and ℓ be non-negative integers with $\ell < 2k$, and let D be a (k,ℓ)-reachable partial orientation on V. Let $u \neq v$ be vertices such that the edge uv is not oriented in D and the graph on V with edge set $\underline{D} \cup \{uv\}$ is (k,ℓ)-sparse. If $\mathrm{peb}_{D,k}(u) + \mathrm{peb}_{D,k}(v) \leq \ell$, then $\mathrm{peb}_{D,k}(w) \geq 1$ for some vertex $w \neq u, v$ that is reachable from u or from v along the arcs of D.*

Proof. Let G be the graph on V with edge set $\underline{D} \cup \{uv\}$, and let $V' = \mathrm{Reach}_D(u) \cup \mathrm{Reach}_D(v)$ be the set of vertices reachable from u or from v along the arcs of D. Then V' contains u and v, and no arc of D leaves V', as shown after Definition 13.12. Since $|V'| \geq 2$ and $\ell < 2k$, we have $\ell \leq k|V'|$. As G is (k,ℓ)-sparse, V' spans at most $k|V'| - \ell$ edges of G.

As in the proof of Lemma 13.18, the arcs of D inside V' have distinct underlying edges, and these are edges of G inside V'. The edge uv is one more, since it is not oriented in D. Hence

$$\mathrm{span}_D(V') + 1 \;\leq\; |E_G[V']| \;\leq\; k|V'| - \ell,$$

and Lemma 13.20 gives the vertex w. □

Lemma 13.22 (A failure witness gives a blocking witness)**.** *Let k and ℓ be non-negative integers with $\ell < 2k$, and let $(D, V', (u, v))$ be a failure witness (Definition 13.11). Let G be a simple graph on V with finitely many edges, whose edges include uv and the edges in $\underline{D}$. Then $\ell \leq k|V'|$ and*

$$|E_G[V']| + \ell \;>\; k|V'|.$$

Proof. The distinct vertices u and v lie in V', so $|V'| \geq 2$ and $\ell < 2k \leq k|V'|$. Since D is (k, ℓ)-reachable, Invariant (2) of Lemma 13.7 gives

$$\mathrm{peb}_{D,k}(V') + \mathrm{span}_D(V') + \mathrm{out}_D(V') = k|V'|.$$

By the definition of a failure witness, $\mathrm{out}_D(V') = 0$ and $\mathrm{peb}_{D,k}(V') \leq \ell$. So $k|V'| \leq \mathrm{span}_D(V') + \ell$.

As in the proof of Lemma 13.18, the arcs of D inside V' have distinct underlying edges, and these are edges of G inside V'. The edge uv is one more, since it lies inside V' and is not oriented in D. Hence $\mathrm{span}_D(V') + 1 \leq |E_G[V']|$, and so $k|V'| < |E_G[V']| + \ell$. Thus V' is a blocking witness for G (Definition 13.13). □

Lemma 13.23 (An accepted edge is independent)**.** *Let V, k, ℓ and the lists $N_D^+(a)$ be as in Definition 13.12. Let D be a (k, ℓ)-reachable partial orientation on V, and let $u \neq v$ be vertices such that the edge uv is not oriented in D. If processing uv at D accepts uv, then the graph on V with edge set $\underline{D} \cup \{uv\}$ is (k, ℓ)-sparse.*

Proof. Let G be this graph, and let D' be the partial orientation that the processing returns. It is (k, ℓ)-reachable (Definition 13.12). By Lemma 13.16 (1), $\underline{D'} = \underline{D} \cup \{uv\} = E(G)$. Let $V' \subseteq V$ with $\ell \leq k|V'|$. By Invariant (4) of Lemma 13.7, $\mathrm{span}_{D'}(V') \leq k|V'| - \ell$. By Lemma 13.18, the left-hand side is $|E_G[V']|$. Hence G is (k, ℓ)-sparse. □

13.6. **Correctness.** We now prove the part of Lee and Streinu's Theorem 8 [LS08, p. 1431] that concerns sparse graphs: the game accepts every edge of G, an outcome they call well-constrained or under-constrained (§3), if and only if G is (k, ℓ)-sparse. One direction is Theorem 13.19. Lee and Streinu prove the other from their Lemma 15 (p. 1433); here it follows from part (2) of the theorem below. We state the theorem for every list of the edges of G and every choice of the lists $N_D^+(a)$, so that it applies to each formal form of the game (Formalization note 13.15).

Theorem 13.24 (Correctness of the pebble game)**.** *Let V be a finite set, let G be a simple graph on V, and let k and ℓ be non-negative integers with $\ell < 2k$. Let the lists $N_D^+(a)$ be as in Definition 13.12, and let the pairs $(u_1, v_1), \ldots, (u_m, v_m)$ list the edges of G, each once, as (u, v) or as (v, u). Then G is (k, ℓ)-sparse if and only if the pebble game on this list from the empty orientation returns a partial orientation. More precisely:*

- *(1) if the game returns a partial orientation D, then $\underline{D} = E(G)$, and G is (k, ℓ)-sparse;*
- *(2) if the game returns a failure witness $(D, V', (u, v))$, then V' is a blocking witness for G (Definition 13.13).*

In particular, G is (k,ℓ)-sparse if and only if the pebble game on G accepts G, and if it rejects G with V', then V' is a blocking witness for G.

Proof. Part (1) is Lemma 13.17 and Theorem 13.19.

For part (2), we first show that every partial orientation that the game reaches orients only edges of G. We argue by induction on the list, as in the proof of Lemma 13.16 (2). The empty orientation orients no edge. When the game accepts u_iv_i, the underlying edge set gains only the edge u_iv_i (Lemma 13.16 (1)), and u_iv_i is an edge of G.

Now suppose that the game returns a failure witness $(D, V', (u,v))$. Then for some i, processing u_iv_i at a partial orientation D_{i-1} that the game has reached rejects u_iv_i. By step 4 of Definition 13.12, $(u,v) = (u_i, v_i)$, so uv is an edge of G. The processing reaches D from D_{i-1} by reversing directed paths in steps 2 and 3. So $\underline{D} = \underline{D_{i-1}}$ (Definition 13.4), and every edge in $\underline{D}$ is an edge of G (`runPebbleGameWith_witness_bridges`). Since G is finite, Lemma 13.22 shows that V' is a blocking witness for G. This is the only step of the proof that uses $\ell < 2k$.

The game returns either a partial orientation or a failure witness. If G is (k,ℓ)-sparse, then no set of vertices is a blocking witness for G, by Definition 3.1. So by (2), the game returns a partial orientation. The converse is (1). The pebble game on G (Definition 13.14) runs on a list of the kind above, which gives the last claim. □

13.7. **The verdict.** We now define the verdicts and the formal pebble game on G, which returns one, and restate Theorem 13.24 for them. The formal pebble game on G uses mathlib's lists (Formalization note 13.15). The computable pebble game on G of Section 14 differs from it only in its lists, which are sorted by a linear order on V. It also returns a verdict, for which the restated theorem holds as well (Theorem 14.7).

Definition 13.25 (Verdict)**.** Let G be a simple graph on V with finitely many edges, and let k and ℓ be non-negative integers. A *verdict* on G is one of the following:

(1) an *acceptance*, given by a (k,ℓ)-reachable partial orientation D on V with $\underline{D} = E(G)$ (Definition 13.6);
(2) a *rejection*, given by a blocking witness V' for G (Definition 13.13).

A verdict *accepts* G if it is an acceptance.

A verdict on G shows whether G is (k,ℓ)-sparse. An acceptance given by D shows that G is (k,ℓ)-sparse, as in the proof of Theorem 13.19. A rejection shows that G is not (k,ℓ)-sparse (the paragraph after Definition 13.13).

Definition 13.26 (The formal pebble game)**.** Let V be a finite set, let G be a simple graph on V, and let k and ℓ be non-negative integers with $\ell < 2k$. The *formal pebble game on* G runs the pebble game from the empty orientation on mathlib's list of the edges of G, with mathlib's lists of out-neighbours as the lists $N^+_D(a)$ (Formalization note 13.15). It returns the verdict on G (Definition 13.25) that is an acceptance, given by D, if the game returns a partial orientation D, and a rejection, given by V', if the game returns a failure witness whose set is V'.

The verdict is well defined. Mathlib's list of the edges of G lists each edge once, as (u, v) or as (v, u) (Lemma 13.17). So the game is the pebble game on G (Definition 13.14). If it returns D, then $\underline{D} = E(G)$ by the same lemma, and D is (k, ℓ)-reachable, since processing an edge returns a (k, ℓ)-reachable orientation when it accepts. If it returns a failure witness, then the witness's pair is an edge of G, and every edge oriented by its partial orientation is an edge of G, as in the proof of Theorem 13.24 (2). Since $\ell < 2k$, the witness's set V' is then a blocking witness for G (Lemma 13.22).

Theorem 13.27 (Correctness of the pebble game, for its verdict)**.** *Let V be a finite set, let G be a simple graph on V, and let k and ℓ be non-negative integers with $\ell < 2k$. Then G is (k, ℓ)-sparse if and only if the verdict that the formal pebble game on G returns accepts G.*

Proof. The formal pebble game on G runs the pebble game from the empty orientation on a list of the edges of G that contains each edge once (Lemma 13.17). It returns an acceptance when this game returns a partial orientation, and a rejection when it returns a failure witness (Definition 13.26). So its verdict accepts G if and only if the game returns a partial orientation. By Theorem 13.24, this holds if and only if G is (k, ℓ)-sparse. □

13.8. **Independence in the count matroid.** We restate Theorem 13.27 for the (k, ℓ)-count matroid of Section 11, whose independent sets are the (k, ℓ)-sparse subsets of $E(K_V)$: the pebble game decides whether the edge set of a graph is independent in it.

Corollary 13.28 (The pebble game decides independence in the count matroid)**.** *Let V be a finite set, let k and ℓ be non-negative integers with $\ell < 2k$, and let G be a simple graph on V. Then $E(G)$ is independent in the (k, ℓ)-count matroid on V (Definition 11.7) if and only if the verdict that the formal pebble game on G returns accepts G.*

Proof. Since G is simple, $E(G) \subseteq E(K_V)$, and the graph on V with edge set $E(G)$ is G. So $E(G)$ is independent in the count matroid if and only if G is (k, ℓ)-sparse (Theorem 11.9). By Theorem 13.27, this holds if and only if the verdict accepts G. □

At $(k, \ell) = (2, 3)$, the count matroid is the planar rigidity matroid $\mathcal{R}(V)$ (Definition 10.12). So the pebble game decides whether $E(G)$ is independent in $\mathcal{R}(V)$, that is, whether $E(G)$ is row-independent at some framework on V in $\mathbb{R}^2$ (Theorem 10.13). A Laman graph is a $(2, 3)$-sparse graph with $2|V| - 3$ edges (Definition 4.1). So the game, together with a count of the edges, decides whether G is Laman, and Section 14 makes this decision computable. This is the setting of Jacobs and Hendrickson's pebble game for rigidity in the plane [JH97].

14. Executable pebble game

The formal pebble game on a graph G (Definition 13.26) cannot be computed, since it uses mathlib's lists of finite sets (Formalization note 13.15). In this section

we define the computable pebble game on G, which uses lists sorted by a linear order on V instead. With it we define procedures that decide, for $\ell < 2k$, whether a finite graph is (k, ℓ)-sparse or (k, ℓ)-tight, and whether it is Laman. These procedures can be run in Lean, and a command-line program uses the computable pebble game to decide whether a graph read from a file is Laman.

In Section 14.1 we define the sorted lists, and in Section 14.2 the computable pebble game. It needs no new correctness argument: its verdict accepts G if and only if G is (k, ℓ)-sparse (Theorem 14.7). We define the three decision procedures in Section 14.3, and describe three ways to run them in Section 14.4. We list the examples on which they are run in Section 14.5, and describe the command-line program in Section 14.6. As for the formal pebble game, we prove no bound on the running time of the computable one (Section 13).

14.1. **Sorted lists.** The formal pebble game uses mathlib's lists in two places. The pebble search follows the arcs of D along the lists $N_D^+(a)$ of out-neighbours (Definition 13.9), and the pebble game on G runs on a list of the edges of G (Definition 13.14). We replace both with lists sorted by a linear order on V.

Definition 14.1 (Sorted out-neighbour list)**.** Let V be a set with a linear order, and let D be a partial orientation on V (Definition 13.1). For $v \in V$, the *sorted out-neighbour list* of v in D is the list of the out-neighbours of v in D, the vertices u with $(v, u) \in D$, in increasing order.

Lemma 14.2 (Membership in the sorted out-neighbour list)**.** *Let V and D be as in Definition 14.1, and let $u, v \in V$. Then u is an entry of the sorted out-neighbour list of v in D if and only if $(v, u) \in D$.*

Proof. Sorting a finite set lists each of its elements, and nothing else. □

So the sorted out-neighbour lists are lists $N_D^+(a)$ as in Section 13.3.

Definition 14.3 (Sorted edge list)**.** Let V be a set with a linear order, and let G be a simple graph on V with finitely many edges. The *sorted edge list* of G is the list of the pairs $(\min e, \max e)$, for the edges e of G, in lexicographic order. Here $\min e$ and $\max e$ are the smaller and the larger end of e.

Lemma 14.4 (Membership in the sorted edge list)**.** *Let V and G be as in Definition 14.3, and let $u, v \in V$. Then (u, v) is an entry of the sorted edge list of G if and only if $u \le v$ and $uv \in E(G)$.*

Proof. Sorting a finite set lists each of its elements, and nothing else. So the entries are the pairs $(\min e, \max e)$ for $e \in E(G)$. Such a pair (u, v) has $u \le v$ and $e = uv$. Conversely, if $u \le v$ and $uv \in E(G)$, then $(u, v) = (\min uv, \max uv)$. □

14.2. **The computable pebble game.** With the sorted lists, the pebble game on a list (Definition 13.14) gives a pebble game on G that can be computed.

Definition 14.5 (The computable pebble game)**.** Let V be a finite set with a linear order, let G be a simple graph on V, and let k and ℓ be non-negative integers with $\ell < 2k$. The *computable pebble game on G* runs the pebble game on the sorted

edge list of G (Definition 14.3) from the empty orientation, with the sorted out-neighbour lists (Definition 14.1) as the lists $N_D^+(a)$. It returns the verdict on G (Definition 13.25) that is an acceptance, given by D, if the game returns a partial orientation D, and a rejection, given by V', if the game returns a failure witness whose set is V'.

Formalization note 14.6. The computable pebble game is formalized as the forms of the pebble game that take the lists as an argument (Formalization note 13.15), applied to the sorted lists. Sorting a finite set by a linear order needs no choice, so the sorted lists, and with them the game, can be computed. Only a definition that can be computed can be run by `#eval` or `native_decide`. A verdict consists of a partial orientation or a set of vertices together with proofs, and proofs are erased in compiled code. So the compiled program computes only the pebble game.

The verdict is well defined, as for the formal pebble game (the paragraph after Definition 13.26). The sorted edge list lists the pairs (u, v) with $u < v$ and $uv \in E(G)$, in increasing order (Lemma 14.4). So it lists each edge of G once, and Theorem 13.24 applies to it. By its part (1), a returned D has $\underline{D} = E(G)$, and D is (k, ℓ)-reachable, as is every orientation that the game reaches. By its part (2), whose proof uses Lemma 13.22 and $\ell < 2k$, a returned V' is a blocking witness for G.

Theorem 14.7 (Correctness of the computable pebble game)**.** *Let V be a finite set with a linear order, let G be a simple graph on V, and let k and ℓ be non-negative integers with $\ell < 2k$. Then G is (k, ℓ)-sparse if and only if the verdict that the computable pebble game on G returns accepts G.*

Proof. The computable pebble game returns an acceptance when the pebble game on the sorted edge list returns a partial orientation, and a rejection when it returns a failure witness (Definition 14.5). So its verdict accepts G if and only if this game returns a partial orientation. The entries of the sorted out-neighbour list of a vertex are its out-neighbours (Lemma 14.2). The sorted edge list lists each edge of G once (Lemma 14.4). So Theorem 13.24 applies, and the game returns a partial orientation if and only if G is (k, ℓ)-sparse. □

14.3. **Deciding sparsity, tightness and the Laman property.** A *decision procedure* for a proposition P returns either a proof of P or a proof of its negation. In Lean it is an instance of `Decidable` P, and a command or a tactic that decides P uses the one registered for P (Section 14.4). We register decision procedures for three properties of a finite graph with a linear order on its vertices. The one for (k, ℓ)-sparsity runs the computable pebble game. Those for (k, ℓ)-tightness and the Laman property add a count of the edges.

Definition 14.8 (Deciding sparsity)**.** Let V be a finite set with a linear order, let G be a simple graph on V, and let k and ℓ be non-negative integers with $\ell < 2k$. The decision procedure for the (k, ℓ)-sparsity of G runs the computable pebble game on G (Definition 14.5). If the verdict accepts G, it returns a proof that G is (k, ℓ)-sparse, and otherwise a proof that G is not, both by Theorem 14.7.

Formalization note 14.9. The procedure takes the hypothesis $\ell < 2k$ as a registered fact, so that a command or a tactic finds the procedure without being given a proof. The fact $3 < 2 \cdot 2$ is registered, so at $(k, \ell) = (2, 3)$ the procedures of this subsection apply with no hypothesis given.

Remark 14.10 (One decision procedure for each property)**.** As V is finite, whether G is (k, ℓ)-sparse can also be decided by checking the inequality of Definition 3.1 for each of the $2^{|V|}$ subsets of V. A procedure of this kind decides the same proposition, but on a sparse graph it checks all $2^{|V|}$ subsets. We register only the procedure of Definition 14.8 for sparsity, and build the procedures for tightness and the Laman property on it. So a command or a tactic that decides one of these properties runs the pebble game.

Definition 14.11 (Deciding tightness)**.** Let V, G, k and ℓ be as in Definition 14.8. The decision procedure for the (k, ℓ)-tightness of G decides whether G is (k, ℓ)-sparse by Definition 14.8, and whether $|E(G)| + \ell = k|V|$ by computing both sides. It returns a proof that G is (k, ℓ)-tight if both hold, and otherwise a proof that G is not (Definition 3.3).

Definition 14.12 (Deciding the Laman property)**.** Let V be a finite set with a linear order, and let G be a simple graph on V. The decision procedure for the Laman property of G is the decision procedure for the $(2, 3)$-tightness of G (Definition 14.11), since $3 < 2 \cdot 2$ and a graph is Laman if and only if it is $(2, 3)$-tight (Definition 4.1).

14.4. **Runtime and backends.** A decision procedure registered for a proposition P, such as "G is Laman" for a given graph G, can be run in three ways.

- The command `#eval decide` P runs the procedure in Lean's interpreter, and prints its answer, `true` or `false`. The examples of Section 14.5 are run in this way.
- The command-line program `lake exe pebble-game` of Section 14.6 runs the computable pebble game, compiled to native code.
- The tactic `native_decide` proves P by running the compiled procedure. The proof rests on an axiom, added for it, that states the program's answer. So it trusts Lean's compiler, and mathlib does not accept such proofs. It proves that the graph obtained from K_4 by deleting an edge is Laman, and that K_4 is not.

The tactic `decide` would instead prove P by evaluating the procedure by reduction, trusting only Lean's type checker. It does not evaluate these procedures. The processing of an edge (Definition 13.12) and the search of Definition 12.2 are defined by well-founded recursion, which `decide` does not unfold. With the option `+kernel`, `decide` leaves the reduction to the type checker itself, which does unfold that recursion but does not evaluate the sort that builds the sorted lists of Section 14.1. It proves that the graph on two vertices with its one edge is Laman, but not that the graph obtained from K_4 by deleting an edge is. A Lean file in the repository, `scripts/DFSTiming.lean`, checks these facts, and those about `native_decide` above. It is not a module of Lean's module system: in a module, the type checker sees the body of a definition from another module only if that module exports it,

and there `decide +kernel` does not prove even that the graph on two vertices with its one edge is Laman.

14.5. **Worked examples.** We run the decision procedures with `#eval` on four graphs, each on the vertices $\{0, \dots, n-1\}$ for some n. The file that does so, `PebbleGame/Examples.lean` in the Lean library, records the expected answer beside each evaluation, and the procedures return it.

(1) The graph on $\{0,1,2,3\}$ obtained from K_4 by deleting the edge 23 has 4 vertices and $5 = 2 \cdot 4 - 3$ edges, and it is Laman (Theorem 5.14). The procedure for the Laman property answers `true`.
(2) The Moser spindle on $\{0, \dots, 6\}$ consists of the triangles 012 and 034, a vertex 5 adjacent to 1 and 2, a vertex 6 adjacent to 3 and 4, and the edge 56. It has 7 vertices and $11 = 2 \cdot 7 - 3$ edges, and the procedure for the Laman property answers `true`.
(3) The complete graph K_4 on $\{0,1,2,3\}$ has 6 edges, more than $2 \cdot 4 - 3 = 5$, so it is not $(2,3)$-sparse. The procedures for sparsity and for the Laman property answer `false`. So the computable pebble game rejects K_4 (Theorem 14.7), and its blocking witness is $\{0,1,2,3\}$. Indeed a blocking witness has at least two vertices (Definition 13.13), and a set X of two or three vertices spans $2|X| - 3$ edges of K_4.
(4) The path on $\{0, \dots, 4\}$ with the edges 01, 12, 23 and 34 is $(2,3)$-sparse, since a set of $s \geq 2$ of its vertices spans at most $s - 1 \leq 2s - 3$ edges. It has 4 edges, fewer than $2 \cdot 5 - 3 = 7$, so it is not $(2,3)$-tight and not Laman. The procedure for sparsity answers `true`, and those for tightness and the Laman property answer `false`.

14.6. **CLI binary.** The command `lake exe pebble-game` *file* runs a program that reads a graph from a file and decides whether it is Laman. The first line of the file is n m, the numbers of vertices and of edges. Each of the next m lines is u v, for an edge between the vertices u and v, where $0 \leq u, v < n$. Numbers on a line are separated by spaces or tabs, a line may end in a carriage return, and blank lines and lines that start with `#` are skipped. If the file does not have this form, the program prints an error message.

The program takes for G the simple graph on $\{0, \dots, n-1\}$ whose edges are the pairs read, leaving out a pair with $u = v$. It runs the computable pebble game on G with $(k, \ell) = (2, 3)$, and prints the following.

- If the verdict is an acceptance, given by D: the line `LAMAN` if $|E(G)| + 3 = 2n$, and `SPARSE_NOT_TIGHT` otherwise; then a line `ARCS` u v for each arc (u, v) of D, in lexicographic order.
- If the verdict is a rejection, given by V': the line `NOT_SPARSE`, then the line `BLOCKING` $|V'|$, then a line `VERTEX` w for each $w \in V'$, in increasing order.

By Theorem 14.7 and Definition 4.1, the first line is `LAMAN` if G is Laman, `SPARSE_NOT_TIGHT` if G is $(2,3)$-sparse but not Laman, and `NOT_SPARSE` if G is not $(2,3)$-sparse. The reading of the file and the printing are not verified. The directory `examples/` holds the four graphs of Section 14.5 as input files, each with its expected output in a comment at the top.

15. Matroid union and Edmonds' partition theorem

This chapter defines the union of finitely many matroids on a common finite set, describes its independent sets, and computes its rank. The rank formula is the matroid-union theorem of Nash-Williams [NW67] and Edmonds. For k copies of one matroid M, its independence criterion is Edmonds' matroid-partition theorem [Edm65, Theorem 1] (Theorem 15.11): a set can be partitioned into k independent sets of M if and only if every subset Y of it has at most $k\,r_M(Y)$ elements. We follow Oxley's presentation [Oxl11, §§11.1–11.3].

In Section 15.1 we show that a non-decreasing integer-valued submodular function f on the subsets of a finite set E induces a matroid M_f on E, and we compute the rank function of M_f when $f(\emptyset) = 0$. In Section 15.2 we define the union $\bigcup_j M_j$ of matroids $M_1, \dots, M_k$ on E as the matroid induced from their direct sum by a bipartite graph. Its independent sets are the sets $I_1 \cup \cdots \cup I_k$ with each I_j independent in M_j. By Rado's theorem on independent transversals, the union is the matroid M_f of the function $f(Y) = \sum_j r_{M_j}(Y)$, and the rank formula for M_f gives its rank.

In Section 16 we apply these results to k copies of the cycle matroid of a multigraph. There the independence criterion becomes (k,k)-sparsity, and it gives the tree-packing theorem of Tutte and Nash-Williams [Tut61, NW61].

The formal proofs of this chapter come from Peter Nelson's matroid package for Lean, `apnelson1/Matroid`, which, like this project, is licensed under Apache-2.0. In the package they rest on a matroid constructor that it has since replaced. We port them onto its current constructor, keeping his authorship, and add the forms of the rank formula proved for Theorem 15.10. The header of each ported file lists the changes.

15.1. Matroids from submodular functions.

Let E be a finite set. The rank function r of a matroid on E is non-decreasing and submodular, and the circuits of the matroid are the minimal non-empty sets C with $r(C) < |C|$. By a result of Edmonds and Rota, every non-decreasing integer-valued submodular function f defines a matroid M_f in the same way, and when $f(\emptyset) = 0$ the rank function of M_f is given by a minimization formula [Oxl11, §11.1]. We follow Oxley's proofs. For M_f, the circuit elimination axiom follows from the submodularity of f [Oxl11, Proposition 11.1.1]. Edmonds [Edm70] studies these functions and the polyhedra they define.

Definition 15.1 (Submodular function)**.** Let E be a finite set. A function f from the subsets of E to the integers is *submodular* if $f(X \cap Y) + f(X \cup Y) \le f(X) + f(Y)$ for all $X, Y \subseteq E$.

Definition 15.2 (Polymatroid rank function)**.** A *polymatroid rank function* on a finite set E is a non-decreasing submodular function f on the subsets of E with $f(\emptyset) = 0$.

Definition 15.3 (Matroid of a submodular function)**.** Let f be a non-decreasing submodular function on the subsets of a finite set E. The matroid *induced* by f is the matroid M_f on E whose circuits are the minimal non-empty sets $C \subseteq E$ with

$f(C) < |C|$. Equivalently, a set $I \subseteq E$ is independent in M_f if and only if $|I'| \leq f(I')$ for every non-empty $I' \subseteq I$.

Definition 15.4 (Matroid of a polymatroid rank function)**.** Let f be a polymatroid rank function on a finite set E. The matroid M_f is the matroid on E induced by f (Definition 15.3). Its independent sets are the sets $I \subseteq E$ with $|I'| \leq f(I')$ for every non-empty $I' \subseteq I$.

Lemma 15.5 (Polymatroid rank formula)**.** *Let f be a polymatroid rank function on a finite set E, and let $X \subseteq E$. Then the rank of X in M_f is*

$$r_{M_f}(X) \;=\; \min_{Y \subseteq X} (f(Y) + |X \setminus Y|).$$

Proof. Let B be a basis of X in M_f, so that $r_{M_f}(X) = |B|$. Let $Y \subseteq X$. The set $B \cap Y$ is independent, so $|B \cap Y| \leq f(B \cap Y)$ (Definition 15.3; if $B \cap Y$ is empty, because $f(\emptyset) = 0$). As f is non-decreasing, $f(B \cap Y) \leq f(Y)$. Also $|B \setminus Y| \leq |X \setminus Y|$. Hence $|B| \leq f(Y) + |X \setminus Y|$.

For the reverse inequality we follow Oxley's proof [Oxl11, Proposition 11.1.7]. If X is independent, then $Y = \emptyset$ attains $|B|$. Otherwise, for each $e \in X \setminus B$ the set $B \cup \{e\}$ is dependent, so it contains a circuit C_e of M_f, and $e \in C_e$. The set $I_e = C_e \setminus \{e\}$ is an independent subset of B, and $f(C_e) \leq |C_e| - 1 = |I_e|$ (Definition 15.3). Let Y be the union of the sets C_e.

We show that $f(Y) \leq |Y \cap B|$, adding the sets C_e one at a time. A single C_e satisfies it, as $C_e \cap B = I_e$. Let A be a union of some of them with $f(A) \leq |A \cap B|$, and let C_a be another. Each C_e lies in $B \cup \{e\}$, so $a \notin A$. Hence $A \cap C_a$ lies in the independent set I_a, and $f(A \cap C_a) \geq |A \cap I_a|$. By submodularity,

$$f(A \cup C_a) \;\leq\; f(A) + f(C_a) - f(A \cap C_a) \;\leq\; |A \cap B| + |I_a| - |A \cap I_a| \;=\; |(A \cup C_a) \cap B|.$$

Finally, Y contains $X \setminus B$, so $X \setminus Y = B \setminus Y$, and $f(Y) + |X \setminus Y| \leq |Y \cap B| + |B \setminus Y| = |B|$. □

15.2. **Matroid union.** Let $M_1, \ldots, M_k$ be matroids on a finite set E. Their direct sum $M_1 \oplus \cdots \oplus M_k$ is a matroid on $\{1, \ldots, k\} \times E$, in which M_j lives on the copy $\{j\} \times E$ of E.

A bipartite graph *matches* a set I of vertices on one side onto a set I' on the other if some bijection from I to I' sends each element of I to a neighbour. Let N be a matroid on one side of a bipartite graph, and let S be the other side. Perfect showed that the subsets of S that the graph matches onto independent sets of N are the independent sets of a matroid on S, the matroid *induced* from N by the graph [Oxl11, Theorem 11.2.12]. The union of the M_j is the matroid on E induced from their direct sum by the graph that joins each element of E to its copies. We show that its independent sets are the unions of independent sets of the M_j. Then we prove Rado's theorem, and from it the rank of the union.

Definition 15.6 (Matroid union)**.** Let $M_1, \ldots, M_k$ be matroids on a finite set E, and let Δ be the bipartite graph that joins each $e \in E$ to its copies $(1, e), \ldots, (k, e)$ in the ground set of the direct sum $M_1 \oplus \cdots \oplus M_k$. The *union* $\bigcup_j M_j$ is the matroid on E induced from $M_1 \oplus \cdots \oplus M_k$ by Δ: a set $I \subseteq E$ is independent in $\bigcup_j M_j$ if and

only if Δ matches I onto an independent set of $M_1 \oplus \cdots \oplus M_k$. For two matroids we write $M_1 \cup M_2$.

Formalization note 15.7. Nelson's package proves that a bipartite graph induces a matroid by checking the augmentation axiom. In the formalization the M_j need not have the same ground set: each is a matroid on a subset of E, and the union is a matroid on all of E. An element of E outside the ground set of M_j has no independent copy in the direct sum, so in the union it behaves as a loop of M_j. This is how Oxley defines the union of matroids on different ground sets [Oxl11, §11.3].

Lemma 15.8 (Independent sets of a matroid union)**.** *Let $M_1, \dots, M_k$ be matroids on a finite set E. A set $I \subseteq E$ is independent in $\bigcup_j M_j$ if and only if $I = I_1 \cup \cdots \cup I_k$ for some sets I_j independent in M_j.*

Proof. Let I be independent in $\bigcup_j M_j$, so that Δ matches I onto an independent set of the direct sum. The matching sends each $e \in I$ to a copy $(j(e), e)$. Its image is independent, so each set $I_j = \{e \in I : j(e) = j\}$ is independent in M_j, and the union of these sets is I.

Conversely, let $I = I_1 \cup \cdots \cup I_k$ with each I_j independent in M_j. For each $e \in I$ choose one index $j(e)$ with $e \in I_{j(e)}$. Each set $\{e \in I : j(e) = j\}$ is a subset of I_j, so $e \mapsto (j(e), e)$ matches I onto an independent set of the direct sum. □

The rank of the union rests on Rado's theorem, which Rado proved in 1942 (see [Oxl11, Theorem 11.2.2]). A *transversal* of a family $(A_i : i \in J)$ of sets is a family $(e_i : i \in J)$ of distinct elements with $e_i \in A_i$ for every i.

Lemma 15.9 (Rado's theorem)**.** *Let M be a matroid on a set S, and let $(A_i : i \in J)$ be a finite family of finite subsets of S. Then the family has a transversal $(e_i : i \in J)$ such that $\{e_i : i \in J\}$ is independent in M if and only if $|K| \le r_M(\bigcup_{i \in K} A_i)$ for every $K \subseteq J$.*

Proof. If such a transversal exists, then for $K \subseteq J$ the set $\{e_i : i \in K\}$ is an independent subset of $\bigcup_{i \in K} A_i$ with $|K|$ elements, so the rank of $\bigcup_{i \in K} A_i$ is at least $|K|$.

For the converse we follow Oxley, who derives Rado's theorem from a theorem of Welsh [Oxl11, Theorem 11.2.3] (`Matroid.generalized_halls_marriage`). Let f be a non-negative, integer-valued, non-decreasing submodular function on the subsets of S, and let the sets A_i be non-empty. Suppose that $f(\bigcup_{i \in K} A_i) \ge |K|$ for every $K \subseteq J$. Then Welsh's theorem gives elements $e_i \in A_i$, not necessarily distinct, with $f(\{e_i : i \in K\}) \ge |K|$ for every $K \subseteq J$.

We apply it to $f = r_M$. The sets A_i are non-empty, since the hypothesis at $K = \{i\}$ gives $r_M(A_i) \ge 1$. For $i \ne i'$, the case $K = \{i, i'\}$ gives $r_M(\{e_i, e_{i'}\}) \ge 2$, so $e_i \ne e_{i'}$. The case $K = J$ then shows that the $|J|$ elements e_i form an independent set. □

Theorem 15.10 (The rank of a matroid union)**.** *Let $M_1, \dots, M_k$ be matroids on a finite set E, with rank functions $r_1, \dots, r_k$, and let $X \subseteq E$. Then the rank of X in*

$\bigcup_j M_j$ *is*

$$\min_{Y \subseteq X} \Big(|X \setminus Y| + \sum_j r_j(Y) \Big).$$

In particular, X is independent in $\bigcup_j M_j$ if and only if $|Y| \leq \sum_j r_j(Y)$ for every $Y \subseteq X$.

Proof. Let Δ be as in Definition 15.6. The neighbours in Δ of a set $Y \subseteq E$ form the set $\{1, \dots, k\} \times Y$, whose rank in the direct sum is $f(Y) = \sum_j r_j(Y)$. As a sum of matroid rank functions, f is a polymatroid rank function (Definition 15.2).

We show that $\bigcup_j M_j$ is the matroid M_f (Definition 15.4; `Matroid.polymatroid_of_adjMap`). Let $I \subseteq E$, and apply Rado's theorem (Lemma 15.9) to the direct sum and the family of the sets $\{1, \dots, k\} \times \{e\}$ for $e \in I$. A transversal of this family that is independent in the direct sum is a matching of I onto an independent set. The condition of Rado's theorem says that $|Y| \leq f(Y)$ for every $Y \subseteq I$. So I is independent in $\bigcup_j M_j$ if and only if it is independent in M_f. The polymatroid rank formula (Lemma 15.5) now gives the rank of X. This is Oxley's proof [Oxl11, Lemma 11.2.13 and Theorem 11.3.1].

The set X is independent exactly when its rank is $|X|$. For $Y \subseteq X$ we have $|X \setminus Y| = |X| - |Y|$, so the rank formula gives the independence criterion. □

By Lemma 15.8, a set is independent in the union of k copies of a matroid M exactly when it is the union of k independent sets of M, which can be taken disjoint. So for k copies of one matroid the independence criterion is Edmonds' matroid-partition theorem [Edm65, Theorem 1].

Theorem 15.11 (Edmonds' matroid-partition theorem)**.** *Let M be a matroid on a finite set E, and let k be a natural number. A set $X \subseteq E$ is independent in the union of k copies of M if and only if $|Y| \leq k\, r_M(Y)$ for every $Y \subseteq X$.*

Proof. The set X is independent exactly when its rank is $|X|$. By Theorem 15.10 with $M_1 = \cdots = M_k = M$, this rank is the minimum of $|X \setminus Y| + k\, r_M(Y)$ over the sets $Y \subseteq X$. For such Y we have $|X \setminus Y| = |X| - |Y|$, so the minimum is $|X|$ if and only if $|Y| \leq k\, r_M(Y)$ for every $Y \subseteq X$. □

16. Body-bar Tay theorem

A body-bar framework consists of rigid bodies joined by bars, each bar attached to its two bodies by ball joints. Its graph is the multigraph with a vertex for each body and an edge for each bar. This chapter proves Tay's theorem [Tay84], which characterizes the graphs of the independent and infinitesimally rigid body-bar frameworks. Let $D = \binom{d+1}{2}$, the dimension of the space of infinitesimal motions of a rigid body in $\mathbb{R}^d$. Then a multigraph G is the graph of an independent and infinitesimally rigid body-bar framework in $\mathbb{R}^d$ if and only if $|E(G)| = D|V(G)| - D$ and every non-empty set E' of bars satisfies $|E'| \leq D|V(E')| - D$, where $V(E')$ is the set of bodies that the bars of E' meet. The graphs of the independent body-bar frameworks are the multigraphs that satisfy the inequalities (Whiteley [Whi88, Theorem 8]). Tay and Whiteley write n for the dimension and d for $n(n+1)/2$.

We write d and D, as Katoh and Tanigawa [KT11] do, because Section 17 and the chapters after it follow their paper.

Whiteley [Whi88] gave another proof of Tay's theorem, by matroid union, and we follow it. Section 16.1 shows that a set of bars is independent in the union of k copies of the cycle matroid if and only if each of its non-empty subsets E' satisfies $|E'| \leq k|V(E')|-k$. This follows from Edmonds' theorem (Section 15). Section 16.2 proves Whiteley's Theorem 1: the general k-frame, a matrix whose nonzero entries are indeterminates, represents this union. Section 16.3 decomposes the bars of a multigraph that satisfies the count into k forests, which are spanning trees when the multigraph is connected and has $k|V(G)| - k$ bars. Section 16.4 defines body-bar frameworks and shows that the rigidity matrix of a framework is the general D-frame with each indeterminate replaced by a coordinate of a bar. Section 16.5 proves Tay's theorem. A decomposition of the bars into D forests gives an independent framework, in which each bar of the j-th forest gets the j-th standard basis vector of $\mathbb{R}^D$. Conversely, the rows of an independent framework satisfy the count, as those of the general D-frame do.

We prove the theorem in this existence form: some framework on G is independent, or independent and infinitesimally rigid. Whiteley also requires each bar to be a line. He shows, by an argument on irreducible varieties [Whi88, §3.1], that for a general choice of the lines the rows have the same independent sets as the rows of the general D-frame. Section 28 proves a generic form of the theorem instead, for bars given by their endpoints, by the coordinate argument of Jackson and Jordán [JJ10].

16.1. **Sparsity and the union of cycle matroids.** Throughout this chapter, G is a finite multigraph: two vertices may be joined by several edges, and loops are allowed. We call its vertices *bodies* and its edges *bars*, and write $V(G)$ and $E(G)$ for these sets. For a set E' of bars, $V(E')$ is the set of bodies incident with a bar of E', and $G \restriction E'$ is the multigraph with the bodies of G and the bars of E'. In this subsection we characterize the sets of bars that are unions of k forests by a count.

A set of bars is a *forest* if it contains no cycle; a loop is a cycle, and so are two bars with the same ends. The forests are the independent sets of a matroid on $E(G)$, the *cycle matroid* of G [Oxl11, §1.1]. We write r for its rank function. Let $c(E')$ be the number of connected components of the multigraph with bodies $V(E')$ and bars E'. Then [Oxl11, (1.3.8)]

$$r(E') \;=\; |V(E')| - c(E').$$

The union $\bigcup_j M_j$ of matroids $M_1, \dots, M_k$ on a common finite set was introduced by Nash-Williams [NW67]. Section 15 defines it as the matroid induced from their direct sum by a bipartite graph (Definition 15.6). A set is independent in the union if and only if it is a union $I_1 \cup \dots \cup I_k$ of sets I_j independent in M_j (Lemma 15.8). For k copies of one matroid M, the rank of the union (Theorem 15.10) shows that a set X is independent in the union if and only if $|Y| \leq k\, r_M(Y)$ for every $Y \subseteq X$. This is Edmonds' matroid-partition theorem [Edm65, Theorem 1] (Theorem 15.11). We apply these results to k copies of the cycle matroid of G, whose independent sets are the unions of k forests.

The count compares each non-empty set of bars with the bodies it meets. For simple graphs, Definition 3.1 instead bounds the number of edges spanned by a set of vertices. Tay's theorem (Theorem 16.27) is about the case $k = \ell$.

Definition 16.1 (Sparse and tight multigraphs)**.** Let k and ℓ be natural numbers. A multigraph G is (k,ℓ)-*sparse* if every non-empty set E' of bars of G satisfies $|E'| \le k\,|V(E')| - \ell$. It is (k,ℓ)-*tight* if moreover $|E(G)| = k\,|V(G)| - \ell$.

The next theorem is the part of Whiteley's Corollary 3 [Whi88] that we use, and we follow his proof.

Theorem 16.2 (Independence in a union of cycle matroids; Whiteley Corollary 3)**.** *Let G be a multigraph, let k be a natural number, and let $E' \subseteq E(G)$. Then E' is independent in the union of k copies of the cycle matroid of G if and only if $G \restriction E'$ is (k,k)-sparse.*

Proof. By Edmonds' matroid-partition theorem (Theorem 15.11), E' is independent in the union if and only if $|Y| \le k\,r(Y)$ for every $Y \subseteq E'$. On the other hand, $G \restriction E'$ is (k,k)-sparse if and only if $|Y| \le k\,|V(Y)| - k$ for every non-empty $Y \subseteq E'$.

Suppose that $|Y| \le k\,r(Y)$ for every $Y \subseteq E'$, and let $Y \subseteq E'$ be non-empty. Then $c(Y) \ge 1$, so $r(Y) \le |V(Y)| - 1$ and $|Y| \le k\,|V(Y)| - k$.

Conversely, let $G \restriction E'$ be (k,k)-sparse, and let $Y \subseteq E'$ be non-empty. Sparsity alone gives $|Y| \le k\,|V(Y)| - k$, which is weaker than $|Y| \le k\,r(Y)$ when $c(Y) > 1$. So we apply it to each component. Let $c = c(Y)$, and let $Y_1, \dots, Y_c$ be the sets of bars of the components of the multigraph with bodies $V(Y)$ and bars Y. Each Y_i is non-empty, as every body in $V(Y)$ meets a bar of Y. Its component is connected, so $r(Y_i) = |V(Y_i)| - 1$, and sparsity gives $|Y_i| \le k\,|V(Y_i)| - k = k\,r(Y_i)$. The sets Y_i partition Y, and the sets $V(Y_i)$ partition $V(Y)$, so

$$|Y| \;=\; \sum_i |Y_i| \;\le\; k \sum_i (|V(Y_i)| - 1) \;=\; k\,(|V(Y)| - c) \;=\; k\,r(Y).$$

□

16.2. **The k-frame matroid as a union of cycle matroids.** Whiteley represents the union of k copies of the cycle matroid of G by a matrix with indeterminate entries, the general k-frame on G [Whi88, Definition 2.2]. Its nonzero entries are indeterminates, placed in the pattern of an incidence matrix of G repeated k times. In this subsection we prove Whiteley's Theorem 1: the matroid of the rows of this matrix is the union of k copies of the cycle matroid (Theorem 16.16).

Fix an orientation of each bar. The *signed incidence vector* a_e of a bar e has one entry for each body: 1 at the head of e, -1 at its tail and 0 elsewhere, so that $a_e = 0$ if e is a loop. Its entries are integers, so we may read it over any ring. Over every field the vectors a_e represent the cycle matroid: a set of bars is a forest if and only if its signed incidence vectors are linearly independent [Oxl11, Lemma 5.1.3]. Reversing a bar changes the sign of a_e, so nothing below depends on the orientation. For a vector w with one entry for each body and for $j \in \{1, \dots, k\}$, we write $w^{(j)}$ for the vector with k blocks that has w in block j and 0 in the other blocks.

Definition 16.3 (The k-frame matroid)**.** Let G be a multigraph and let $k \geq 1$. Let $R = \mathbb{Q}[X_{e,j}]$ be the polynomial ring over $\mathbb{Q}$ in one indeterminate $X_{e,j}$ for each bar e and each $j \in \{1, \dots, k\}$, and let K be its field of fractions. The *k-frame row* of a bar e is

$$\rho_e \;=\; \sum_{j=1}^{k} X_{e,j}\, a_e^{(j)} \;=\; (X_{e,1}\, a_e, \dots, X_{e,k}\, a_e),$$

where a_e is the signed incidence vector of e. The *k-frame matroid* $F(G, X)$ is the matroid on $E(G)$ in which a set of bars is independent if and only if their k-frame rows are linearly independent over K.

Whiteley's matrix groups its columns by body. Grouped by block instead, as in the proof of his Theorem 1, its rows are the rows ρ_e. By Theorem 16.2, it suffices to show that a set E' of bars is independent in $F(G, X)$ if and only if $G \restriction E'$ is (k, k)-sparse (Lemma 16.15).

Suppose that the rows of E' are linearly independent, and let $Y \subseteq E'$. The rows of the bars in Y lie in a subspace of dimension $k\, r(Y)$, so $|Y| \leq k\, r(Y)$ (Lemma 16.8). Conversely, suppose that $G \restriction E'$ is (k, k)-sparse. Then E' is the union of k disjoint forests. Setting $X_{e,j} = 1$ when e lies in the j-th forest, and every other indeterminate to 0, turns the rows of E' into a block-diagonal matrix whose blocks are incidence matrices of forests. Its rows are linearly independent, and hence so are the rows of E' (Lemma 16.14).

The first direction rests on three facts of linear algebra. For a subspace W of a vector space U, we write W^k for the subspace of U^k of the vectors whose k blocks all lie in W.

Lemma 16.4 (The k-frame rows lie in a product of incidence spans)**.** *Let A be the span over K of the signed incidence vectors of the bars of G. Then the k-frame row of every bar lies in A^k, and hence so does the span of all k-frame rows.*

Proof. Block j of ρ_e is $X_{e,j}\, a_e$, which lies in A. □

Formalization note 16.5. The formalization proves the lemma for every set Y of bars, with the span A_Y of $\{a_e : e \in Y\}$ in place of A: the k-frame row of every bar of Y lies in A_Y^k, and so does the span of these rows. The statement is the case $Y = E(G)$, and the proof of Lemma 16.8 uses the general case. The first pinned theorem is the second statement, the second is the first, and the third pin defines A_Y^k.

Lemma 16.6 (The dimension of a product subspace)**.** *Let W be a finite-dimensional subspace of a vector space U over K. Then* $\dim W^k = k \dim W$.

Proof. Taking a vector of W^k to its k blocks is a linear isomorphism from W^k onto the product of k copies of W. □

Lemma 16.7 (The span of a represented set has dimension its rank)**.** *Let M be a matroid of finite rank, represented over a field by vectors v_e: a set I is independent in M if and only if the vectors v_e, $e \in I$, are linearly independent. Then for every set Y of elements of M, the span of $\{v_e : e \in Y\}$ has dimension $r_M(Y)$. In particular, for every set Y of bars of G, the span over K of $\{a_e : e \in Y\}$ has dimension $r(Y)$.*

Proof. Let I be a basis of Y in M. For $e \in Y \setminus I$ the set $I \cup \{e\}$ is dependent, so v_e lies in the span of $\{v_f : f \in I\}$. Hence this span is the span of $\{v_e : e \in Y\}$. The vectors v_f, $f \in I$, are linearly independent, so it has dimension $|I| = r_M(Y)$. □

Lemma 16.8 (Independent k-frame rows satisfy the count)**.** *Let $E' \subseteq E(G)$ be a set of bars whose k-frame rows are linearly independent over K. Then $|Y| \leq k\, r(Y)$ for every $Y \subseteq E'$.*

Proof. Let $Y \subseteq E'$, and let A_Y be the span over K of $\{a_e : e \in Y\}$. The $|Y|$ rows ρ_e with $e \in Y$ are linearly independent. They lie in A_Y^k, by Lemma 16.4 with Y in place of $E(G)$ (the note after it). By Lemma 16.6 and Lemma 16.7, $\dim A_Y^k = k \dim A_Y = k\, r(Y)$. Hence $|Y| \leq k\, r(Y)$.

Whiteley proves this direction from a nonzero minor of order $|E'|$. Some monomial of the minor is nonzero, and setting its indeterminates to 1 and the others to 0 exhibits E' as a union of k forests. The dimension bound gives the count without expanding the determinant. □

Substituting numbers for the indeterminates is a ring homomorphism on R, but it is not defined on all of K. So for the converse we work over R, which contains the entries of every k-frame row.

Lemma 16.9 (A forest packing gives independent block vectors)**.** *Let $F_1, \dots, F_k$ be forests of G. Then the vectors $a_e^{(j)}$, one for each pair (j, e) with $e \in F_j$, are linearly independent over K.*

Proof. For each j, the vectors a_e with $e \in F_j$ are linearly independent, because F_j is a forest. A linear relation among the vectors $a_e^{(j)}$ is, block by block, a linear relation among the vectors a_e with $e \in F_j$ for each j, so it is trivial. □

Lemma 16.10 (Independence over K and over R)**.** *Let $E' \subseteq E(G)$ be a set of bars. Then the k-frame rows of the bars in E' are linearly independent over K if and only if they are linearly independent over R.*

Proof. A nontrivial linear relation over R is one over K. Conversely, multiplying a nontrivial linear relation over K by a common denominator of its coefficients gives one over R. □

Lemma 16.11 (A (k,k)-sparse set of bars is a union of k disjoint forests)**.** *Let $E' \subseteq E(G)$ be a set of bars such that $G \restriction E'$ is (k,k)-sparse. Then $E' = F_1 \cup \cdots \cup F_k$ for some pairwise disjoint forests $F_1, \dots, F_k$ of G.*

Proof. By Theorem 16.2, E' is independent in the union of k copies of the cycle matroid. By Lemma 15.8, it is the union of k sets independent in the cycle matroid, that is, of k forests $F_1, \dots, F_k$. The sets $F_j \setminus (F_1 \cup \cdots \cup F_{j-1})$ are pairwise disjoint, have the same union, and are forests, since a subset of a forest is a forest. □

Lemma 16.12 (Specializing the k-frame rows on a forest packing)**.** *Let $F_1, \dots, F_k$ be pairwise disjoint forests of G, and let $\varphi\colon R \to \mathbb{Q}$ be the ring homomorphism with $\varphi(X_{e,j}) = 1$ if $e \in F_j$ and $\varphi(X_{e,j}) = 0$ otherwise. Let $e \in F_{j_0}$. Then applying φ to each entry of ρ_e gives $a_e^{(j_0)}$.*

Proof. Block j of ρ_e is $X_{e,j}\, a_e$. The entries of a_e are 0 and ± 1, so φ maps this block to $\varphi(X_{e,j})\, a_e$. Here $\varphi(X_{e,j_0}) = 1$. For $j \neq j_0$ we have $e \notin F_j$, as the forests are disjoint, so $\varphi(X_{e,j}) = 0$. □

Formalization note 16.13. The formalization defines the k-frame row twice, with entries in R (`Graph.kFrameRowR`) and with entries in K, and proves that the second is the image of the first under the inclusion of R in K (`Graph.kFrameRow_eq_map_kFrameRowR`).

Lemma 16.14 (Sparse sets of bars have independent k-frame rows)**.** *Let $E' \subseteq E(G)$ be a set of bars such that $G \restriction E'$ is (k,k)-sparse. Then the k-frame rows of the bars in E' are linearly independent over K.*

Proof. We follow the converse half of Whiteley's proof [Whi88, Theorem 1]. By Lemma 16.11, E' is the union of pairwise disjoint forests $F_1, \ldots, F_k$. Let φ be the homomorphism of Lemma 16.12. It maps the row of each bar $e \in F_j$ to $a_e^{(j)}$. These vectors are linearly independent over K (Lemma 16.9), and their entries are rational, so they are linearly independent over $\mathbb{Q}$.

Let P be the matrix over R whose rows are the k-frame rows of the bars in E'. The matrix $\varphi(P)$ has linearly independent rows, so it has a nonzero minor of order $|E'|$. The same minor of P is a polynomial, and φ maps it to that nonzero minor of $\varphi(P)$. Hence it is nonzero, and the rows of P are linearly independent over R (`Matrix.linearIndependent_rows_of_specialized_submatrix_det_nezero`). By Lemma 16.10, they are linearly independent over K. □

Lemma 16.15 (Independence in the k-frame matroid by the count)**.** *Let $E' \subseteq E(G)$ be a set of bars. Then E' is independent in $F(G,X)$ if and only if $G \restriction E'$ is (k,k)-sparse.*

Proof. The set E' is independent in $F(G,X)$ if and only if its k-frame rows are linearly independent over K. If they are, then $|Y| \leq k\, r(Y)$ for every $Y \subseteq E'$ (Lemma 16.8). So E' is independent in the union of k copies of the cycle matroid, by Edmonds' matroid-partition theorem (Theorem 15.11). Then $G \restriction E'$ is (k,k)-sparse by Theorem 16.2. The converse is Lemma 16.14. □

Theorem 16.16 (The k-frame matroid is a union of cycle matroids; Whiteley Theorem 1)**.** *Let G be a multigraph and let $k \geq 1$. Then the k-frame matroid $F(G,X)$ is the union of k copies of the cycle matroid of G.*

Proof. Both matroids have ground set $E(G)$, so it suffices to show that they have the same independent sets. By Lemma 16.15, a set $E' \subseteq E(G)$ is independent in $F(G,X)$ if and only if $G \restriction E'$ is (k,k)-sparse. By Theorem 16.2, the same holds in the union. □

Formalization note 16.17. In the formalization the bars of G form a subset $E(G)$ of a type of edge labels, and the union of the cycle matroids is a matroid on all the labels, in which a label outside $E(G)$ is a loop. This is how the formalization treats unions of matroids on different ground sets (Definition 15.6). The formal theorem therefore states that $F(G,X)$ is the restriction of that union to $E(G)$.

16.3. **Decompositions into forests and spanning trees.** Applied to the set of all bars, Theorem 16.2 characterizes the multigraphs whose bars form a union of k forests. Whiteley [Whi88, remark after Corollary 3] attributes this theorem to Tutte [Tut61] and to Nash-Williams [NW61, NW64], and its first proof by matroid union to Edmonds [Edm65]. Oxley derives it from Edmonds' theorem, as we do [Oxl11, Proposition 11.3.14].

Theorem 16.18 (Decomposition into forests; Tutte, Nash-Williams)**.** *Let G be a multigraph and let k be a natural number. Then $E(G)$ is the union of k pairwise disjoint forests if and only if G is (k,k)-sparse.*

Proof. If $E(G)$ is the union of k forests, then it is independent in the union of k copies of the cycle matroid (Lemma 15.8). As $G \restriction E(G) = G$, the multigraph G is then (k,k)-sparse (Theorem 16.2). Conversely, if G is (k,k)-sparse, then Lemma 16.11, applied to $E' = E(G)$, gives the k pairwise disjoint forests. □

Corollary 16.19 (Tight multigraphs are unions of spanning trees)**.** *Let G be a connected (k,k)-tight multigraph. Then $E(G)$ is the union of k pairwise disjoint spanning trees of G.*

Proof. By Theorem 16.18, $E(G)$ is the union of pairwise disjoint forests $F_1, \dots, F_k$. Since G is connected, $r(E(G)) = |V(G)| - 1$, so the tightness of G gives $|E(G)| = k\,r(E(G))$. Each F_i is independent in the cycle matroid, so $|F_i| \le r(E(G))$, and the numbers $|F_i|$ sum to $|E(G)|$. Hence $|F_i| = r(E(G))$ for every i, so each F_i is a basis of the cycle matroid, that is, a maximal forest of G (`Graph.isMaximalAcyclicSet_of_isForestPacking_of_isTight`). A maximal forest of a connected multigraph is a spanning tree. □

16.4. **Body-bar frameworks and the rigidity matrix.** An infinitesimal motion of a rigid body in $\mathbb{R}^d$ is a screw. The screws form a vector space of dimension $D = \binom{d+1}{2}$, with d dimensions of translations and $\binom{d}{2}$ of rotations. A bar e joining bodies u and v imposes one linear condition on their screws $m(u)$ and $m(v)$, of the form $\langle b_e, m(u) - m(v)\rangle = 0$ for a vector $b_e \in \mathbb{R}^D$. Whiteley [Whi88, §3] takes for b_e the vector of Plücker coordinates of the line of the bar, with the screws written in dual coordinates. Following Tay [Tay84, Definition 5.1], we allow an arbitrary vector $b_e \in \mathbb{R}^D$ on each bar.

Definition 16.20 (Body-bar framework)**.** Let d be a natural number and let $D = \binom{d+1}{2}$. A *body-bar framework* in $\mathbb{R}^d$ on a multigraph G is a pair (G, b), where b assigns a vector $b_e \in \mathbb{R}^D$ to each bar e of G.

Formalization note 16.21. The formalization calls the dimension n, as Tay and Whiteley do, and writes D as `Graph.bodyBarDim` $n = n(n+1)/2$.

Definition 16.22 (Body-bar rigidity map)**.** Let (G, b) be a body-bar framework in $\mathbb{R}^d$, and let $D = \binom{d+1}{2}$. Orient each bar e of G from a body u_e, its tail, to a body v_e, its head. A *motion* of (G, b) assigns a vector $m(v) \in \mathbb{R}^D$ to each body v. The *rigidity map* of (G, b) is the linear map from motions to $\mathbb{R}^{E(G)}$ given by

$$m \;\longmapsto\; \big(\langle b_e,\, m(u_e) - m(v_e)\rangle\big)_{e \in E(G)}.$$

The *row* of a bar e is the linear functional $m \mapsto \langle b_e, m(u_e) - m(v_e)\rangle$. Reversing e changes the sign of its row, so the rank of the rigidity map does not depend on the orientation. We identify a linear functional on motions with its vector of coefficients. This vector has D blocks, one for each coordinate of $\mathbb{R}^D$, and each block has one entry for each body. So we can use the notation a_e and $w^{(c)}$ of Section 16.2, with $k = D$ and the orientation above. As a_e has 1 at v_e and -1 at u_e, the row of e is

$$-\sum_{c=1}^{D} (b_e)_c \, a_e^{(c)}$$

(`Graph.BodyBarFramework.rigidityRow_eq`). Up to sign, it is the D-frame row ρ_e of Definition 16.3 with each indeterminate $X_{e,c}$ replaced by the coordinate $(b_e)_c$. In Whiteley's terms, the rigidity matrix of (G, b) is the D-frame on G at b [Whi88, §3].

Let G have at least one body. A motion that assigns the same vector to every body lies in the kernel of the rigidity map, and these constant motions form a space of dimension D. Whiteley calls them trivial: each corresponds to one screw motion of the whole space. The motions form a space of dimension $D|V(G)|$, so the rigidity map has rank at most $D|V(G)| - D$, with equality exactly when every motion in its kernel is constant.

Definition 16.23 (Infinitesimal rigidity of a body-bar framework)**.** Let $D = \binom{d+1}{2}$. A body-bar framework (G, b) in $\mathbb{R}^d$ is *infinitesimally rigid* if its rigidity map has rank $D|V(G)| - D$.

Definition 16.24 (Independence of a body-bar framework)**.** A body-bar framework (G, b) is *independent* if its rigidity map has rank $|E(G)|$, that is, if the rows of its bars are linearly independent.

Infinitesimal rigidity is the analogue for body-bar frameworks of infinitesimal rigidity for bar-joint frameworks (Definition 6.5). Independence is the analogue of row-independence (Definition 9.3). Following Whiteley [Whi88, §2.3], we call a framework that is independent and infinitesimally rigid *isostatic*. The rank of its rigidity map is then $|E(G)| = D|V(G)| - D$.

16.5. **Tay's theorem.** For the existence direction we follow Whiteley's proof of his Theorem 8 [Whi88]. Its framework comes from the specialization in the proof of Lemma 16.14. For the converse, Whiteley derives the count from the independence of the rows of the general D-frame [Whi88, proof of Corollary 3]. We bound the dimension of the span of the rows directly instead, as in Lemma 16.8.

Proposition 16.25 (Frameworks on sparse and tight multigraphs)**.** *Let $D = \binom{d+1}{2}$, and let G be a (D, D)-sparse multigraph. Then there is an independent body-bar framework in $\mathbb{R}^d$ on G. If G is (D, D)-tight, then there is a body-bar framework in $\mathbb{R}^d$ on G that is independent and infinitesimally rigid.*

Proof. By Theorem 16.18, $E(G)$ is the union of pairwise disjoint forests $F_1, \dots, F_D$. For a bar e, let $j(e)$ be the index with $e \in F_{j(e)}$, and let b_e be the $j(e)$-th standard basis vector of $\mathbb{R}^D$. Up to sign, the row of e is then $a_e^{(j(e))}$ (Section 16.4). So the rows of the bars of F_j lie in block j, where up to sign they are the signed incidence

vectors of the bars of the forest F_j, and these are linearly independent over $\mathbb{R}$. Hence all the rows are linearly independent, as in the proof of Lemma 16.9, and (G, b) is independent (`Graph.BodyBarFramework.stdFramework_finrank_range`). If G is (D, D)-tight, the rank of the rigidity map is $|E(G)| = D|V(G)| - D$, so (G, b) is also infinitesimally rigid. □

Lemma 16.26 (Independent frameworks have sparse graphs)**.** *Let $D = \binom{d+1}{2}$, and let (G, b) be an independent body-bar framework in $\mathbb{R}^d$. Then G is (D, D)-sparse.*

Proof. Let E' be a non-empty set of bars. The rows of the bars in E' are linearly independent, so their span has dimension $|E'|$. Let $A_{E'}$ be the span over $\mathbb{R}$ of the vectors a_e with $e \in E'$. Up to sign, the row of a bar $e \in E'$ is $\sum_c (b_e)_c \, a_e^{(c)}$, which lies in $A_{E'}^D$. As in the proof of Lemma 16.8, with $\mathbb{R}$ in place of K, the space $A_{E'}^D$ has dimension $D\,r(E')$ (`Graph.BodyBarFramework.finrank_rigidityRow_span_le`). Hence $|E'| \leq D\,r(E')$. As E' is non-empty, $c(E') \geq 1$, so $r(E') \leq |V(E')| - 1$ and $|E'| \leq D|V(E')| - D$. □

Theorem 16.27 (Tay's theorem; Whiteley Theorem 8)**.** *Let $D = \binom{d+1}{2}$, and let G be a multigraph. Then there is an independent body-bar framework in $\mathbb{R}^d$ on G if and only if G is (D, D)-sparse. There is a body-bar framework in $\mathbb{R}^d$ on G that is independent and infinitesimally rigid if and only if G is (D, D)-tight.*

Proof. If there is an independent framework on G, then G is (D, D)-sparse by Lemma 16.26. If the framework is also infinitesimally rigid, then the rank of its rigidity map is both $|E(G)|$ and $D|V(G)| - D$, so G is (D, D)-tight. The converses are Proposition 16.25. □

The first statement is Whiteley's Theorem 8 [Whi88]. The second is Tay's Theorem 5.3 [Tay84] in the existence form. By Theorem 16.18, the multigraph G is (D, D)-sparse if and only if $E(G)$ is the union of D disjoint forests. By Corollary 16.19, the bars of a connected (D, D)-tight multigraph form the union of D disjoint spanning trees, and Tay states his theorem in this form.

Each standard basis vector of $\mathbb{R}^D$ satisfies the quadratic Plücker relations, so it is the vector of Plücker coordinates of a line in projective d-space [Whi88, proof of Theorem 8]. The frameworks of Proposition 16.25 therefore have a line on each bar. So the theorem also holds for Whiteley's body-bar frameworks, whose bars are lines.

17. Body-hinge Tay–Whiteley theorem

A body-hinge framework in $\mathbb{R}^d$ consists of rigid bodies joined by hinges. A hinge is a $(d-2)$-dimensional affine subspace shared by two bodies: a pin joint in the plane, a line in 3-space, a plane in 4-space [Whi88, §3.3]. The two bodies can rotate relative to each other only about the hinge. This chapter proves the theorem of Tay [Tay89] and Whiteley [Whi88] on these frameworks. It characterizes the graphs of the independent body-hinge frameworks, and those of the independent and infinitesimally rigid ones. As with Tay's theorem in Section 16, we prove it in the existence form.

As in Section 16, let $D = \binom{d+1}{2}$. The relative screws of two bodies form a space of dimension D, and a hinge between them leaves a space of dimension one, the rotations about the hinge. So the hinge acts as $D-1$ bars, all meeting the hinge, whose Plücker vectors are linearly independent (Whiteley [Whi88, proof of Theorem 10]). Replacing each hinge by such a bundle of bars turns a body-hinge framework on a multigraph G into a body-bar framework on the multigraph $(D-1)\cdot G$, in which each edge of G is repeated $D-1$ times. We apply Tay's theorem (Theorem 16.27) to $(D-1)\cdot G$. The result, Theorem 17.5, is that G has an independent body-hinge framework if and only if the edges of $(D-1)\cdot G$ form a union of D disjoint forests, and an independent and infinitesimally rigid one if and only if $(D-1)\cdot G$ is (D,D)-tight.

17.1. **Body-hinge frameworks and the hinge-to-bar reduction.** We define the multigraph $(D-1)\cdot G$ and body-hinge frameworks, and we apply Tay's theorem to $(D-1)\cdot G$.

Definition 17.1 (Edge multiplication)**.** Let G be a multigraph and let m be a natural number. The multigraph $m\cdot G$ has the vertices of G, and for each edge e of G it has m edges $(e,1),\dots,(e,m)$, each with the ends of e.

So $|E(m\cdot G)| = m\,|E(G)|$ (`Graph.edgeMultiply_edgeSet_ncard`). For $m \geq 1$, a set of edges of G meets the same vertices as the set of their copies in $m\cdot G$ (`Graph.spanningVerts_edgeMultiply`). Katoh and Tanigawa write $\tilde{G}$ for $(D-1)\cdot G$ [KT11, §1], and so do the later chapters.

Definition 17.2 (Body-hinge framework)**.** Let $D = \binom{d+1}{2}$, and let G be a multigraph. A *body-hinge framework* in $\mathbb{R}^d$ on G assigns to each edge e of G a $(d-2)$-dimensional affine subspace of $\mathbb{R}^d$, the hinge of e, and to the $D-1$ copies of e in $(D-1)\cdot G$ a basis of the space spanned by the Plücker vectors of the lines that meet the hinge. Its rigidity map is that of the body-bar framework on $(D-1)\cdot G$ with these vectors (Definition 16.22). It is independent, or infinitesimally rigid, if this body-bar framework is.

Formalization note 17.3. The formalization records a body-hinge framework on G by its vectors alone. It is a body-bar framework on $(D-1)\cdot G$, with no hinge and no condition on the $D-1$ vectors of an edge. So Lemma 17.4 and Theorem 17.5 are proved for these bundles of bars, and the frameworks they provide need not come from hinges. For multigraphs, the formalization has the theorem in this form. The rigidity statement for frameworks with hinges (Definition 19.3), at every generic choice of the hinges, is Corollary 28.44, and it is for multigraphs, with parallel edges and loops allowed. It also asks that every body be a vertex of G and that the field be infinite, and nothing of the edge labels, as the formalization note after it describes.

Lemma 17.4 (Tay's theorem for the multiplied multigraph)**.** *Let $D = \binom{d+1}{2}$, and let G be a multigraph. Then there is an independent body-hinge framework in $\mathbb{R}^d$ on G if and only if $(D-1)\cdot G$ is (D,D)-sparse. There is a body-hinge framework in $\mathbb{R}^d$ on G that is independent and infinitesimally rigid if and only if $(D-1)\cdot G$ is (D,D)-tight.*

Proof. By Definition 17.2, a body-hinge framework on G is independent, or infinitesimally rigid, exactly when the body-bar framework on $(D-1)\cdot G$ with the same vectors is. In the formalization every body-bar framework on $(D-1)\cdot G$ arises in this way, as the formalization note after Definition 17.2 explains (`Graph.BodyHingeFramework.exists_toBodyBar_iff`). So the two statements are those of Theorem 16.27 for the multigraph $(D-1)\cdot G$. □

Since $|E((D-1)\cdot G)| = (D-1)|E(G)|$, the multigraph $(D-1)\cdot G$ is (D,D)-tight if and only if $(D-1)|E(G)| = D|V(G)| - D$ and $(D-1)|E'| \le D|V(E')| - D$ for every non-empty set E' of edges of G. These inequalities suffice, because a set of edges of $(D-1)\cdot G$ meets the same vertices as the set of all copies of the edges of G that it uses, and is no larger.

17.2. **The Tay–Whiteley theorem.** Tay [Tay89, §7] and Whiteley [Whi88, Theorem 10] characterized, independently, the graphs of the infinitesimally rigid body-hinge frameworks. Whiteley's introduction also states the characterization of the graphs of the independent ones.

Theorem 17.5 (Body-hinge frameworks; Tay, Whiteley)**.** *Let $D = \binom{d+1}{2}$, and let G be a multigraph. Then there is an independent body-hinge framework in $\mathbb{R}^d$ on G if and only if $E((D-1)\cdot G)$ is the union of D pairwise disjoint forests. There is a body-hinge framework in $\mathbb{R}^d$ on G that is independent and infinitesimally rigid if and only if $(D-1)\cdot G$ is (D,D)-tight.*

Proof. By Lemma 17.4, there is an independent body-hinge framework on G if and only if $(D-1)\cdot G$ is (D,D)-sparse. By Theorem 16.18, this holds if and only if $E((D-1)\cdot G)$ is the union of D pairwise disjoint forests. The second statement is that of Lemma 17.4. □

If G is connected, then so is $(D-1)\cdot G$. By Corollary 16.19, the second condition then makes $E((D-1)\cdot G)$ the union of D disjoint spanning trees. Katoh and Tanigawa state the theorem for infinitesimal rigidity alone (KT Proposition 1.1): G has an infinitesimally rigid body-hinge framework in $\mathbb{R}^d$ if and only if $\tilde{G}$ contains D edge-disjoint spanning trees. For frameworks with hinges at a generic choice of points, this form is Corollary 28.44, which also asks that the field be infinite. The formalization also has the theorem above, for multigraphs, proved for bundles of bars as the note after the definition of body-hinge frameworks explains.

Katoh and Tanigawa call a body-hinge framework a panel-hinge framework if, for each body, the hinges at that body lie in a common hyperplane [KT11, §1]. Tay and Whiteley [TW84] conjectured that a graph has an infinitesimally rigid body-hinge framework in $\mathbb{R}^d$ if and only if it has an infinitesimally rigid panel-hinge framework in $\mathbb{R}^d$ (KT Conjecture 1.2). This is the molecular conjecture. It does not follow from Theorem 17.5, which places no condition on the hinges at a body. Katoh and Tanigawa proved it, and we prove its case of simple graphs in Theorem 23.56, following them.

18. Grassmann–Cayley extensor algebra

This chapter sets up the part of the Grassmann–Cayley algebra that Katoh and Tanigawa use [KT11, §2.1], and proves their Lemma 2.1. A point of K^d is written in homogeneous coordinates in K^{d+1}, and the affine subspace spanned by points $p_1, \dots, p_k$ is represented by its extensor, the product of their homogeneous coordinate vectors in the exterior algebra of K^{d+1}. KT Lemma 2.1 (Lemma 18.9) says that if $p_1, \dots, p_{d+1}$ are affinely independent points of K^d, then the extensors of the $D = \binom{d+1}{2}$ sets of $d-1$ of these points are linearly independent. Section 19 builds the rigidity matrix of a body-hinge framework from the extensors of its hinges, and the proof of KT Theorem 5.5 uses Lemma 2.1 in its last case (Section 23).

Nothing in the chapter uses an inner product, an order or a hypothesis on the characteristic, so we work over an arbitrary field K; Katoh and Tanigawa work over $\mathbb{R}$. Section 18.1 shows that points are affinely independent if and only if their homogeneous coordinate vectors are linearly independent (Lemma 18.2). Section 18.2 defines extensors and their join in the exterior algebra, following Barnabei, Brini and Rota [BBR85]. It relates extensors to the Plücker coordinates through which Katoh and Tanigawa define them, and defines the extensor of an affine subspace. Section 18.3 proves KT Lemma 2.1.

18.1. **Homogeneous coordinates and affine independence.** As in Section 16 and Section 17, d is the dimension and $D = \binom{d+1}{2}$; over $\mathbb{R}$, D is the dimension of the space of infinitesimal motions of a rigid body in $\mathbb{R}^d$. Following Katoh and Tanigawa, we give each point of K^d a homogeneous coordinate vector in K^{d+1}.

Definition 18.1 (Homogeneous coordinates)**.** Let $p \in K^d$. The *homogeneous coordinate vector* of p is the vector $\bar{p} = (p, 1) \in K^{d+1}$, obtained by appending the coordinate 1.

Recall that points $p_1, \dots, p_k$ of K^d are affinely independent if the only scalars $w_1, \dots, w_k$ with $\sum_i w_i = 0$ and $\sum_i w_i p_i = 0$ are $w_1 = \cdots = w_k = 0$.

Lemma 18.2 (Affine independence in homogeneous coordinates)**.** *Let $p_1, \dots, p_k \in K^d$. Then $p_1, \dots, p_k$ are affinely independent if and only if $\bar{p}_1, \dots, \bar{p}_k$ are linearly independent. In particular, $d+1$ points $p_1, \dots, p_{d+1}$ of K^d are affinely independent if and only if the $(d+1) \times (d+1)$ matrix with rows $\bar{p}_1, \dots, \bar{p}_{d+1}$ has nonzero determinant.*

Proof. Let $w_1, \dots, w_k \in K$. The first d coordinates of $\sum_i w_i \bar{p}_i$ are those of $\sum_i w_i p_i$, and its last coordinate is $\sum_i w_i$. So $\sum_i w_i \bar{p}_i = 0$ if and only if $\sum_i w_i p_i = 0$ and $\sum_i w_i = 0$. The linear relations among $\bar{p}_1, \dots, \bar{p}_k$ are therefore exactly the families w in the definition of affine independence, and one kind of independence holds if and only if the other does. The second statement follows, since the rows of a square matrix are linearly independent if and only if its determinant is nonzero. □

18.2. **Extensors, the join, and Plücker coordinates.** Let $\bigwedge K^{d+1}$ be the exterior algebra of K^{d+1}, and let $\bigwedge^j K^{d+1}$ be its graded piece of degree j, which has dimension $\binom{d+1}{j}$. An extensor is a product of vectors in this algebra. We follow Barnabei, Brini and Rota [BBR85, §3].

Definition 18.3 (Extensor)**.** Let $v_1, \dots, v_j \in K^{d+1}$. The *extensor* of the family $(v_1, \dots, v_j)$ is the product $v_1 \wedge \cdots \wedge v_j$ in $\bigwedge K^{d+1}$. A *j-extensor* is an element of $\bigwedge K^{d+1}$ of this form.

A j-extensor lies in $\bigwedge^j K^{d+1}$. The extensor of $(v_1, \dots, v_j)$ is linear in each v_i and alternating: it is zero when two of the v_i are equal, and it changes sign when two of them are exchanged. So it does not change when a multiple of one v_i is added to another, and it is zero when $v_1, \dots, v_j$ are linearly dependent. Conversely, let $v_1, \dots, v_j$ be linearly independent, and extend them to a basis of K^{d+1}. The products of basis vectors over the increasing sequences of indices form a basis of $\bigwedge K^{d+1}$, and $v_1 \wedge \cdots \wedge v_j$ is one of them. So the extensor of $(v_1, \dots, v_j)$ is nonzero if and only if $v_1, \dots, v_j$ are linearly independent (`extensor_ne_zero_iff_linearIndependent`), as Katoh and Tanigawa recall [KT11, p. 651]. Barnabei, Brini and Rota call only a nonzero product an extensor; like Katoh and Tanigawa, we allow the zero product.

Barnabei, Brini and Rota call the product of the exterior algebra the join.

Definition 18.4 (Join)**.** Let $A, B \in \bigwedge K^{d+1}$. The *join* of A and B is their product $A \vee B = A \wedge B$ in $\bigwedge K^{d+1}$.

The join is associative, and the join of two extensors is the extensor of the concatenated family:

$$(v_1 \wedge \cdots \wedge v_m) \vee (w_1 \wedge \cdots \wedge w_n) \;=\; v_1 \wedge \cdots \wedge v_m \wedge w_1 \wedge \cdots \wedge w_n$$

(`join_extensor`). The extensor of $(v_1, \dots, v_j)$ is therefore the join $v_1 \vee \cdots \vee v_j$ of its vectors, and from now on we write it so, as Katoh and Tanigawa do. Two bases of one j-dimensional subspace of K^{d+1} have extensors that differ by a nonzero scalar, and a nonzero j-extensor determines the subspace spanned by its vectors [BBR85, Proposition 3.4]. Let F and G be nonzero extensors of subspaces X and Y. Then $F \vee G = 0$ if $X \cap Y \neq 0$, and otherwise $F \vee G$ is an extensor of $X + Y$ [BBR85, Proposition 3.5].

Katoh and Tanigawa define extensors through coordinates, which we define next.

Definition 18.5 (Plücker coordinates)**.** Let $v_1, \dots, v_j \in K^{d+1}$, and let $A(v)$ be the $j \times (d+1)$ matrix whose i-th row is v_i. For $1 \le i_1 < \cdots < i_j \le d+1$, the *Plücker coordinate* $P_{i_1,\dots,i_j}$ of $(v_1, \dots, v_j)$ is $(-1)^{1+i_1+\cdots+i_j}$ times the determinant of the $j \times j$ submatrix of $A(v)$ on the columns $i_1, \dots, i_j$. The *Plücker coordinate vector* of $(v_1, \dots, v_j)$ is the vector of its $\binom{d+1}{j}$ Plücker coordinates.

The sign is Katoh and Tanigawa's [KT11, p. 651]. Katoh and Tanigawa work with the Plücker coordinate vector and call the extensor its symbolic version. For $j = d+1$ there is one Plücker coordinate, $\pm \det A(v)$ (`pluckerCoord_univ`). For homogenized points, $\det A(v)$ is the determinant of Lemma 18.2.

Lemma 18.6 (Plücker expansion)**.** *Let $v_1, \dots, v_j \in K^{d+1}$. The products $e_{i_1} \wedge \cdots \wedge e_{i_j}$ with $i_1 < \cdots < i_j$ form a basis of $\bigwedge^j K^{d+1}$, and the coordinate of $v_1 \vee \cdots \vee v_j$ at $e_{i_1} \wedge \cdots \wedge e_{i_j}$ is $(-1)^{1+i_1+\cdots+i_j} P_{i_1,\dots,i_j}$, the Plücker coordinate up to Katoh and Tanigawa's sign.*

Proof. Write each v_i in the standard basis and expand $v_1 \wedge \cdots \wedge v_j$ by multilinearity and alternation. The coefficient of $e_{i_1} \wedge \cdots \wedge e_{i_j}$ is the minor of $A(v)$ on the columns $i_1, \dots, i_j$. The sign squares to 1, so this minor is the coordinate $P_{i_1,\dots,i_j}$ times the sign. □

Formalization note 18.7. The pin is the library's basis of the exterior power induced by the standard basis, and the expansion is its coordinate formula (`exteriorPower.basis_repr_appl` for the minor. Katoh and Tanigawa's sign is the one in Definition 18.5, so the statement carries it.

Katoh and Tanigawa represent an affine subspace by the extensor of the homogeneous coordinate vectors of points that span it.

Definition 18.8 (Extensor of an affine subspace)**.** Let $p_1, \dots, p_k \in K^d$. The *extensor* of $p_1, \dots, p_k$ is

$$C(p_1, \dots, p_k) \;=\; \bar{p}_1 \vee \cdots \vee \bar{p}_k \;\in\; \bigwedge^{k} K^{d+1}.$$

If $p_1, \dots, p_k$ are affinely independent and span the affine subspace U, we also write $C(U)$ for $C(p_1, \dots, p_k)$ and call it a k-extensor *supporting* U.

By Lemma 18.2, the points $p_1, \dots, p_k$ are affinely independent if and only if $\bar{p}_1, \dots, \bar{p}_k$ are linearly independent. By the criterion after Definition 18.3, this holds if and only if $C(p_1, \dots, p_k) \neq 0$ (`affineSubspaceExtensor_ne_zero_iff`). Now let $p_1, \dots, p_k$ be affinely independent with affine span U, and let W be the subspace of K^{d+1} spanned by $\bar{p}_1, \dots, \bar{p}_k$. A point p lies in U if and only if $\bar{p} \in W$, so U and W determine each other. By Proposition 3.4 of Barnabei, Brini and Rota [BBR85], W determines its nonzero extensors up to a nonzero scalar, and each of them determines W. So $C(U)$ is determined by U up to a nonzero scalar, and it determines U.

18.3. **Independence of the $(d-1)$-extensors (KT Lemma 2.1).** This subsection proves KT Lemma 2.1. Katoh and Tanigawa call it a fundamental fact about extensors and prove it for completeness [KT11, pp. 651–652]; we follow their proof.

Lemma 18.9 (Extensor independence; KT Lemma 2.1)**.** *Let $p_1, \dots, p_{d+1} \in K^d$ be affinely independent. Then the $D = \binom{d+1}{2}$ extensors*

$$\bar{p}_1 \vee \cdots \vee \widehat{\bar{p}_a} \vee \cdots \vee \widehat{\bar{p}_b} \vee \cdots \vee \bar{p}_{d+1} \qquad (1 \le a < b \le d+1)$$

are linearly independent in $\bigwedge^{d-1} K^{d+1}$, where a hat marks an omitted vector.

Proof. For $a < b$, write E_{ab} for the extensor in the statement: the join of the $\bar{p}_i$ with $i \neq a, b$, in increasing order of i. Suppose that $\sum_{a<b} \lambda_{ab} E_{ab} = 0$ for scalars λ_{ab}, and fix a pair $a < b$. We join the relation on the left with $\bar{p}_a \vee \bar{p}_b$, which gives

$$\sum_{c<c'} \lambda_{cc'}\, \bar{p}_a \vee \bar{p}_b \vee E_{cc'} \;=\; 0.$$

Every term with $\{c, c'\} \neq \{a, b\}$ vanishes. Indeed, one of a and b, say a, is not in $\{c, c'\}$, so $\bar{p}_a$ is one of the vectors joined in $E_{cc'}$. Then $\bar{p}_a \vee \bar{p}_b \vee E_{cc'}$ is the join of a family in which $\bar{p}_a$ occurs twice, and it is zero because the join is alternating (Definition 18.3).

The remaining term is $\lambda_{ab}\,\bar{p}_a \vee \bar{p}_b \vee E_{ab}$. Its join is that of the family obtained from $\bar{p}_1, \dots, \bar{p}_{d+1}$ by a bijection of $\{1, \dots, d+1\}$: the first two places go to a and b, and the remaining $d-1$ places go, in order, to the elements of $\{1, \dots, d+1\} \setminus \{a, b\}$ in increasing order. The vectors $\bar{p}_1, \dots, \bar{p}_{d+1}$ are linearly independent (Lemma 18.2), so the reordered family is linearly independent too, and its join is nonzero (Definition 18.3). Hence $\lambda_{ab} = 0$. As the pair $a < b$ was arbitrary, every λ_{ab} is zero. □

Formalization note 18.10. The formalization writes $d = e + 1$, so that the points lie in K^{e+1} and each E_{ab} is an e-extensor, and it indexes the extensors by the pairs $a < b$. The three declarations are E_{ab} (`omitTwoExtensor`), the lemma as stated (`omitTwoExtensor_linearIndependent`), and the lemma for any $d + 1$ linearly independent vectors of K^{d+1} (`omitTwoExtensor_linearIndependent_of_li`), from which the lemma as stated follows by Lemma 18.2.

The proof uses the points only through the linear independence of $\bar{p}_1, \dots, \bar{p}_{d+1}$, so the lemma holds for any linearly independent vectors $v_1, \dots, v_{d+1}$ of K^{d+1}. Since $\bigwedge^{d-1} K^{d+1}$ has dimension $\binom{d+1}{d-1} = D$, the D extensors then form a basis of it. So a linear functional on $\bigwedge^{d-1} K^{d+1}$ that vanishes on all of them is zero. Katoh and Tanigawa use the lemma in this way in the last case of the proof of KT Theorem 5.5: in KT Claim 6.12 for $d = 3$, and in KT Lemma 6.13 for general d [KT11, §6.4]. There they choose $d + 1$ affinely independent points such that the affine subspace spanned by any $d - 1$ of them lies in one of d given panels. Case III of Section 23 uses the lemma for linearly independent vectors (Lemma 23.169).

19. The panel-hinge rigidity matrix $R(G, p)$

This chapter defines the rigidity matrix $R(G, p)$ of a body-hinge framework (G, p), following Katoh and Tanigawa [KT11, §§2.2–2.4], and proves the lemmas on its rank that their proof of the molecular conjecture uses. In Section 17 a hinge was replaced by $D - 1$ bars, where $D = \binom{d+1}{2}$, and the rigidity of a body-hinge framework on G was that of the resulting body-bar framework on $\tilde{G} = (D - 1) \cdot G$. Here each edge e has a hinge $p(e)$, an affine subspace of dimension $d - 2$, and the hinge constrains the screw centers of the two bodies of e through its extensor (Section 18.2). As in Section 18, we work over an arbitrary field K.

Section 19.1 defines $R(G, p)$ and the space $Z(G, p)$ of infinitesimal motions, and shows that $\operatorname{rank} R(G, p) \le D(|V| - 1)$. The next chapter defines the deficiency $\operatorname{def}(\tilde{G})$ and sharpens the bound to $\operatorname{rank} R(G, p) \le D(|V| - 1) - \operatorname{def}(\tilde{G})$ at every realization (Section 20.3). Equality holds at a generic realization. This rank formula is Jackson and Jordán's [JJ10, Theorem 6.1], which Katoh and Tanigawa state as their Proposition 2.3. We prove it in a conditional form, from a bound on the space of motions, in the chapter on the algebraic induction (Proposition 23.19), and at every generic hinge-point assignment of a multigraph in a later chapter (Theorem 28.40, Proposition 28.42). By the theorem of Tutte and Nash-Williams [Tut61, NW61], $\operatorname{def}(\tilde{G}) = 0$ exactly when $\tilde{G}$ contains D edge-disjoint spanning trees [KT11, p. 655]. So the formula implies Tay and Whiteley's theorem in the form that KT Proposition 1.1 states (Section 17.2).

Section 19.2 proves KT Lemmas 5.1–5.3 on the rank of $R(G,p)$. Section 19.3 proves the rank-addition lemmas that the algebraic induction (Section 23) and the induction of the pencil chapters (Section 30) apply when they split a graph into smaller ones.

19.1. **Body-hinge frameworks and the rigidity matrix.** This subsection defines the rigidity matrix of a body-hinge framework, following Katoh and Tanigawa [KT11, §§2.2–2.4]. They identify an infinitesimal motion of a rigid body in $\mathbb{R}^d$ with a vector of $\mathbb{R}^D$, its screw center, where $D = \binom{d+1}{2}$. A hinge joining two bodies constrains the difference of their screw centers to the span of a $(d-1)$-extensor supporting the hinge. We take this constraint as the definition, over a field K. Screw centers lie in the following space, as do the supporting extensors of Definition 18.8.

Definition 19.1 (Screw space)**.** The *screw space* is $\bigwedge^{d-1} K^{d+1}$. It has dimension $\binom{d+1}{d-1} = D$, where $D = \binom{d+1}{2}$.

Formalization note 19.2. The dimension is written $d = k+1$, so that the screw space is $\bigwedge^{k} K^{k+2}$. The pins are the screw space, the number D and the dimension of the screw space.

Definition 19.3 (Hinge constraint via the supporting extensor)**.** A *body-hinge framework* (G,p) in K^d is a multigraph $G = (V,E)$ together with a map p that assigns to each edge e a $(d-2)$-dimensional affine subspace $p(e)$ of K^d, the *hinge* of e. Let $C(p(e)) \in \bigwedge^{d-1} K^{d+1}$ be a $(d-1)$-extensor supporting $p(e)$ (Definition 18.8). A *screw assignment* is a map S from V to the screw space $\bigwedge^{d-1} K^{d+1}$. It satisfies the *hinge constraint* at an edge e with ends u and v if

$$S(u) - S(v) \in \operatorname{span} C(p(e)).$$

The hinge $p(e)$ is spanned by $d-1$ affinely independent points, so $C(p(e)) \neq 0$, and $p(e)$ determines $C(p(e))$ up to a nonzero scalar (the remark after Definition 18.8). So the hinge constraint does not depend on the choice of $C(p(e))$. It is symmetric in u and v, since $S(v) - S(u) = -(S(u) - S(v))$.

Formalization note 19.4. The formalization records a body-hinge framework by its multigraph and, for each edge e, an element $C(e)$ of the screw space in place of $C(p(e))$. It does not require $C(e)$ to be nonzero or to come from points, so the same definitions apply to hinges given as the meet of two panels (Definition 23.5). A lemma that needs $C(e) \neq 0$ assumes it. Screw assignments are defined on a fixed set of bodies, which may contain bodies that lie on no edge of G (Section 19.3).

Katoh and Tanigawa write the hinge constraint as linear equations [KT11, §2.4]. For this we identify $\bigwedge^{d-1} K^{d+1}$ with K^D by the basis of the products $e_{i_1} \wedge \cdots \wedge e_{i_{d-1}}$ with $i_1 < \cdots < i_{d-1}$ (Lemma 18.6), and write $x \cdot y$ for the dot product of K^D.

Definition 19.5 (Hinge-row block)**.** Let (G,p) be a body-hinge framework and let e be an edge of G. Let $r_1(p(e)), \dots, r_{D-1}(p(e))$ be a basis of the orthogonal complement of $\operatorname{span} C(p(e))$ in K^D, and let $r(p(e))$ be the $(D-1) \times D$ matrix with these rows.

Then a screw assignment S satisfies the hinge constraint at e, with ends u and v, if and only if

$$r_i(p(e)) \cdot (S(u) - S(v)) = 0 \qquad (1 \le i \le D-1).$$

The orthogonal complement has dimension $D-1$ because $C(p(e)) \neq 0$. The equivalence holds because a subspace of K^D is the orthogonal complement of its orthogonal complement, as the dot product is nondegenerate. Through the dot product, the rows of $r(p(e))$ form a basis of the linear forms on K^D that vanish on $\operatorname{span} C(p(e))$. A linear form vanishes on $\operatorname{span} C(p(e))$ if and only if it vanishes at $C(p(e))$, as the span consists of the multiples of $C(p(e))$; we then say that it vanishes on $C(p(e))$.

Formalization note 19.6. The first pin is the space of linear forms on the screw space that vanish on $\operatorname{span} C(p(e))$, with no basis chosen, and the second is the equivalence. The third gives this space the dimension $D-1$ when $C(p(e)) \neq 0$, and the fourth says that a form lies in it if and only if it vanishes at $C(p(e))$.

The rigidity matrix collects these equations for all the edges.

Definition 19.7 (Panel-hinge rigidity matrix)**.** Let (G, p) be a body-hinge framework. Its *rigidity matrix* $R(G, p)$ is the $(D-1)|E| \times D|V|$ matrix with a block of $D-1$ rows for each edge and a block of D columns for each body. Orient each edge e from one end u to the other end v. The rows of e have $r(p(e))$ in the columns of u, $-r(p(e))$ in the columns of v, and zeros elsewhere (Definition 19.5). An *infinitesimal motion* of (G, p) is a screw assignment that satisfies the hinge constraint at every edge, that is, a vector of $\ker R(G, p) \subseteq K^{D|V|}$. We write $Z(G, p)$ for the space of infinitesimal motions.

The space $Z(G, p)$ depends neither on the choice of the rows of the matrices $r(p(e))$ nor on the orientation of the edges [KT11, p. 654]. So neither does the rank of $R(G, p)$, which is

$$\operatorname{rank} R(G, p) \;=\; D|V| - \dim Z(G, p).$$

Formalization note 19.8. The formalization fixes no basis of the screw space. A row of $R(G, p)$ is the linear form $S \mapsto r(S(u) - S(v))$ on screw assignments, for an edge e with ends u and v and a linear form r on the screw space that vanishes on $C(p(e))$. The rank of $R(G, p)$ is the dimension of the span of these forms, and $Z(G, p)$ is the space on which they all vanish. The first two pins are the infinitesimal motions and $Z(G, p)$, the next two the rows of $R(G, p)$, and the last the description of $Z(G, p)$ by the rows.

We record three facts about the rows of $R(G, p)$. First, the rows of an edge with distinct ends are linearly independent.

Lemma 19.9 (The rows of an edge are independent)**.** *Let (G, p) be a body-hinge framework, let $u \neq v$ be bodies, and let $r_1, \dots, r_n \in K^D$ be linearly independent. Then the linear forms $S \mapsto r_i \cdot (S(u) - S(v))$ on screw assignments are linearly independent. In particular, if e is an edge with ends u and v and $C(p(e)) \neq 0$, then the $D-1$ rows of e are linearly independent.*

Proof. The map $S \mapsto S(u) - S(v)$ takes the screw assignments onto K^D, as $u \neq v$. So a linear relation among the forms gives the same relation among the r_i, and its coefficients are zero. If $C(p(e)) \neq 0$, the rows of e are the forms for $r_i = r_i(p(e))$, $1 \leq i \leq D-1$, which are linearly independent (Definition 19.5). □

Formalization note 19.10. The first pin is the claim for linear forms, and the second its form for rows.

Second, the rows of several edges are jointly independent when each of the edges has an end of its own.

Lemma 19.11 (Independent rows at edges with ends of their own)**.** *Let (G, p) be a body-hinge framework. Let $u_1, \dots, u_m$ be distinct bodies, and let $w_1, \dots, w_m$ be bodies none of which is equal to any u_j. For each j, let $r_{j,1}, \dots, r_{j,n_j} \in K^D$ be linearly independent. Then the linear forms $S \mapsto r_{j,i} \cdot (S(u_j) - S(w_j))$ on screw assignments are linearly independent. In particular, if e_j is an edge with ends u_j and w_j and $C(p(e_j)) \neq 0$ for each j, then the $(D-1)m$ rows of $e_1, \dots, e_m$ are linearly independent.*

Proof. Take a linear combination of the forms that vanishes, and evaluate it at the screw assignment with value x at u_j and 0 elsewhere. A form of index $j' \neq j$ vanishes there, as $u_{j'} \neq u_j$ and $w_{j'} \neq u_j$. So the coefficients c_i of the forms of index j give $\sum_i c_i r_{j,i} \cdot x = 0$ for every $x \in K^D$, and every c_i is zero, as the $r_{j,i}$ are linearly independent. If $C(p(e_j)) \neq 0$, the rows of e_j are the forms for $r_{j,i} = r_i(p(e_j))$, $1 \leq i \leq D-1$, which are linearly independent (Definition 19.5). □

For example, the edges of a star at a body w satisfy the hypothesis if their other ends are distinct bodies other than w.

Formalization note 19.12. The first pin is the claim for linear forms, the second its form for rows, and the third the case of a star.

Third, the rows of the edges of a forest are jointly independent. A forest need not satisfy the hypothesis of Lemma 19.11: in a path with three edges, each end of the middle edge is also an end of another edge.

Lemma 19.13 (Independent rows of a forest)**.** *Let (G, p) be a body-hinge framework, and let T be a forest with finitely many edges that is a subgraph of G, with $C(p(e)) \neq 0$ at each edge e of T. Then $R(G, p)$ has $(D-1)|E(T)|$ linearly independent rows.*

Proof. We show that the rows of the edges of T are linearly independent, by induction on the number of edges of T; there is nothing to prove when T has none. Otherwise T has a pendant edge e: an edge with distinct ends x and y, where x is an end of no other edge of T (the first edge of a longest path, say). Deleting e leaves a forest $T - e$, whose rows are linearly independent by induction. Each of them is a form $S \mapsto r \cdot (S(u) - S(w))$ for an edge of $T - e$ with ends u and w, neither of which is x, so it vanishes on the screw assignments that vanish off x. On such an assignment, with value s at x, the $D-1$ rows of e take the values $r_i(p(e)) \cdot s$, and these forms in s are linearly independent (Definition 19.5). So the rows of e and those of $T - e$ are linearly independent together (Lemma 23.127), and they are the rows of T. □

Formalization note 19.14. The pin asks the set of edge labels to be finite, so that every forest in G has finitely many edges.

The next definitions are Katoh and Tanigawa's [KT11, §§2.3–2.4], who take that of a generic framework from Jackson and Jordán [JJ10, §6]. A body-hinge framework on G is also called a *realization* of G.

Definition 19.15 (Trivial motions; infinitesimal rigidity)**.** Let (G, p) be a body-hinge framework. A screw assignment S is *trivial* if $S(u) = S(v)$ for all bodies u and v, and the trivial screw assignments form the space of *trivial motions*. The framework (G, p) is *infinitesimally rigid* if all its infinitesimal motions are trivial.

Katoh and Tanigawa call an infinitesimal motion trivial under the same condition [KT11, §2.3]. By Lemma 19.18 below, every trivial screw assignment is an infinitesimal motion.

Definition 19.16 (Degree of freedom; generic realization)**.** Let (G, p) be a body-hinge framework. The *degree of freedom* of (G, p) is $D(|V| - 1) - \operatorname{rank} R(G, p)$. The framework (G, p) is *generic* if $R(G, p)$ and each of its edge-induced submatrices (the rows of the edges of a subset of E) have the largest rank over all realizations of G.

Formalization note 19.17. The formalization writes $d = k + 1$, as in the note after Definition 19.3. The degree of freedom is formalized as stated, in the integers, with V the vertex set of G; the screw assignments are defined on all the bodies, and the rows of $R(G, p)$ involve only those of $V(G)$. It is 0 exactly when (G, p) is infinitesimally rigid on $V(G)$, for $V(G)$ nonempty (the second pin). Genericity is formalized for the frameworks whose hinges are spanned by points (the hinge-point assignments q of Definition 28.27), as `IsMaxRankHingePoints`. The realizations of G compared with q are the hinge-point assignments whose hinge at every link between distinct bodies is nondegenerate, and q itself must have this property. Nondegeneracy is not asked at a loop, whose hinge constrains no motion, so a framework with a degenerate hinge at a loop counts as generic, where Katoh and Tanigawa's multigraphs have no loops. The cheapest witness is a single loop, where every assignment is generic. Allowing degenerate hinges at the links between distinct bodies would not match the source, as a degenerate hinge welds its two bodies. Every assignment that is generic for row independence is generic in this sense, so one exists over an infinite field, and each has $\operatorname{def}(\tilde{G})$ degrees of freedom (Lemma 28.34, Corollary 28.41).

By the next lemma, the trivial motions form a D-dimensional subspace of $Z(G, p)$ when V is nonempty. So the degree of freedom is $\dim Z(G, p) - D \geq 0$, and it is 0 if and only if (G, p) is infinitesimally rigid.

Lemma 19.18 (The D trivial motions and the rank bound)**.** *Let (G, p) be a body-hinge framework with V nonempty, and for $1 \leq i \leq D$ let S_i^* be the screw assignment whose value at every body is the i-th basis vector of K^D. Then $S_1^*, \ldots, S_D^*$ are infinitesimal motions, and they form a basis of the space of trivial motions. Hence $\operatorname{rank} R(G, p) \leq D(|V| - 1)$, with equality if and only if (G, p) is infinitesimally rigid.*

Proof. A constant screw assignment S has $S(u) - S(v) = 0$ at every edge, so it is an infinitesimal motion. The trivial motions are the constant assignments

(Definition 19.15). Hence they form a subspace of $Z(G,p)$ (`trivialMotions_le_infinitesimalMotions`). As V is nonempty, sending $x \in K^D$ to the constant assignment with value x is an isomorphism from K^D onto them (`trivialMotions_eq_range_const`), and it sends the standard basis to $S_1^*, \dots, S_D^*$. So the trivial motions form a D-dimensional subspace of $Z(G,p)$. The screw assignments form a space of dimension $D|V|$ (`finrank_screwAssignment`), so $\operatorname{rank} R(G,p) = D|V| - \dim Z(G,p) \le D(|V|-1)$. Equality holds if and only if $Z(G,p)$ is the space of trivial motions, that is, if and only if (G,p) is infinitesimally rigid. □

Formalization note 19.19. The first pin says that the trivial screw assignments are infinitesimal motions, the second that they form a space of dimension D, and the third that $Z(G,p)$ is this space exactly when (G,p) is infinitesimally rigid. The fourth states the equality case for the dimension of $Z(G,p)$, which is $D|V| - \operatorname{rank} R(G,p)$: $\dim Z(G,p) = D$ exactly when (G,p) is infinitesimally rigid.

19.2. **Rank lemmas.** This subsection proves two rank lemmas of Katoh and Tanigawa [KT11, §5], and compares two frameworks on one graph. Deleting the columns of one body does not change the rank of $R(G,p)$ (KT Lemma 5.1). Two edges between the same two bodies, with linearly independent supporting extensors, force the two bodies to have the same screw center in every infinitesimal motion (KT Lemma 5.3). The last lemma says that if every hinge of one framework has a supporting extensor in the span of the other's, then every infinitesimal motion of the first is one of the second. Katoh and Tanigawa's remaining lemma of their §5, KT Lemma 5.2, rotates a panel without decreasing the rank; the formalization takes its steps with the realizations of the algebraic induction (the passage before Lemma 23.78).

Lemma 19.20 (Deleting one body's columns preserves rank; KT Lemma 5.1)**.** *Let (G,p) be a body-hinge framework, let v be a body, and let* $\operatorname{pinned}(v) = \{S \in Z(G,p) : S(v) = 0\}$. *Then $Z(G,p)$ is the direct sum of the space of trivial motions and* $\operatorname{pinned}(v)$, *so* $\dim \operatorname{pinned}(v) + D = \dim Z(G,p)$. *Equivalently, deleting the D columns of v from $R(G,p)$ does not change its rank.*

Proof. A trivial motion that vanishes at v vanishes everywhere. An infinitesimal motion S is the sum of the trivial motion with value $S(v)$ and the screw assignment $u \mapsto S(u) - S(v)$, which vanishes at v. The latter is again an infinitesimal motion, since it has the same differences $S(u) - S(w)$ as S. So $Z(G,p)$ is the direct sum, and the trivial motions have dimension D (Lemma 19.18).

Deleting the columns of v leaves the matrix whose null space consists of the restrictions to $V \setminus \{v\}$ of the motions in $\operatorname{pinned}(v)$. Its rank is therefore $D(|V| - 1) - \dim \operatorname{pinned}(v) = D|V| - \dim Z(G,p)$, the rank of $R(G,p)$. □

Katoh and Tanigawa prove the lemma by adding to $R(G,p)$ the D rows that tie down the body v [KT11, p. 668]. They cite White and Whiteley [WW87, §2] for the fact that fixing one body does not change the space of nontrivial motions. The direct sum above expresses that fact.

Lemma 19.21 (Two hinges of parallel edges give the full block; KT Lemma 5.3)**.** *Let (G,p) be a body-hinge framework, and let e and f be edges of G that both*

join the bodies $u \neq v$, with $C(p(e))$ and $C(p(f))$ linearly independent. Then every screw assignment S that satisfies the hinge constraints at e and f has $S(u) = S(v)$. Equivalently, the rows of $R(G,p)$ at e and f, restricted to the D columns of v, have rank D.

Proof. The difference $S(u)-S(v)$ lies in $\operatorname{span} C(p(e)) \cap \operatorname{span} C(p(f))$ (Definition 19.3). Two lines spanned by linearly independent vectors meet only in 0, so $S(u) = S(v)$. In the columns of v, the rows at e and f are $\pm r(p(e))$ and $\pm r(p(f))$. Their common null space in K^D is the same intersection (Definition 19.5), so they have rank D. □

Katoh and Tanigawa state the lemma for the graph with the two bodies u and v and the two edges e and f, with distinct hinges $p(e) \neq p(f)$, and their proof shows that distinct hinges satisfy the hypothesis above [KT11, p. 670]. By the remark after Definition 18.8, $C(p(e))$ is nonzero, it is determined by $p(e)$ up to a nonzero scalar, and it determines $p(e)$. So $C(p(e))$ and $C(p(f))$ are proportional only if $p(e) = p(f)$, and two nonzero vectors that are not proportional are linearly independent. For example, two hinges spanned by $d-1$ of $d+1$ affinely independent points, omitting different pairs of them, have linearly independent extensors by KT Lemma 2.1 (Lemma 18.9). Distinct hinges can lie in a common hyperplane, so the two-body graph has an infinitesimally rigid panel-hinge realization. This graph is the base case of the algebraic induction (Lemma 23.30).

Lemma 19.22 (Smaller hinge spans give fewer infinitesimal motions)**.** *Let (G,p) and (G,p') be body-hinge frameworks on the same multigraph G, and suppose that $\operatorname{span} C(p'(e)) \subseteq \operatorname{span} C(p(e))$ at every edge e. Then $Z(G,p') \subseteq Z(G,p)$. In particular, $\dim Z(G,p') \leq \dim Z(G,p)$ and $\operatorname{rank} R(G,p) \leq \operatorname{rank} R(G,p')$.*

Proof. Let $S \in Z(G,p')$, and let e be an edge with ends u and v. Then $S(u) - S(v)$ lies in $\operatorname{span} C(p'(e))$ (Definition 19.3), hence in $\operatorname{span} C(p(e))$, so S satisfies the hinge constraint of (G,p) at e. So $Z(G,p') \subseteq Z(G,p)$, and $\dim Z(G,p') \leq \dim Z(G,p)$. The rank of $R(G,p)$ is $D|V| - \dim Z(G,p)$ (the remark after Definition 19.7), and likewise for p', which gives the bound on the ranks. □

Formalization note 19.23. The first pin is the inclusion, and the second the bound on the dimensions, for a finite set of bodies; the bound on the ranks follows by the remark after Definition 19.7. Both pins assume the inclusion of spans at every edge label, also outside $E(G)$. The formalization lets $C(p(e))$ be zero (the note after Definition 19.3), so the inclusion can be strict, at an edge with $C(p'(e)) = 0 \neq C(p(e))$. When all the extensors are nonzero, the hypothesis says that the spans are equal, and then $Z(G,p') = Z(G,p)$.

19.3. **Rank-addition lemmas.** This subsection computes or bounds the rank of $R(G,p)$ from the ranks of pieces of G. In the first lemmas the pieces share at most two bodies. If two sides are joined by at most one edge, the rank of the whole is at least the sum of their ranks plus $D-1$ for the joining edge; this is the rank step in the proof of KT Lemma 6.1. If two sides share one body, their ranks add. If they share two bodies u and v, the rank of the whole is the sum of the ranks of the sides minus a correction, which depends on the motions of v relative to u that each side allows. We compute the correction for a path attached at its two ends. The next

lemmas bound the rank from below when a rigid subgraph is contracted to one body (KT Lemma 6.2) and when a body of degree two is restored after splitting off (KT Lemmas 6.8 and 6.10), from the block-triangular shape of the rigidity matrix. The last one deletes the columns of the bodies of a contracted subgraph, which Case I of the algebraic induction does when it compares the rows outside the subgraph with a realization of the contraction (KT Claim 6.4, Section 23.6).

Throughout, (G, p) is a body-hinge framework. A subgraph H of G keeps the hinges of p at its edges and the bodies of (G, p): its screw assignments are those of (G, p), maps $B \to \bigwedge^{d-1} K^{d+1}$ on a finite set B of bodies that contains $V(G)$. We write $R(H, p)$ and $Z(H, p)$ for its rigidity matrix and its space of infinitesimal motions, defined as before. A body of $B \setminus V(H)$ lies on no edge of H, so no hinge constraint of H involves it. Its columns in $R(H, p)$ are zero, and the rank of $R(H, p)$ does not depend on these bodies. As after Definition 19.7, we regard the rows of $R(H, p)$ as linear forms on screw assignments: the rows of an edge e with ends u and v are the forms $S \mapsto r_i(p(e)) \cdot (S(u) - S(v))$. The *row space* of $R(H, p)$ is the span of its rows, and its dimension is the rank of $R(H, p)$. For a set V_1 of bodies, $G[V_1]$ is the subgraph of G induced on V_1.

Lemma 19.24 (Rank addition at a cut edge; from the proof of KT Lemma 6.1)**.** *Let (G, p) be a body-hinge framework with $C(p(e)) \neq 0$ at every edge e. Let V_1 be a nonempty proper subset of $V(G)$, let $V_2 = V(G) \setminus V_1$, and let C be a set of at most one edge of G, each with one end in V_1 and the other in V_2. Then*

$$\operatorname{rank} R(G[V_1], p) + (D-1)\,|C| + \operatorname{rank} R(G[V_2], p) \;\leq\; \operatorname{rank} R(G, p).$$

In particular, if an edge of G joins V_1 to a body $w \in V_2$, then

$$\operatorname{rank} R(G[V_1], p) + D - 1 \;\leq\; \operatorname{rank} R(G[V_1 \cup \{w\}], p).$$

Proof. Let $\mathcal{R}_1$, $\mathcal{R}_2$ and $\mathcal{R}$ be the row spaces of $R(G[V_1], p)$, $R(G[V_2], p)$ and $R(G, p)$. The rows of $G[V_1]$ and of $G[V_2]$ are rows of G, so $\mathcal{R}_1 + \mathcal{R}_2 \subseteq \mathcal{R}$. A row of $G[V_1]$ depends only on the screws at the bodies of V_1, and a row of $G[V_2]$ only on those at the bodies of V_2. Let $\varphi \in \mathcal{R}_1 \cap \mathcal{R}_2$, let S be a screw assignment, and let S' agree with S on V_1 and vanish elsewhere. As $\varphi \in \mathcal{R}_1$, it takes the same value at S and at S'. As $\varphi \in \mathcal{R}_2$ and S' vanishes on V_2, that value is 0. Hence $\mathcal{R}_1 \cap \mathcal{R}_2 = 0$, which proves the bound when C is empty.

Otherwise $C = \{e\}$, where e has ends $u \in V_1$ and $v \in V_2$. By Lemma 19.9, the rows of e span a subspace $\mathcal{R}_e$ of $\mathcal{R}$ of dimension $D - 1$. Its members are the forms $S \mapsto r(S(u) - S(v))$, where r is a linear form on K^D that vanishes on $C(p(e))$ (Definition 19.5). For $x \in K^D$, let S^x be the screw assignment equal to x on V_1 and to 0 elsewhere. A row of an edge inside V_1 or inside V_2 vanishes at S^x, and the member of $\mathcal{R}_e$ given by r takes the value $r(x)$ there. So a member of $\mathcal{R}_e$ that lies in $\mathcal{R}_1 + \mathcal{R}_2$ has $r(x) = 0$ for every x, and it is zero. Therefore $\mathcal{R}_1$, $\mathcal{R}_e$ and $\mathcal{R}_2$ are independent. Their sum lies in $\mathcal{R}$ and has dimension $\operatorname{rank} R(G[V_1], p) + (D-1) + \operatorname{rank} R(G[V_2], p)$.

For the particular case, apply the bound to $G[V_1 \cup \{w\}]$, V_1 and the given edge. The first side is $G[V_1]$, and the rank of the second side, the single body w, is at least 0. □

Katoh and Tanigawa prove this bound for a cut edge uv in the proof of KT Lemma 6.1 [KT11, p. 672]. They delete the columns of u from the rows of $G[V_1]$, which keeps the rank of these rows by KT Lemma 5.1 (Lemma 19.20). They then read the bound off the block-triangular shape of the resulting matrix.

When two sides share a body, we compare their row spaces through their motion spaces, by the following lemma.

Lemma 19.25 (The row space is the annihilator of the motions)**.** *Let (G, p) be a body-hinge framework with finitely many bodies. Then the row space of $R(G, p)$ is the space of linear forms on screw assignments that vanish on $Z(G, p)$. In particular, two body-hinge frameworks on the same bodies with the same infinitesimal motions have the same row space, and so the same rank.*

Proof. The space $Z(G, p)$ consists of the screw assignments at which every row of $R(G, p)$ vanishes (Definition 19.7). The screw assignments form a finite-dimensional space. In the dual of a finite-dimensional space, every subspace is the space of the forms that vanish wherever all its members vanish. Applied to the row space, this gives the first claim, and the second follows from it. □

Lemma 19.26 (Rank addition at a cut vertex)**.** *Let (G, p) be a body-hinge framework, let v be a body, and let V_1, V_2 be sets of bodies with $V_1 \cap V_2 \subseteq \{v\}$ such that every edge of G lies inside V_1 or inside V_2. Then*

$$\operatorname{rank} R(G, p) = \operatorname{rank} R(G[V_1], p) + \operatorname{rank} R(G[V_2], p).$$

Proof. Every edge of G is an edge of $G[V_1]$ or of $G[V_2]$, so the row space of $R(G, p)$ is the sum of the row spaces of $R(G[V_1], p)$ and $R(G[V_2], p)$. We show that these two row spaces meet in 0; then the ranks add. By Lemma 19.25, a form in both vanishes on $Z(G[V_1], p)+Z(G[V_2], p)$, so it suffices that this sum contains every screw assignment S. Let T be the assignment equal to $S(v)$ on V_1 and to S elsewhere. It is constant on V_1, so it is a motion of $G[V_1]$. The difference $S - T$ vanishes outside V_1 and at v, hence on V_2, so it is a motion of $G[V_2]$. □

Two sides that share two bodies u and v interact through the motions of v relative to u.

Definition 19.27 (Relative screws and joint rows)**.** Let (G, p) be a body-hinge framework and u, v two bodies. The *relative screws* of u, v in (G, p) form the subspace

$$\rho_{uv}(G, p) = \{\, S(v) - S(u) : S \in Z(G, p) \,\} \subseteq K^D.$$

For a subspace $U \subseteq K^D$, the *joint rows* at u, v confining the relative screw to U are the linear forms $S \mapsto r(S(u) - S(v))$ on screw assignments, one for each linear form r on K^D that vanishes on U.

The joint rows at u, v confining the relative screw to U all vanish at a screw assignment S if and only if $S(u) - S(v) \in U$, as for the hinge-row block (Definition 19.5). For an edge e with ends u and v, the joint rows confining the relative screw to $\operatorname{span} C(p(e))$ span the same space as the rows of e in $R(G, p)$. At $U = 0$ the joint rows force $S(u) = S(v)$. If $u \neq v$, the map $S \mapsto S(u) - S(v)$ takes the screw assignments onto K^D, so the joint rows span a space of dimension $D - \dim U$.

The next lemma glues two subgraphs that share two bodies. We state it for any two subgraphs whose edges together are those of G, not only for induced subgraphs: Lemma 19.32 below glues $G[V_1]$ to a path, which is not an induced subgraph when its ends are adjacent.

Lemma 19.28 (Rank gluing at two shared bodies)**.** *Let (G,p) be a body-hinge framework and let $u \neq v$ be bodies. Let V_1, V_2 be sets of bodies with $V_1 \cap V_2 \subseteq \{u,v\}$, and let G_1, G_2 be subgraphs of G such that every edge of G is an edge of G_1 or of G_2, and every edge of G_i lies inside V_i. Let $\rho_i = \rho_{uv}(G_i,p)$ (Definition 19.27). Then the row spaces of $R(G_1,p)$ and $R(G_2,p)$ meet in the span of the joint rows at u,v confining the relative screw to $\rho_1+\rho_2$, and*

$$\operatorname{rank} R(G,p) = \operatorname{rank} R(G_1,p) + \operatorname{rank} R(G_2,p) + \dim(\rho_1+\rho_2) - D.$$

Proof. Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be the row spaces of $R(G_1,p)$ and $R(G_2,p)$, and let $\delta(S) = S(u) - S(v)$. By Lemma 19.25, a form lies in $\mathcal{R}_1 \cap \mathcal{R}_2$ if and only if it vanishes on $Z(G_1,p) + Z(G_2,p)$.

This sum contains every screw assignment S with $S(u) = S(v)$. Indeed, let T be the assignment equal to $S(v)$ on V_2 and to S elsewhere. It is constant on V_2, so it is a motion of G_2. The difference $S - T$ vanishes outside V_2, and on $V_1 \cap V_2 \subseteq \{u,v\}$ because $S(u) = S(v)$. So it vanishes on V_1, and it is a motion of G_1. As δ maps $Z(G_i,p)$ onto $-\rho_i = \rho_i$, the sum $Z(G_1,p) + Z(G_2,p)$ is therefore the space of screw assignments S with $\delta(S) \in \rho_1 + \rho_2$.

Since $u \neq v$, δ is onto K^D, so every form that vanishes on the kernel of δ is $r \circ \delta$ for a linear form r on K^D. Such a form vanishes on $Z(G_1,p) + Z(G_2,p)$ exactly when r vanishes on $\rho_1 + \rho_2$, that is, when it is a joint row at u,v confining the relative screw to $\rho_1 + \rho_2$. This proves the first claim. These joint rows span a space of dimension $D - \dim(\rho_1 + \rho_2)$, by the remarks after Definition 19.27.

Every edge of G is an edge of G_1 or of G_2, with the same hinge, so the row space of $R(G,p)$ is $\mathcal{R}_1 + \mathcal{R}_2$. The rank formula follows from $\dim(\mathcal{R}_1 + \mathcal{R}_2) = \dim \mathcal{R}_1 + \dim \mathcal{R}_2 - \dim(\mathcal{R}_1 \cap \mathcal{R}_2)$. □

Corollary 19.29 (Rank gluing at a vertex 2-cut)**.** *Let (G,p) be a body-hinge framework, let $u \neq v$ be bodies, and let V_1, V_2 be sets of bodies with $V_1 \cap V_2 = \{u,v\}$ such that every edge of G lies inside V_1 or inside V_2. Let $\rho_i = \rho_{uv}(G[V_i],p)$. Then the row spaces of $R(G[V_1],p)$ and $R(G[V_2],p)$ meet in the span of the joint rows at u,v confining the relative screw to $\rho_1+\rho_2$, and*

$$\operatorname{rank} R(G,p) = \operatorname{rank} R(G[V_1],p) + \operatorname{rank} R(G[V_2],p) + \dim(\rho_1+\rho_2) - D.$$

Proof. Apply Lemma 19.28 to $G_i = G[V_i]$. The edges of $G[V_i]$ lie inside V_i, and every edge of G is an edge of $G[V_1]$ or of $G[V_2]$. □

We apply Lemma 19.28 to a path attached to $G[V_1]$ at two bodies $a \neq b$; Definition 30.63 calls such a path an open ear on V_1. The rank and the relative screws of a path come first.

Lemma 19.30 (The rank of a path)**.** *Let (P,p) be a body-hinge framework whose edges are those of a path $a = x_0 - x_1 - \cdots - x_k - x_{k+1} = b$ ($k \geq 0$) on distinct bodies, and let $C_i = C(p(x_i x_{i+1})) \neq 0$ for $0 \leq i \leq k$. Then* $\operatorname{rank} R(P,p) = (D-1)(k+1)$.

Proof. The matrix $R(P,p)$ has $D-1$ rows for each of its $k+1$ edges (Definition 19.5), so its rank is at most $(D-1)(k+1)$. Conversely, for $0 \le i \le k$ the edge x_ix_{i+1} is the only edge of P that joins x_{i+1} to $\{x_0, \dots, x_i\}$. So adding x_{i+1} to $\{x_0, \dots, x_i\}$ raises the rank of the subgraph of P induced on these bodies by at least $D-1$ (Lemma 19.24). The subgraph induced on $\{x_0\}$ has no edge, and the subgraph induced on all the bodies of the path is P. □

Lemma 19.31 (The relative screws of a path)**.** *Let (P,p) be a body-hinge framework whose edges are those of a path $a = x_0 - x_1 - \cdots - x_k - x_{k+1} = b$ ($k \ge 0$) on distinct bodies, and let $C_i = C(p(x_ix_{i+1}))$ for $0 \le i \le k$. Then $\rho_{ab}(P,p) = \operatorname{span}(C_0, \dots, C_k)$.*

Proof. Let S be an infinitesimal motion of (P,p). Each difference $S(x_{i+1}) - S(x_i)$ lies in $\operatorname{span} C_i$ (Definition 19.3), and these differences add up to $S(b) - S(a)$, which therefore lies in $\operatorname{span}(C_0, \dots, C_k)$. Conversely, fix i and let S be the screw assignment equal to C_i at $x_{i+1}, \dots, x_{k+1}$ and to 0 elsewhere. Its difference across the edge x_ix_{i+1} is C_i, and across every other edge of P it is 0. So S is an infinitesimal motion with $S(b) - S(a) = C_i$, and the subspace $\rho_{ab}(P,p)$ contains every C_i. □

Lemma 19.32 (Rank addition along an open ear)**.** *Let (G,p) be a body-hinge framework and V_1 a set of bodies. Let $a - x_1 - \cdots - x_k - b$ ($k \ge 0$) be a path in G with ends $a \ne b$ in V_1 and distinct bodies $x_1, \dots, x_k$ outside V_1, such that every edge of G off the path lies inside V_1. Write $x_0 = a$ and $x_{k+1} = b$, suppose that $C_i = C(p(x_ix_{i+1})) \ne 0$ for $0 \le i \le k$, and let $\Lambda = \operatorname{span}(C_0, \dots, C_k)$ and $\rho = \rho_{ab}(G[V_1], p)$. Then*

$$\operatorname{rank} R(G,p) = \operatorname{rank} R(G[V_1],p) + (D-1)(k+1) + \dim(\rho + \Lambda) - D.$$

Proof. Let P be the path, as a subgraph of G, and let V_2 be the set of its bodies. Then $V_1 \cap V_2 = \{a, b\}$, and every edge of G is an edge of P or of $G[V_1]$. By Lemma 19.28 at $u = a$ and $v = b$, with $G_1 = G[V_1]$ and $G_2 = P$,

$$\operatorname{rank} R(G,p) = \operatorname{rank} R(G[V_1],p) + \operatorname{rank} R(P,p) + \dim(\rho + \rho_{ab}(P,p)) - D.$$

The path has rank $(D-1)(k+1)$ (Lemma 19.30) and relative screws $\rho_{ab}(P,p) = \Lambda$ (Lemma 19.31). □

We also need notation for deleting columns. For a set W of bodies and a screw assignment S, let S^W agree with S off W and vanish on W. For a linear form φ on screw assignments, let $\pi_W(\varphi)$ be the form $S \mapsto \varphi(S^W)$. If φ is a row of a rigidity matrix, then $\pi_W(\varphi)$ is the same row with zeros in the columns of W, and we say that the columns of W are deleted from it. The map π_W is linear.

In the proof of KT Lemma 6.2 [KT11, pp. 673–674], where H is a rigid subgraph of G, Katoh and Tanigawa read the rank of $R(G,p)$ off its block-triangular shape (KT eq. (6.3)). They compare the rank of the rows of the edges of G not in H, with the columns of $V(H)$ deleted, with that of a realization of the contraction $G/E(H)$, in which H becomes one body (Definition 21.3; KT eqs. (6.4) and (6.5)). The next lemma is the count behind this comparison, for any linear map π in place of $\pi_{V(H)}$.

Lemma 19.33 (Block-triangular rank addition; from the proof of KT Lemma 6.2)**.** *Let F, F_H and F_c be body-hinge frameworks on one set of bodies, with row spaces $\mathcal{R}$, $\mathcal{R}_H$ and $\mathcal{R}_c$, and let π be a linear map from the space of linear forms on screw*

assignments to itself. Suppose that $\mathcal{R}_H \subseteq \mathcal{R} \cap \ker \pi$*, that* $\pi(\mathcal{R}_c) \subseteq \pi(\mathcal{R})$*, and that* π *is injective on* $\mathcal{R}_c$*. Then*

$$\operatorname{rank} R(F_H) + \operatorname{rank} R(F_c) \;\leq\; \operatorname{rank} R(F).$$

Proof. By rank-nullity for π on $\mathcal{R}$, $\dim \mathcal{R} = \dim \pi(\mathcal{R}) + \dim(\mathcal{R} \cap \ker \pi)$. The first hypothesis gives $\dim \mathcal{R}_H \leq \dim(\mathcal{R} \cap \ker \pi)$. The other two give $\dim \mathcal{R}_c = \dim \pi(\mathcal{R}_c) \leq \dim \pi(\mathcal{R})$. Adding the two bounds proves the lemma. □

In the proof of KT Lemma 6.2 (Lemma 23.102), F realizes G, F_H is its restriction to H, F_c realizes the contraction $G/E(H)$, and $\pi = \pi_{V(H)}$. The first hypothesis holds because a row of an edge of H depends only on the screws at the bodies of $V(H)$, so $\pi_{V(H)}$ maps it to 0. The second holds because the edges of $G \setminus E(H)$ have the same hinges in both realizations (KT eq. (6.4)), so that their rows agree once the columns of $V(H)$ are deleted.

In KT Lemmas 6.8 and 6.10 a body v of degree two is restored after splitting off. Katoh and Tanigawa bring the rigidity matrix to a block-triangular form by column and row operations (KT eqs. (6.16) and (6.29), pp. 679 and 686). In this form a block of rows has nonzero entries only in the columns of v, and they read the rank off it. The next lemma gives such a count for two families of linear forms in the row space of a rigidity matrix, which need not be its rows: one family is independent on the screws of v, and the other vanishes on them.

Lemma 19.34 (Rank addition at one body; from the proofs of KT Lemmas 6.8 and 6.10)**.** *Let* F *be a body-hinge framework, let* v *be a body, and let* $(\varphi_i)_{i \in I}$ *and* $(\psi_j)_{j \in J}$ *be finite families of linear forms in the row space of* $R(F)$*. Suppose that the restrictions of the* φ_i *to the screw assignments that vanish off* v *are linearly independent, and that the* ψ_j *are linearly independent and vanish on these screw assignments. Then* $|I| + |J| \leq \operatorname{rank} R(F)$*. In particular, if* ω *is a further form in the row space such that the restrictions of the* φ_i *and of* ω *to these screw assignments are linearly independent, then* $|I| + 1 + |J| \leq \operatorname{rank} R(F)$*.*

Proof. Suppose that $\sum_i a_i \varphi_i + \sum_j b_j \psi_j = 0$. Every ψ_j vanishes on the screw assignments that vanish off v, so $\sum_i a_i \varphi_i$ vanishes on them, and every a_i is 0 by the first hypothesis. Then $\sum_j b_j \psi_j = 0$, and every b_j is 0. So the $|I| + |J|$ forms are linearly independent (Lemma 23.127). They lie in the row space of $R(F)$, whose dimension is therefore at least $|I| + |J|$. The second claim is the first, applied to the family of the φ_i and ω (Lemma 23.159). □

The independence of the two families is Lemma 23.127 of the algebraic induction, for KT Lemma 6.8, and with ω added it is Lemma 23.159, for KT eq. (6.29). In Case II (KT Lemma 6.8) the φ_i are the $D - 1$ rows of an edge at v, and the ψ_j are rows of the graph obtained by splitting off v, which lie in the row space of $R(F)$ without being its rows. In KT Lemma 6.10, ω is the further row that the row operations of eq. (6.29) produce.

Case I of the algebraic induction contracts a proper rigid subgraph H of G (Definition 20.17). Let r be a body of H. The contraction $G/E(H)$ has the edges of $G \setminus E(H)$, with every end in $V(H)$ replaced by r, so its bodies meet $V(H)$ only

in r. Deleting the columns of $V(H)$ from its rows therefore deletes those of r. By KT Lemma 5.1 this does not change the rank, as Katoh and Tanigawa note at eq. (6.5) (p. 674). The next lemma states this in the form of the third hypothesis of Lemma 19.33, for $\pi = \pi_{V(H)}$.

Lemma 19.35 (Deleting the columns of a set that meets $V(G)$ in one body; KT eq. (6.5))**.** *Let (G, p) be a body-hinge framework, and let t be a set of bodies with $V(G) \cap t = \{r\}$. Then every screw assignment is the sum of an infinitesimal motion of (G, p) and a screw assignment that vanishes on t. Hence π_t is injective on the row space of $R(G, p)$, and deleting the columns of t from $R(G, p)$ does not change its rank.*

Proof. Let S be a screw assignment, and let T be equal to $S(r)$ on $V(G)$ and to S elsewhere. Both ends of every edge lie in $V(G)$, where T is constant, so T is an infinitesimal motion. The difference $S - T$ vanishes at r and off $V(G)$. As $t \setminus \{r\}$ lies off $V(G)$, the difference vanishes on t. This proves the first claim.

Let φ be a form in the row space with $\pi_t(\varphi) = 0$. Then φ vanishes on the screw assignments that vanish on t, and, like every row of $R(G, p)$, on $Z(G, p)$ (Definition 19.7). By the first claim it vanishes everywhere, so $\varphi = 0$. Hence π_t is injective on the row space, and the row space has the same dimension as its image, the row space of $R(G, p)$ with the columns of t deleted. □

The motion T plays the part of the trivial motion with value $S(r)$ in the proof of KT Lemma 5.1 (Lemma 19.20). In Case I the rows of $G \setminus E(H)$ are compared with those of $G/E(H)$ after this deletion (KT Claim 6.4 and eq. (6.9)), at every deficiency in Lemma 23.96 and when $G/E(H)$ is rigid in Lemma 23.98.

20. The matroid $M(\tilde{G})$, deficiency, and k-dof graphs

This chapter develops the combinatorics of Katoh and Tanigawa's proof [KT11, §2.5 and §3]. Throughout, G is a multigraph, $D = \binom{d+1}{2}$, and $\tilde{G} = (D-1) \cdot G$ is the multigraph of Section 17, in which each edge of G is repeated $D - 1$ times. We study two numbers attached to $\tilde{G}$: the rank of a matroid $M(\tilde{G})$ on its edges, the union of D copies of its cycle matroid, and the deficiency $\operatorname{def}(\tilde{G})$, a maximum over the partitions of $V(G)$. Katoh and Tanigawa's Proposition 2.3, which they credit to Jackson and Jordán [JJ10], relates both to the rank of the rigidity matrix $R(G, p)$ of a generic body-hinge framework (G, p) (Section 19):

$$\operatorname{rank} R(G,p) \;=\; D\,(|V(G)| - 1) - \operatorname{def}(\tilde{G}) \;=\; \operatorname{rank} M(\tilde{G}).$$

This chapter proves the second equality, KT eq. (2.4), and the inequality $\leq$ in the first at every realization. The first is the rank formula of Jackson and Jordán. We prove it in a conditional form, from a bound on the space of motions, in the chapter on the algebraic induction (Proposition 23.19), and at every generic hinge-point assignment of a multigraph in a later chapter (Theorem 28.40, Proposition 28.42).

Section 20.1 defines $M(\tilde{G})$, the deficiency, and the k-dof-graphs, the multigraphs with $\operatorname{def}(\tilde{G}) = k$. Section 20.2 proves KT eq. (2.4) from the rank of a matroid union (Section 15), and characterizes deficiency zero by D edge-disjoint spanning

trees (KT Proposition 2.2). Section 20.3 bounds the rank of $R(G,p)$ by the deficiency. Section 20.4 proves Katoh and Tanigawa's lemmas on k-dof-graphs from their §3, and computes the deficiency at cuts, cut vertices and ears. The combinatorial induction of the next chapter (Section 21) uses these results.

20.1. **The matroid $M(\tilde{G})$, the deficiency and k-dof-graphs.** This subsection follows Katoh and Tanigawa [KT11, §2.5 and the start of §3]. We define a matroid $M(\tilde{G})$ on the edges of $\tilde{G}$, the deficiency $\mathrm{def}(\tilde{G})$, a maximum over the partitions of $V(G)$, and the partitions that attain it. Every base B of $M(\tilde{G})$ has $|B| + \mathrm{def}(\tilde{G}) = D(|V(G)| - 1)$ (KT eq. (2.4)), which we prove in Section 20.2. Then we define the k-dof-graphs, the multigraphs G with $\mathrm{def}(\tilde{G}) = k$, and the minimal ones.

We use the notation of Section 16.1: $V(F)$ is the set of vertices incident with an edge of a set F of edges, and $\tilde{G} \restriction F$ is the multigraph with the vertices of $\tilde{G}$ and the edges of F. Katoh and Tanigawa define $M(\tilde{G})$ as the matroid on $E(\tilde{G})$ induced by the submodular function $F \mapsto D(|V(F)| - 1)$ (Definition 15.3). A set F is independent in it if and only if $\tilde{G} \restriction F$ is (D,D)-sparse (Definition 16.1). They remark that this matroid is the union of D copies of the cycle matroid [KT11, p. 655], and we take the union as the definition.

Definition 20.1 (The matroid $M(\tilde{G})$)**.** Let G be a multigraph. The matroid $M(\tilde{G})$ is the union of D copies of the cycle matroid of $\tilde{G}$ (Definition 15.6). A set F of edges of $\tilde{G}$ is independent in $M(\tilde{G})$ if and only if $\tilde{G} \restriction F$ is (D,D)-sparse.

Proof. Apply Theorem 16.2 to the multigraph $\tilde{G}$, with $k = D$. □

Formalization note 20.2. As for the k-frame matroid (Formalization note 16.17), the formal union is a matroid on all pairs (e,i) of an edge label e and an index $1 \le i \le D-1$, in which a pair outside $E(\tilde{G})$ is a loop. The matroid $M(\tilde{G})$ is its restriction to $E(\tilde{G})$.

So $M(\tilde{G})$ is Katoh and Tanigawa's matroid. By Lemma 15.8, its independent sets are the unions of D forests of $\tilde{G}$. They are also the sets F such that the multigraph $\tilde{G} \restriction F$ has an independent body-bar framework in $\mathbb{R}^d$, by Tay's theorem (Theorem 16.27).

Definition 20.3 (The D-deficiency)**.** Let G be a multigraph. For a partition P of $V(G)$ into nonempty parts, let $d_G(P)$ be the number of edges of G that join different parts of P. The *D-deficiency* of P is

$$\mathrm{def}_{\tilde{G}}(P) \;=\; D\,(|P| - 1) \;-\; (D-1)\,d_G(P),$$

and the *deficiency* $\mathrm{def}(\tilde{G})$ is the largest value of $\mathrm{def}_{\tilde{G}}(P)$ over the partitions P of $V(G)$.

Formalization note 20.4. In the formalization a partition of $V(G)$ is given by a labeling of the vertices whose labels are again vertices. Its parts are the sets of vertices of G with a common label. So $|P|$ is the number of labels used on $V(G)$, and an edge joins different parts when its ends have different labels. The values of

the labelings are integers, and the deficiency is their supremum, which is attained because there are finitely many labelings.

Katoh and Tanigawa [KT11, p. 655] write $\mathrm{def}_{\tilde{G}}(P) = D(|P|-1) - d_{\tilde{G}}(P)$, where $d_{\tilde{G}}(P) = (D-1)\,d_G(P)$ is the number of edges of $\tilde{G}$ that join different parts of P. If $V(G)$ is nonempty, then the partition $\{V(G)\}$ has value 0, so $\mathrm{def}(\tilde{G}) \geq 0$. By the theorem of Tutte and Nash-Williams [Tut61, NW61] (KT Proposition 2.2), $\mathrm{def}(\tilde{G}) = 0$ if and only if $\tilde{G}$ contains D edge-disjoint spanning trees. We prove this for $D \geq 2$ in Lemma 20.14.

Several proofs below start from a partition of $V(G)$ of largest value.

Definition 20.5 (Tight partitions)**.** Let G be a multigraph. Following Jackson and Jordán [JJ08a, §2], a partition P of $V(G)$ is *tight* if it attains the deficiency, that is, if $\mathrm{def}_{\tilde{G}}(P) = \mathrm{def}(\tilde{G})$.

Lemma 20.6 (Tight partitions exist)**.** *Let G be a multigraph whose vertex set is finite and nonempty. Then some partition P of $V(G)$ is tight, that is,* $\mathrm{def}_{\tilde{G}}(P) = \mathrm{def}(\tilde{G})$.

Proof. A finite set has finitely many partitions, so the maximum that defines $\mathrm{def}(\tilde{G})$ is attained. □

The deficiency depends on the dimension d through $D = \binom{d+1}{2}$. When d varies, we write $\mathrm{def}_d(G)$ for $\mathrm{def}(\tilde{G})$ at this D. So $\mathrm{def}_2(G)$ and $\mathrm{def}_3(G)$ are the deficiencies at $D = 3$ and $D = 6$, which Section 30 compares.

Lemma 20.7 (On a connected graph the deficiency does not increase with the dimension)**.** *Let G be a connected multigraph, and let $d \leq d'$ be natural numbers. Then* $\mathrm{def}_{d'}(G) \leq \mathrm{def}_d(G)$. *In particular* $\mathrm{def}_3(G) \leq \mathrm{def}_2(G)$.

Proof. Let $D \leq D'$ be the values of $\binom{d+1}{2}$ and $\binom{d'+1}{2}$, and let P be a partition of $V(G)$. At least $|P|-1$ edges of G join different parts of P. Indeed, once these edges are deleted, each component lies in one part, so there are at least $|P|$ components. Deleting an edge raises the number of components by at most one, and G is connected. Hence the value of P at D' minus its value at D is

$$(D'-D)(|P|-1-d_G(P)) \;\leq\; 0.$$

So the value of P at D' is at most $\mathrm{def}_d(G)$, and the lemma follows. □

By Jackson and Jordán's rank formula (KT Proposition 2.3; Proposition 28.42), a generic realization of G has $\mathrm{def}(\tilde{G})$ degrees of freedom (Definition 19.16, Corollary 28.41). This is why Katoh and Tanigawa speak of k-dof-graphs, for k degrees of freedom [KT11, §3].

Definition 20.8 (k-dof-graphs and minimal k-dof-graphs)**.** Let G be a multigraph and let k be an integer. Then G is a *k-dof-graph* if $\mathrm{def}(\tilde{G}) = k$. It is a *minimal* k-dof-graph if moreover, for every edge e of G, every base of $M(\tilde{G})$ meets the set $\tilde{e}$ of the $D-1$ copies of e in $\tilde{G}$.

Katoh and Tanigawa call a k-dof-graph minimal if removing any edge gives a graph that is not a k-dof-graph, and observe that this is the condition above [KT11, pp. 656–657]. Indeed, removing an edge e of G leaves the deficiency unchanged if and only if some base of $M(\tilde{G})$ avoids $\tilde{e}$. This follows from KT eq. (2.4) (Theorem 20.12 below), since $M(\widetilde{G-e})$ is the restriction of $M(\tilde{G})$ to $E(\tilde{G}) \setminus \tilde{e}$ (Lemma 20.24).

20.2. **The deficiency and the rank of $M(\tilde{G})$.** This subsection proves KT eq. (2.4). Let $D \geq 1$, and let G have at least one vertex. Every base of $M(\tilde{G})$ has $\operatorname{rank} M(\tilde{G})$ elements, so eq. (2.4) says that

$$\operatorname{def}(\tilde{G}) \;=\; D\,(|V(G)| - 1) - \operatorname{rank} M(\tilde{G}).$$

We call the right-hand side the *corank* of $M(\tilde{G})$. Katoh and Tanigawa observe eq. (2.4) from the equivalence of (ii) and (iii) in their Proposition 2.3. They note that the deficiency is the corank $|E(\tilde{G})| - \operatorname{rank} M(\tilde{G})$ in the usual sense when $\tilde{G}$ has $D(|V(G)| - 1)$ edges [KT11, pp. 655–656].

Since the independent sets of $M(\tilde{G})$ are the unions of D forests of $\tilde{G}$, eq. (2.4) is the classical formula for the largest size of such a union, applied to $\tilde{G}$ (see Jackson and Jordán [JJ10, Theorem 4.1(a)]). We derive it from the rank of a matroid union (Theorem 15.10). For a partition P of $V(G)$, the edges of $\tilde{G}$ inside the parts of P give $\operatorname{rank} M(\tilde{G}) + \operatorname{def}_{\tilde{G}}(P) \leq D(|V(G)| - 1)$ (Lemma 20.11). Conversely, let Y be a set of edges at which the rank formula attains its minimum. Then equality holds at the partition of $V(G)$ into the vertex sets of the components of $\tilde{G} \restriction Y$ (Theorem 20.12). We begin with an upper bound on the rank that follows from sparsity alone. Finally we characterize deficiency zero by spanning trees.

Lemma 20.9 (An upper bound on the rank of $M(\tilde{G})$)**.** *Let G be a multigraph with at least one vertex. Then*

$$\operatorname{rank} M(\tilde{G}) \;\leq\; D\,(|V(G)| - 1).$$

Proof. Let B be a base of $M(\tilde{G})$. Then $\tilde{G} \restriction B$ is (D, D)-sparse (Definition 20.1). If B is empty, the bound holds as $|V(G)| \geq 1$. Otherwise sparsity, applied to B itself, gives $|B| \leq D\,|V(B)| - D \leq D\,(|V(G)| - 1)$. □

For a set Y of edges of $\tilde{G}$, the multigraph $\tilde{G} \restriction Y$ has all the vertices of G, and each vertex outside $V(Y)$ is a component of its own. So the rank of Y in the cycle matroid of $\tilde{G}$ is $|V(G)| - c$, where c is the number of components of $\tilde{G} \restriction Y$ (Section 16.1).

Lemma 20.10 (The rank of the edges inside the parts of a partition)**.** *Let G be a multigraph, and let r be the rank function of the cycle matroid of $\tilde{G}$. Let P be a partition of $V(G)$, and let Y be a set of edges of $\tilde{G}$ such that the ends of each edge of Y lie in one part of P. Then $r(Y) + |P| \leq |V(G)|$.*

Proof. The ends of each edge of Y lie in one part of P, so each component of $\tilde{G} \restriction Y$ lies in one part. Each part is nonempty, so it contains a component, and $\tilde{G} \restriction Y$ has at least $|P|$ components. Hence $r(Y) \leq |V(G)| - |P|$. □

The proof of Lemma 20.11 applies the lemma to the set of all the edges of $\tilde{G}$ inside the parts of P, and the formal proof of Theorem 20.12 to the set Y_0 of its proof.

Lemma 20.11 (Weak duality: $\operatorname{def}(\tilde{G}) \le \operatorname{corank} M(\tilde{G})$)**.** *Let $D \ge 1$, and let G be a multigraph with at least one vertex. Then every partition P of $V(G)$ satisfies*

$$\operatorname{rank} M(\tilde{G}) + \operatorname{def}_{\tilde{G}}(P) \;\le\; D\,(|V(G)| - 1).$$

In particular, $\operatorname{rank} M(\tilde{G}) + \operatorname{def}(\tilde{G}) \le D\,(|V(G)| - 1)$.

Proof. Let r be as in Lemma 20.10, and let Y be the set of edges of $\tilde{G}$ whose ends lie in one part of P. The edges of $\tilde{G}$ outside Y are the $D-1$ copies of each of the $d_G(P)$ edges of G that join different parts of P. The matroid $M(\tilde{G})$ is the union of D copies of the cycle matroid of $\tilde{G}$ (Definition 20.1). So the rank of a matroid union (Theorem 15.10) gives the first inequality of

$$\operatorname{rank} M(\tilde{G}) \;\le\; |E(\tilde{G}) \setminus Y| + D\,r(Y) \;\le\; (D-1)\,d_G(P) + D\,(|V(G)| - |P|).$$

The second holds by Lemma 20.10. As $\operatorname{def}_{\tilde{G}}(P) = D(|P|-1) - (D-1)\,d_G(P)$, this is the inequality at P. Taking P of largest value gives the inequality for $\operatorname{def}(\tilde{G})$. □

Theorem 20.12 (The deficiency is the corank of $M(\tilde{G})$; KT eq. (2.4))**.** *Let $D \ge 1$, and let G be a multigraph with at least one vertex. Then every base B of $M(\tilde{G})$ satisfies*

$$|B| + \operatorname{def}(\tilde{G}) \;=\; D\,(|V(G)| - 1).$$

That is, $\operatorname{rank} M(\tilde{G}) + \operatorname{def}(\tilde{G}) = D\,(|V(G)| - 1)$.

Proof. By Lemma 20.11, $\operatorname{rank} M(\tilde{G}) + \operatorname{def}(\tilde{G}) \le D\,(|V(G)| - 1)$. For the reverse inequality we find a partition P of $V(G)$ with $\operatorname{rank} M(\tilde{G}) + \operatorname{def}_{\tilde{G}}(P) \ge D\,(|V(G)| - 1)$.

Let r be the rank function of the cycle matroid of $\tilde{G}$. By the rank of a matroid union (Theorem 15.10), applied to the D copies of this matroid, some set Y_0 of edges of $\tilde{G}$ satisfies

$$\operatorname{rank} M(\tilde{G}) \;=\; |E(\tilde{G}) \setminus Y_0| + D\,r(Y_0).$$

Let P be the partition of $V(G)$ into the vertex sets of the components of $\tilde{G} \restriction Y_0$. Then $r(Y_0) = |V(G)| - |P|$, as before Lemma 20.10. The ends of an edge of Y_0 lie in one component of $\tilde{G} \restriction Y_0$, so in one part of P. Hence the $D-1$ copies of each of the $d_G(P)$ edges of G that join different parts of P lie in $E(\tilde{G}) \setminus Y_0$, and

$$\operatorname{rank} M(\tilde{G}) \;\ge\; (D-1)\,d_G(P) + D\,(|V(G)| - |P|) \;=\; D\,(|V(G)| - 1) - \operatorname{def}_{\tilde{G}}(P).$$

As $\operatorname{def}_{\tilde{G}}(P) \le \operatorname{def}(\tilde{G})$, the reverse inequality follows. Every base B of $M(\tilde{G})$ has $|B| = \operatorname{rank} M(\tilde{G})$. □

Formalization note 20.13. The first pin is the identity for the rank, the second the identity for a base, and the third the reverse inequality. In the third, the partition into components is the labeling that sends each vertex to a chosen vertex of its component (Formalization note 20.4). By Lemma 20.10, applied to Y_0, this labeling has at most as many parts as there are components. It has at least as many, as different components get different labels.

The characterization of deficiency zero announced in Section 20.1 follows from KT eq. (2.4).

Lemma 20.14 (Deficiency zero and D edge-disjoint spanning trees; KT Proposition 2.2)**.** *Let $D \geq 2$, and let G be a multigraph with at least one vertex. Then* $\mathrm{def}(\tilde{G}) = 0$ *if and only if $\tilde{G}$ contains D edge-disjoint spanning trees.*

Proof. Suppose that $\tilde{G}$ contains D edge-disjoint spanning trees. Each has $|V(G)|-1$ edges, so their union F has $D(|V(G)|-1)$ edges. As a union of D forests, F is independent in $M(\tilde{G})$ (Lemma 15.8). As $\mathrm{def}(\tilde{G}) \geq 0$, KT eq. (2.4) (Theorem 20.12) gives $\mathrm{rank}\, M(\tilde{G}) \leq D(|V(G)|-1)$. So F is a base, and KT eq. (2.4) gives $\mathrm{def}(\tilde{G}) = 0$.

Conversely, let $\mathrm{def}(\tilde{G}) = 0$, and let B be a base of $M(\tilde{G})$. Then $|B| = D(|V(G)|-1)$ by KT eq. (2.4). The multigraph $H = \tilde{G} \restriction B$ is (D,D)-sparse (Definition 20.1), so it is (D,D)-tight. It is also connected. Otherwise $V(G)$ is the union of two disjoint nonempty sets V_1 and V_2 with no edge of H between them. For $i = 1, 2$, the edges of H inside V_i number at most $D|V_i| - D$: by sparsity if there are any, and as $|V_i| \geq 1$ if there are none. So $|B| \leq D|V(G)| - 2D$, against $|B| = D|V(G)| - D$. By Corollary 16.19, B is the union of D disjoint spanning trees of H, and these are spanning trees of $\tilde{G}$. □

20.3. **The deficiency and the rank of $R(G,p)$.** This subsection bounds the rank of the rigidity matrix $R(G,p)$ of a body-hinge framework (G,p) (Section 19) by the deficiency of $\tilde{G}$: first for a framework on G, then for one that also has bodies on no edge of G.

Lemma 20.15 (The deficiency bound)**.** *Let (G,p) be a body-hinge framework with V nonempty, and suppose that $C(p(e)) \neq 0$ at every edge e of G whose two ends differ. Then*

$$\dim Z(G,p) \;\geq\; D + \mathrm{def}(\tilde{G}),$$

that is, $\mathrm{rank}\, R(G,p) \leq D(|V|-1) - \mathrm{def}(\tilde{G})$.

Proof. Let P be a partition of V, and let W_P be the space of screw assignments that are constant on each part of P. It has dimension $D|P|$. On W_P, the hinge constraint at an edge inside a part always holds. At an edge e that joins two parts it consists of $D-1$ linear equations, since $C(p(e)) \neq 0$ (Definition 19.5); a loop joins no two parts, so the hypothesis is not asked of it. So the infinitesimal motions in W_P form a subspace of dimension at least

$$D|P| - (D-1)\, d_G(P) \;=\; D + \mathrm{def}_{\tilde{G}}(P).$$

A partition P with $\mathrm{def}_{\tilde{G}}(P) = \mathrm{def}(\tilde{G})$ gives the bound. □

This bound holds at every realization, generic or not, and it is one of the two inequalities of the rank formula in Proposition 28.42. Jackson and Jordán derive its rank form from a count of independent rows [JJ10, §4]; we count motions instead.

Later chapters consider a framework on a subgraph G of a fixed graph together with all the bodies of the fixed graph, as Section 19.3 does. Its screw assignments are maps $B \to \bigwedge^{d-1} K^{d+1}$ on a finite set B of bodies that contains $V(G)$, and $R(G,p)$

and $Z(G,p)$ are defined as before. A body of $B \setminus V(G)$ lies on no edge of G, so no hinge constraint involves it.

Lemma 20.16 (The relative deficiency bound)**.** *Let (G,p) be a body-hinge framework whose screw assignments are defined on a finite set B of bodies with $\emptyset \neq V(G) \subseteq B$, and suppose that $C(p(e)) \neq 0$ at every edge e of G that is not a loop. Then*

$$D\,(|B \setminus V(G)| + 1) + \operatorname{def}(\tilde{G}) \;\leq\; \dim Z(G,p),$$

that is, $\operatorname{rank} R(G,p) \leq D(|V(G)| - 1) - \operatorname{def}(\tilde{G})$.

Proof. Let P be a partition of $V(G)$ with $\operatorname{def}_{\tilde{G}}(P) = \operatorname{def}(\tilde{G})$, and let P' be the partition of B whose parts are those of P and the singletons $\{b\}$ for $b \in B \setminus V(G)$. No edge meets these singletons, so exactly $d_G(P)$ edges join different parts of P'. The count in the proof of Lemma 20.15, applied to P', gives

$$\dim Z(G,p) \;\geq\; D|P'| - (D-1)\,d_G(P) \;=\; D\,|B \setminus V(G)| + D + \operatorname{def}_{\tilde{G}}(P).$$

The rank form follows, as $\operatorname{rank} R(G,p) = D|B| - \dim Z(G,p)$. □

20.4. **Rigid subgraphs and structural lemmas.** This subsection proves the properties of k-dof-graphs that Katoh and Tanigawa establish in their §3, and some deficiency computations that later chapters use. We define rigid subgraphs, and show that a 0-dof-graph is 2-edge-connected (KT Lemma 3.1) and that minimality passes to subgraphs (KT Lemma 3.3). Then we compare the deficiency of a multigraph with those of its pieces: at a cut of at most one edge (KT Lemma 3.6), at a cut vertex, and along an ear. We also define the largest value of a partition that keeps two given vertices in one part, and show that every multigraph has a minimal spanning subgraph of the same deficiency. Finally we treat the circuits of $M(\tilde{G})$ (the first step of KT Lemma 3.4), show that the two-vertex double edge and a short cycle are 0-dof-graphs, and list the minimal k-dof-graphs with at most two vertices.

Katoh and Tanigawa's combinatorial induction contracts proper rigid subgraphs (Section 21).

Definition 20.17 (Rigid and proper rigid subgraphs)**.** Let H be a subgraph of a multigraph G. Then H is a *rigid subgraph* of G if H is a 0-dof-graph. It is a *proper* rigid subgraph if moreover $1 < |V(H)| < |V(G)|$.

These are Katoh and Tanigawa's definitions [KT11, pp. 658–659]. When $V(H)$ is nonempty, H is rigid exactly when $\tilde{H}$ contains D edge-disjoint spanning trees (KT Proposition 2.2). A subgraph with one vertex is always rigid, as its only partition has one part; the condition $|V(H)| > 1$ excludes these.

Lemma 20.18 (A 0-dof-graph is 2-edge-connected; KT Lemma 3.1)**.** *Let $D \geq 1$, let G be a 0-dof-graph, and let V' be a set of vertices of G with $\emptyset \neq V' \neq V(G)$. Then at least two edges of G join V' to $V(G) \setminus V'$.*

Proof. Let $P = \{V', V(G) \setminus V'\}$. If at most one edge joined V' to $V(G) \setminus V'$, then $d_G(P) \leq 1$, and

$$\operatorname{def}(\tilde{G}) \;\geq\; \operatorname{def}_{\tilde{G}}(P) \;=\; D - (D-1)\,d_G(P) \;\geq\; 1,$$

contradicting $\operatorname{def}(\tilde{G}) = 0$. □

Formalization note 20.19. The formal statement fixes $a \in V'$ and $b \in V(G) \setminus V'$, and counts the edges whose ends get different labels under the labeling that sends V' to a and every other vertex to b (Formalization note 20.4). A second pin defines this labeling, and a third shows that it has two parts.

We name the edges of such a cut, with Katoh and Tanigawa's notation [KT11, p. 654], and define 2-edge-connectivity by the condition of Lemma 20.18.

Definition 20.20 (Cuts and 2-edge-connectivity)**.** Let G be a multigraph. For a set V' of vertices of G, let $\delta_G(V')$ be the set of edges of G that join V' to $V(G) \setminus V'$, and let $d_G(V') = |\delta_G(V')|$. Then G is 2-*edge-connected* if $d_G(V') \geq 2$ for every set V' with $\emptyset \neq V' \neq V(G)$.

Lemma 20.21 (2-edge-connectivity of 0-dof-graphs, and the degrees)**.** *Let G be a multigraph. If $D \geq 1$ and G is a 0-dof-graph, then G is 2-edge-connected. If G is 2-edge-connected and has at least two vertices, then every vertex of G has degree at least 2.*

Proof. The first claim is Lemma 20.18. For the second, let v be a vertex of G. As G has another vertex, $\emptyset \neq \{v\} \neq V(G)$, so $d_G(\{v\}) \geq 2$. The edges of $\delta_G(\{v\})$ are the edges at v that are not loops, so $d_G(\{v\})$ is at most the degree of v. □

Formalization note 20.22. The formalization states Lemma 20.18 with a labeling, as its formalization note explains. The formal proof identifies $\delta_G(V')$ with the set of edges whose ends get different labels under that labeling (`cutEdges_eq_crossingEdges_cutLabeling`). For $V' = \{v\}$, these are the edges at v that are not loops (`crossingEdges_cutLabeling_singleton_subset`), and their number is at most the degree of v (`crossingEdges_cutLabeling_singleton_ncard_le`).

Lemma 20.23 (A 0-dof-graph is connected, of minimum degree at least two)**.** *Let $D \geq 1$, and let G be a 0-dof-graph with at least one vertex. Then G is connected. If G has at least two vertices, then every vertex of G has degree at least 2.*

Proof. By Lemma 20.21, G is 2-edge-connected, and this gives the bound on the degrees. If G were disconnected, the vertex set V' of one of its components would have $\emptyset \neq V' \neq V(G)$ and $d_G(V') = 0$. □

Next we show that minimality passes to subgraphs. If H is a subgraph of G, then $\tilde{H}$ is a subgraph of $\tilde{G}$, and their matroids are related as follows.

Lemma 20.24 (The matroid of a subgraph is a restriction)**.** *Let H be a subgraph of a multigraph G. Then $M(\tilde{H})$ is the restriction of $M(\tilde{G})$ to $E(\tilde{H})$:*

$$M(\tilde{G}) \,|\, E(\tilde{H}) \;=\; M(\tilde{H}).$$

Proof. Both matroids have the ground set $E(\tilde{H})$. Let $F \subseteq E(\tilde{H})$. By Definition 20.1, F is independent in $M(\tilde{G})$ if and only if $\tilde{G} \restriction F$ is (D, D)-sparse, and in $M(\tilde{H})$ if and only if $\tilde{H} \restriction F$ is. Each edge of $\tilde{H}$ has the same ends in $\tilde{H}$ as in $\tilde{G}$, and sparsity depends only on the edges and their ends (Definition 16.1). So the two conditions agree. □

Lemma 20.25 (Minimality passes to subgraphs; KT Lemma 3.3)**.** *Let G be a minimal k-dof-graph, and let H be a subgraph of G with $\operatorname{def}(\tilde{H}) = k'$. Then H is a minimal k'-dof-graph. In particular, a rigid subgraph of a minimal k-dof-graph is a minimal 0-dof-graph.*

Proof. We follow Katoh and Tanigawa's proof [KT11, p. 657]. As H is a k'-dof-graph, we must show that a base B' of $M(\tilde{H})$ meets $\tilde{e}$ for every edge e of H. By Lemma 20.24, B' is a maximal subset of $E(\tilde{H})$ that is independent in $M(\tilde{G})$. Extend B' to a base B of $M(\tilde{G})$. Then $B \cap E(\tilde{H})$ is independent and contains B', so it equals B'. As G is minimal, B meets $\tilde{e}$, and $\tilde{e} \subseteq E(\tilde{H})$ because e is an edge of H. Hence B' meets $\tilde{e}$. □

Katoh and Tanigawa prove their Lemma 3.6 by writing $M(\tilde{G})$ as a direct sum, using their Lemmas 3.1 and 3.4; we compare partitions directly. For a set X of vertices of G, $G[X]$ is the subgraph induced by X, and $\tilde{G}[X] = (D-1) \cdot G[X]$. We first compute the value of a partition none of whose parts meets both sides of a cut.

Lemma 20.26 (The value of a separated partition)**.** *Let G be a multigraph, let V' be a set of vertices of G with $\emptyset \neq V' \neq V(G)$, and let $V'' = V(G) \setminus V'$. Let P be a partition of $V(G)$ that is* separated*, that is, each of its parts lies in V' or in V''. Let P' and P'' be the partitions of V' and of V'' formed by these parts. Then*

$$\operatorname{def}_{\tilde{G}}(P) \;=\; \operatorname{def}_{\tilde{G}[V']}(P') + \operatorname{def}_{\tilde{G}[V'']}(P'') + D - (D-1)\, d_G(V').$$

Proof. An edge of G joins different parts of P if and only if it joins different parts of P', or of P'', or lies in $\delta_G(V')$. As $|P| - 1 = (|P'| - 1) + (|P''| - 1) + 1$, the identity follows. □

Lemma 20.27 (The deficiency at a cut of at most one edge; KT Lemma 3.6)**.** *Let G be a multigraph.*

(1) Let V' be a set of vertices of G with $V' \neq V(G)$ and $d_G(V') \leq 1$, and let $V'' = V(G) \setminus V'$. Then

$$\operatorname{def}(\tilde{G}) \;=\; \operatorname{def}(\tilde{G}[V']) + \operatorname{def}(\tilde{G}[V'']) + D - (D-1)\, d_G(V').$$

So the cut adds 1 to the sum if $d_G(V') = 1$, and D if $d_G(V') = 0$.

(2) If G is a minimal k-dof-graph that is not 2-edge-connected, then for some nonempty set V' as in (1), $G[V']$ and $G[V'']$ are minimal k'- and k''-dof-graphs, where $k' = \operatorname{def}(\tilde{G}[V'])$, $k'' = \operatorname{def}(\tilde{G}[V''])$, and $k = k' + k'' + D - (D-1)\, d_G(V')$.

(3) If G has no loops, V' is as in (1), and an edge of G joins V' to a vertex $w \notin V'$, then $\operatorname{def}(\tilde{G}[V' \cup \{w\}]) = \operatorname{def}(\tilde{G}[V']) + 1$.

Proof. (1) If $V' = \emptyset$, then $V'' = V(G)$ and $d_G(V') = 0$, and the only partition of $\emptyset$ has no part, so $\operatorname{def}(\tilde{G}[V']) = -D$ and (1) holds. Let $V' \neq \emptyset$. The union of a partition P' of V' and a partition P'' of V'' is a separated partition of $V(G)$, whose value is given by Lemma 20.26. Taking P' and P'' of largest value gives the inequality $\geq$ in (1).

For $\leq$, let P be any partition of $V(G)$, and split each part Q that meets both V' and V'' into $Q \cap V'$ and $Q \cap V''$. The result is a separated partition. If s parts are split, the number of parts rises by s. Let c be the number of edges that join different parts of the new partition but not of P. Each of them lies in $\delta_G(V')$, so $c \leq 1$, and $c = 0$ if $s = 0$. So the value changes by $Ds - (D-1)c \geq 0$. Hence the value of P is at most that of a separated partition (`exists_sides_separated_partitionDef_le`). By Lemma 20.26, the value of a separated partition is at most the right-hand side of (1).

(2) As G is not 2-edge-connected, some set V' with $\emptyset \neq V' \neq V(G)$ has $d_G(V') \leq 1$. The formula for k is (1), and $G[V']$ and $G[V'']$ are minimal by KT Lemma 3.3 (Lemma 20.25). This is the first step of Katoh and Tanigawa's proof of their Lemma 6.1 [KT11, p. 672].

(3) Apply (1) to the multigraph $G[V' \cup \{w\}]$ and the set $\{w\}$. Since $d_G(V') \leq 1$, the given edge is the only edge of $G[V' \cup \{w\}]$ at w. The multigraph $G[\{w\}]$ has no edge, as G has no loops, and its deficiency is 0, as its only partition has one part. The other side is $G[V']$, and $D - (D-1) = 1$. □

Formalization note 20.28. The three pins are statements (1), (2) and (3). Partitions are labelings in the formalization (Formalization note 20.4). The value of a labeling depends only on its labels on $V(G)$ (`partitionDef_congr`), and it does not change when the labels are renamed injectively (`partitionDef_comp_of_injOn`). The proof of (1) uses these two facts to give the two sides disjoint sets of labels before it combines their partitions.

Lemma 20.29 (The deficiency at a cut vertex)**.** *Let G be a multigraph without loops, and let V_1 and V_2 be sets of vertices of G with $V_1 \cup V_2 = V(G)$ and $V_1 \cap V_2 = \{v\}$, such that every edge of G lies inside V_1 or inside V_2. Then*

$$\operatorname{def}(\tilde{G}) = \operatorname{def}(\tilde{G}[V_1]) + \operatorname{def}(\tilde{G}[V_2]).$$

Proof. Let P be a partition of $V(G)$. For $i = 1, 2$, the nonempty sets $Q \cap V_i$ with $Q \in P$ form a partition P_i of V_i. Let I be the set of parts of P that meet both V_1 and V_2, so that $|P_1| + |P_2| = |P| + |I|$. An edge of G inside V_i joins different parts of P if and only if it joins different parts of P_i. No edge lies inside both V_1 and V_2, since it would be a loop at v. Hence

$$\operatorname{def}_{\tilde{G}}(P) \;=\; \operatorname{def}_{\tilde{G}[V_1]}(P_1) + \operatorname{def}_{\tilde{G}[V_2]}(P_2) + D\,(1 - |I|).$$

The part of P that contains v lies in I, so $|I| \geq 1$, and the value of P is at most the right-hand side of the lemma.

Conversely, let P_1 and P_2 be partitions of V_1 and V_2 of largest value. Let P consist of the union of their two parts that contain v, together with their other parts. Then P is a partition of $V(G)$, the partitions of V_1 and V_2 that it gives as above are P_1 and P_2, and $|I| = 1$. By the identity above, the value of P is the right-hand side. □

Formalization note 20.30. With partitions as labelings (Formalization note 20.4), I is the set of labels used both on V_1 and on V_2. The third pin is the identity of the proof for two sets V_1 and V_2 that meet in two vertices u and v not joined by an edge; a cut vertex is the case $u = v$. The second pin is the lemma, and the first is

its inequality $\geq$. The proof of the first renames the labels of two attaining labelings injectively (`partitionDef_comp_of_injOn`), so that their sets of labels meet only in the label of v.

Let V_1 be a set of vertices of G. A path of G whose ends lie in V_1, possibly equal, and whose other vertices lie outside V_1 is an ear on V_1; Section 30.6 calls it open if its ends differ and closed otherwise. The next two lemmas compare the deficiency of G with that of $G[V_1]$ when G consists of $G[V_1]$ and one ear.

Lemma 20.31 (The deficiency along an ear)**.** *Let $D \geq 1$, let G be a multigraph, and let V_1 be a set of vertices of G. Let $a - x_1 - \cdots - x_k - b$ ($k \geq 1$) be a path of G with $a, b \in V_1$, possibly $a = b$, and distinct vertices $x_1, \ldots, x_k$ outside V_1, such that $V(G) = V_1 \cup \{x_1, \ldots, x_k\}$ and every edge of G off the path lies inside V_1. Then*

$$\operatorname{def}(\tilde{G}) \;\geq\; \operatorname{def}(\tilde{G}[V_1]) + k + 1 - D.$$

Proof. Let $V'' = \{x_1, \ldots, x_k\}$, and let P'' be the partition of V'' into single vertices. Take a partition P_1 of V_1 of largest value, and let $P = P_1 \cup P''$. Each part of P lies in V_1 or in V'', so P is separated in the sense of Lemma 20.26, with $V' = V_1$. That lemma gives

$$\operatorname{def}_{\tilde{G}}(P) \;=\; \operatorname{def}(\tilde{G}[V_1]) + \operatorname{def}_{\tilde{G}[V'']}(P'') + D - (D-1)\, d_G(V_1).$$

The edges of $G[V'']$ are among the $k-1$ edges $x_i x_{i+1}$ of the path. As $D \geq 1$, the value of P'' is therefore at least $D(k-1) - (D-1)(k-1) = k-1$. The edges of $\delta_G(V_1)$ are among the two end edges ax_1 and $x_k b$ of the path, so $D - (D-1)\, d_G(V_1) \geq 2 - D$. Since $\operatorname{def}(\tilde{G}) \geq \operatorname{def}_{\tilde{G}}(P)$, the bound follows. □

In the proof of Lemma 20.31, the vertices of the ear form parts of their own. If they can instead join a part that contains both ends of the ear, the deficiency does not drop.

Lemma 20.32 (The deficiency along an ear whose ends share a part)**.** *Let G be a multigraph, and let V_1 be a set of vertices of G. Let $a - x_1 - \cdots - x_k - b$ ($k \geq 0$) be a walk of G with $a, b \in V_1$, possibly $a = b$, and vertices $x_1, \ldots, x_k$ outside V_1, not necessarily distinct, such that $V(G) = V_1 \cup \{x_1, \ldots, x_k\}$ and every edge of G off the walk lies inside V_1. Suppose that some partition of V_1 of value $\operatorname{def}(\tilde{G}[V_1])$ has a and b in one part. Then*

$$\operatorname{def}(\tilde{G}[V_1]) \;\leq\; \operatorname{def}(\tilde{G}).$$

Proof. Let P_1 be such a partition, and let P be obtained from it by adding $x_1, \ldots, x_k$ to the part that contains a and b. Then $|P| = |P_1|$. Every edge of the walk lies inside that part. Every other edge of G lies inside V_1, where P and P_1 agree, so it joins different parts of P if and only if it joins different parts of P_1. Hence the value of P is that of P_1, which is $\operatorname{def}(\tilde{G}[V_1])$. □

The hypothesis of Lemma 20.32 says that the largest value of a partition of V_1 with a and b in one part is $\operatorname{def}(\tilde{G}[V_1])$. We give this largest value a name.

Definition 20.33 (The deficiency with two vertices in one part)**.** Let G be a multigraph, and let u and v be vertices of G. The *merged deficiency* of $\tilde{G}$ at u and v is

$$\operatorname{def}(\tilde{G}; u, v) \;=\; \max_{P} \operatorname{def}_{\tilde{G}}(P),$$

the maximum over the partitions P of $V(G)$ that have u and v in one part.

So $\operatorname{def}_{\tilde{G}}(P) \le \operatorname{def}(\tilde{G}; u, v)$ for every partition P with u and v in one part, and $\operatorname{def}(\tilde{G}; u, v) \le \operatorname{def}(\tilde{G})$. We use the merged deficiency when an edge joining u and v is added: the new edge lowers the value of each partition that separates u and v by $D-1$, and leaves the value of every other partition unchanged (Lemma 21.36).

Katoh and Tanigawa reduce their Theorem 5.6, on the realizations of an arbitrary multigraph, to minimal k-dof-graphs, by deleting edges while the deficiency stays the same [KT11, p. 670].

Lemma 20.34 (A minimal spanning subgraph of the same deficiency; KT Theorem 5.6, first step)**.** *Let $D \ge 1$, and let G be a multigraph with at least one vertex. Then G has a spanning subgraph that is a minimal* $\operatorname{def}(\tilde{G})$*-dof-graph.*

Proof. Among the sets F of edges of G with $\operatorname{def}(\widetilde{G \restriction F}) = \operatorname{def}(\tilde{G})$, such as $E(G)$, choose one, F_0, of least size. Then $G' = G \restriction F_0$ is a spanning subgraph of G with $\operatorname{def}(\widetilde{G'}) = \operatorname{def}(\tilde{G})$.

Suppose that a base B of $M(\widetilde{G'})$ misses $\tilde{e}$ for some edge e of G', and let $G'' = G \restriction (F_0 \setminus \{e\})$. Then $B \subseteq E(\widetilde{G''})$, so B is independent in $M(\widetilde{G''})$ (Lemma 20.24). The multigraphs G' and G'' have the vertices of G, so KT eq. (2.4) (Theorem 20.12) gives

$$\operatorname{def}(\widetilde{G''}) \;=\; D\,(|V(G)| - 1) - \operatorname{rank} M(\widetilde{G''}) \;\le\; D\,(|V(G)| - 1) - |B| \;=\; \operatorname{def}(\widetilde{G'}).$$

As $D \ge 1$, deleting an edge does not lower the value of a partition, so $\operatorname{def}(\widetilde{G''}) \ge \operatorname{def}(\widetilde{G'})$. Hence $F_0 \setminus \{e\}$ keeps the deficiency of G, against the choice of F_0. So every base of $M(\widetilde{G'})$ meets $\tilde{e}$ for every edge e of G', and G' is minimal. □

Lemma 20.35 (A circuit minus an edge is a maximal sparse subset; KT Lemma 3.4, first step)**.** *Let G be a multigraph, let X be a circuit of $M(\tilde{G})$, and let $e \in X$. Then $\tilde{G} \restriction (X - e)$ is (D, D)-sparse, and $X - e$ is a basis of X, that is, a maximal subset of X that is independent in $M(\tilde{G})$.*

Proof. As X is a minimal dependent set, $X - e$ is independent, so $\tilde{G} \restriction (X - e)$ is (D, D)-sparse (Definition 20.1). As X itself is dependent, no larger subset of X is independent. □

Katoh and Tanigawa's Lemma 3.4 says more: $G[V(X)]$ is a rigid subgraph of G, and $X - e$ is the union of D edge-disjoint spanning trees on $V(X)$ [KT11, p. 658]. We prove this in the next chapter (Lemma 21.2), using KT eq. (2.4).

Katoh and Tanigawa prove their Theorem 5.5 by induction on the number of vertices, from the case of two vertices [KT11, p. 671]. The next lemma computes the deficiency of the two-vertex double edge, and the one after it that of a short cycle,

which the next chapter uses (Section 21). The third lists the minimal k-dof-graphs with two vertices, and with one.

Lemma 20.36 (The two-vertex double edge is a 0-dof-graph)**.** *Let $D \geq 2$, and let H be a multigraph with two vertices $x \neq y$ and two edges $e_1 \neq e_2$, each joining x and y. Then H is a 0-dof-graph.*

Proof. The partitions of $V(H)$ are $\{V(H)\}$, of value 0, and $\{\{x\}, \{y\}\}$. Both edges join the two parts of the second, so its value is $D - 2(D-1) = 2 - D \leq 0$ (Definition 20.3). Hence $\operatorname{def}(\tilde{H}) = 0$. □

Lemma 20.37 (A short cycle is a 0-dof-graph)**.** *Let $3 \leq m \leq D$, and let C be a multigraph whose vertices are m distinct vertices $v_0, \dots, v_{m-1}$ and whose edges are m edges $f_0, \dots, f_{m-1}$, where f_i joins v_i and v_{i+1}, with indices modulo m. Then C is a 0-dof-graph.*

Proof. Let P be a partition of $V(C)$ into at least two parts. Each part is a proper subset of $V(C)$, so it has a vertex v_i whose successor v_{i+1} lies in another part. The edge f_i joins two parts, and different parts give different edges. So at least $|P|$ edges join different parts, and the value of P (Definition 20.3) is at most $D(|P|-1)-(D-1)|P| = |P| - D \leq 0$, as $|P| \leq m \leq D$. The partition into one part has value 0, so $\operatorname{def}(\tilde{C}) = 0$. □

Formalization note 20.38. The pin takes the vertices and the edges of C as two maps on $\{0, \dots, m-1\}$, and it does not ask that the vertices be distinct.

Lemma 20.39 (The minimal k-dof-graphs with at most two vertices; KT p. 671)**.** *Let $D \geq 2$, and let G be a minimal k-dof-graph with one or two vertices. Then exactly one of the following holds: (i) G has no edge, and $k = D(|V(G)|-1)$; (ii) G has two vertices and one edge, which joins them, and $k = 1$; (iii) G has two vertices and two edges, each of which joins them, and $k = 0$.*

Proof. We first compute three deficiencies. If a multigraph H has no edge, then every partition P of $V(H)$ has value $D(|P| - 1) \leq D(|V(H)| - 1)$, with equality for the partition into single vertices, so $\operatorname{def}(\tilde{H}) = D(|V(H)| - 1)$ (`deficiency_of_edgeSet_empty`). Let H have two vertices x and y. Its partitions are $\{V(H)\}$, of value 0, and $\{\{x\}, \{y\}\}$. If H has one edge, joining x and y, the second has value $D-(D-1) = 1$, so $\operatorname{def}(\tilde{H}) = 1$ (`deficiency_of_single_edge`). If H has two edges, each joining x and y, the second has value $D-2(D-1) = 2-D \leq 0$, so $\operatorname{def}(\tilde{H}) = 0$. This last computation is Lemma 20.36.

A minimal k-dof-graph has no loops. Indeed, let p be a copy of a loop of G. Then $\tilde{G} \upharpoonright \{p\}$ is not (D, D)-sparse, as $1 > D \cdot 1 - D$, so no base of $M(\tilde{G})$ contains p (Definition 20.1). As every base of $M(\tilde{G})$ meets the copies of each edge of G, G has no loop. Hence if G has one vertex, it has no edge, and (i) holds.

Let G have two vertices x and y, so that every edge of G joins them. Suppose that G had three edges e_1, e_2, e_3, and let $H = G \upharpoonright \{e_1, e_2\}$. Then $\operatorname{def}(\tilde{H}) = 0$, so a base B_H of $M(\tilde{H})$ has D elements by KT eq. (2.4) (Theorem 20.12). It is independent in $M(\tilde{G})$ (Lemma 20.24), so $\operatorname{rank} M(\tilde{G}) \geq D$. By KT eq. (2.4) for G,

$k = D - \operatorname{rank} M(\tilde{G}) \leq 0$. As $k \geq 0$, it follows that $k = 0$ and that B_H is a base of $M(\tilde{G})$. But $B_H \subseteq E(\tilde{H})$ misses $\tilde{e}_3$, against the minimality of G. So G has at most two edges (`edgeSet_ncard_le_two_of_isMinimalKDof_of_ncard_two`), and the three computations give (i), (ii) and (iii). Katoh and Tanigawa deduce that G has at most two edges from their Lemma 3.2: a minimal k-dof-graph is not 3-edge-connected. □

21. The combinatorial induction: graph operations and KT Theorem 4.9

This chapter proves Katoh and Tanigawa's Lemmas 3.4 and 3.5 and the results of their §4 [KT11] on minimal k-dof-graphs, with the induction and some lemmas of their §6. Two operations reduce a minimal k-dof-graph G with at least three vertices to a smaller one. If G has a proper rigid subgraph, contracting it gives a minimal k-dof-graph (KT Lemma 3.5). If G has none, splitting off at a vertex of degree two gives a minimal 0-dof-graph when $k = 0$, and a minimal $(k-1)$-dof-graph when $k > 0$ (KT Lemma 4.8). Such a vertex exists when G is 2-edge-connected (KT Lemma 4.6), and every 0-dof-graph is 2-edge-connected. So every minimal 0-dof-graph with at least two vertices reduces to the two-vertex double edge, which is KT Theorem 4.9. Katoh and Tanigawa prove their Theorem 5.5 by an induction on the number of vertices over minimal k-dof-graphs of every deficiency k. The algebraic induction (Section 23) constructs the panel-hinge realizations that this induction needs.

Throughout, $D = \binom{d+1}{2}$, and $\tilde{G} = (D-1) \cdot G$ is the multigraph of Section 17, in which each edge of G is repeated $D - 1$ times. The matroid $M(\tilde{G})$, the deficiency, k-dof-graphs and rigid subgraphs are those of Section 20.

Section 21.1 proves KT Lemmas 3.4 and 3.5. By the first, the ends of the edges of a circuit of $M(\tilde{G})$ induce a rigid subgraph. We prove the second for the corresponding contraction of the matroid $M(\tilde{G})$. Section 21.2 defines removal, splitting off and edge-splitting at a vertex of degree two. Section 21.3 proves KT Lemmas 4.1 and 4.2, which compare the independent sets of $M(\tilde{G})$ before and after splitting off and edge-splitting. We prove KT Lemma 4.1 for bases, and show that a base has the partition into forests that its proof assumes (Remark 21.15). Section 21.4 proves KT Lemmas 4.3 to 4.8. We bound the deficiency after splitting off and after removal by comparing partitions of $V(G)$, and find a vertex of degree two when G has no proper rigid subgraph. Section 21.5 proves KT Theorem 4.9 and the induction over all deficiencies.

The last two subsections concern a minimal 0-dof-graph G with at least three vertices and no proper rigid subgraph. Section 21.6 shows that G is simple, and that it has two adjacent vertices of degree two when $d \geq 3$. The last subsection proves KT Lemma 4.6 for G (Lemma 21.66): G is a cycle on at most d vertices, or it has a chain of length d whose interior vertices have degree two. Case III of the algebraic induction starts from this dichotomy.

21.1. **Circuits and the contraction of a rigid subgraph.** This subsection proves Katoh and Tanigawa's Lemmas 3.4 and 3.5 [KT11, §3.2]. By KT Lemma 3.4, the ends of the edges of a circuit of $M(\tilde{G})$ induce a rigid subgraph. Lemma 20.35 is its

first step, and we complete it with KT eq. (2.4). By KT Lemma 3.5, contracting a rigid subgraph of a minimal k-dof-graph gives a minimal k-dof-graph. We define the contraction, and prove the deficiency and minimality statements of KT Lemma 3.5 for the corresponding contraction of the matroid $M(\tilde{G})$, then for the graph, the form that the algebraic induction uses.

Definition 21.1 (The subgraph induced by the ends of a set of edges)**.** Let G be a multigraph, let X be a set of edges of $\tilde{G}$, and let $V(X)$ be the set of vertices incident with an edge of X. Then $G[V(X)]$ is the subgraph of G induced by $V(X)$. Its vertices are those of $V(X)$, and its edges are the edges of G with both ends in $V(X)$.

Lemma 21.2 (A circuit induces a rigid subgraph; KT Lemma 3.4)**.** *Let $D \geq 1$, let G be a multigraph, let X be a circuit of $M(\tilde{G})$, and let $e \in X$. Then $|X - e| = D(|V(X)| - 1)$, and $X - e$ is the union of D edge-disjoint spanning trees on $V(X)$. Moreover, $G[V(X)]$ is a rigid subgraph of G.*

Proof. For the count we follow Katoh and Tanigawa [KT11, p. 658]. As X is a circuit, $X - e$ is independent, so $\tilde{G} \restriction (X - e)$ is (D, D)-sparse (Lemma 20.35). If $X - e$ is not empty, sparsity applied to $X - e$ itself gives

$$|X - e| \;\leq\; D\,(|V(X - e)| - 1) \;\leq\; D\,(|V(X)| - 1).$$

If $X - e$ is empty, the bound holds as $V(X)$ is not empty. Conversely, X is dependent, so some nonempty $F \subseteq X$ has $|F| > D(|V(F)| - 1)$ (Definition 20.1). If F were a proper subset of X, it would miss some $f \in X$ and lie in the independent set $X - f$, whose sparsity gives $|F| \leq D(|V(F)| - 1)$. Hence $F = X$, and $|X| > D(|V(X)| - 1)$, that is, $|X - e| \geq D(|V(X)| - 1)$. So $|X - e| = D(|V(X)| - 1)$.

If $X - e$ is empty, then $|V(X)| = 1$, and the D spanning trees are empty. Otherwise both inequalities above are equalities, so $V(X - e) = V(X)$, and the multigraph on $V(X)$ with the edges $X - e$ is (D, D)-tight. It is connected, by the count in the proof of Lemma 20.14. By Corollary 16.19, $X - e$ is the union of D edge-disjoint spanning trees on $V(X)$.

Let $H = G[V(X)]$. The ends of each edge of X lie in $V(X)$, so $X \subseteq E(\tilde{H})$. Hence $X - e$ is independent in $M(\tilde{G}) \,|\, E(\tilde{H})$, which is $M(\tilde{H})$ (Lemma 20.24). So $\operatorname{rank} M(\tilde{H}) \geq |X - e| = D(|V(H)| - 1)$, and the reverse inequality is Lemma 20.9. By KT eq. (2.4) (Theorem 20.12), $\operatorname{def}(\tilde{H}) = 0$. As H is a subgraph of G, it is a rigid subgraph of G. □

If G has no proper rigid subgraph (Definition 20.17), the lemma shows that every circuit X of $M(\tilde{G})$ with $|V(X)| \geq 2$ has $V(X) = V(G)$. Katoh and Tanigawa use it in this form in the proofs of their Lemmas 4.5 and 4.8 [KT11, pp. 663 and 666].

Katoh and Tanigawa's induction also contracts proper rigid subgraphs [KT11, p. 659]. We define the contraction for any subgraph, which we contract to one of its vertices.

Definition 21.3 (Contracting a subgraph)**.** Let H be a subgraph of a multigraph G, and let r be a vertex of H. The *contraction* $G/E(H)$ of H to r is obtained from

G by deleting the edges of H and identifying every vertex of H with r. Its vertices are r and the vertices of G not in H. Its edges are the edges of G not in H, and each end of such an edge that lies in $V(H)$ is replaced by r.

So $G/E(H)$ has $|V(G)| - |V(H)| + 1$ vertices, and

$$D\left(|V(G/E(H))| - 1\right) \;=\; D\left(|V(G)| - |V(H)|\right).$$

Katoh and Tanigawa [KT11, pp. 658–659] prove their Lemma 3.5 with partitions of bases into forests. We work instead with the contraction of $M(\tilde{G})$ by $E(\tilde{H})$. For a matroid M on a set E and a set $T \subseteq E$, the contraction M/T is the matroid on $E \setminus T$ whose rank function is $X \mapsto r_M(X \cup T) - r_M(T)$ [Oxl11, §3.1 and Proposition 3.1.6]. The ground set of $M(\tilde{G})/E(\tilde{H})$ consists of the copies of the edges of G not in H, which are the edges of $\widetilde{G/E(H)}$. We first compute its rank.

Lemma 21.4 (The rank of a rigid subgraph; KT Lemma 3.5, first step)**.** *Let $D \geq 1$, and let H be a rigid subgraph of a multigraph G, with at least one vertex. Then* $\operatorname{rank} M(\tilde{H}) = D(|V(H)| - 1)$.

Proof. By KT eq. (2.4) (Theorem 20.12), $\operatorname{rank} M(\tilde{H}) + \operatorname{def}(\tilde{H}) = D(|V(H)| - 1)$, and $\operatorname{def}(\tilde{H}) = 0$ as H is rigid. □

Lemma 21.5 (The rank of $M(\tilde{G})/E(\tilde{H})$)**.** *Let H be a subgraph of a multigraph G. Then*

$$\operatorname{rank}(M(\tilde{G})/E(\tilde{H})) + \operatorname{rank} M(\tilde{H}) = \operatorname{rank} M(\tilde{G}).$$

Proof. A matroid M on E and a set $T \subseteq E$ satisfy $\operatorname{rank}(M/T) = \operatorname{rank} M - r_M(T)$, by Proposition 3.1.6 of Oxley [Oxl11] at $X = E \setminus T$. Take $M = M(\tilde{G})$ and $T = E(\tilde{H})$. Then $r_M(T)$ is the rank of $M(\tilde{G}) \,|\, E(\tilde{H})$, which is $M(\tilde{H})$ (Lemma 20.24). □

Lemma 21.6 (Contracting a rigid subgraph keeps the deficiency; KT Lemma 3.5)**.** *Let $D \geq 1$, and let H be a rigid subgraph of a multigraph G, with at least one vertex. Then*

$$D(|V(G)| - |V(H)|) - \operatorname{rank}(M(\tilde{G})/E(\tilde{H})) = \operatorname{def}(\tilde{G}).$$

Proof. By Lemma 21.5 and Lemma 21.4,

$$\operatorname{rank}(M(\tilde{G})/E(\tilde{H})) \;=\; \operatorname{rank} M(\tilde{G}) - D\left(|V(H)| - 1\right).$$

The vertex set of G contains that of H, so it is not empty, and KT eq. (2.4) (Theorem 20.12) gives $\operatorname{rank} M(\tilde{G}) = D(|V(G)|-1) - \operatorname{def}(\tilde{G})$. Substituting this into the first identity gives the lemma. This is Katoh and Tanigawa's count [KT11, p. 658]. □

The next lemma is the analogue of Lemma 20.25 for the contraction.

Lemma 21.7 (Minimality passes to the contraction; KT Lemma 3.5)**.** *Let G be a minimal k-dof-graph, and let H be a subgraph of G. Then every base of $M(\tilde{G})/E(\tilde{H})$ meets $\tilde{e}$ for every edge e of G that is not an edge of H.*

Proof. Let B' be a base of $M(\tilde{G})/E(\tilde{H})$, and let J be a maximal subset of $E(\tilde{H})$ that is independent in $M(\tilde{G})$. By Corollary 3.1.8 of Oxley [Oxl11], $B' \cup J$ is a base

of $M(\tilde{G})$. Let e be an edge of G that is not an edge of H. As G is minimal, $B' \cup J$ meets $\tilde{e}$. The copies of e are not in $E(\tilde{H})$, which contains J, so B' meets $\tilde{e}$. This is Katoh and Tanigawa's argument [KT11, p. 659]. □

Lemma 21.8 (The contraction of a proper rigid subgraph; KT Lemma 3.5)**.** *Let $D \geq 1$, let G be a minimal k-dof-graph, and let H be a proper rigid subgraph of G. Then*

$$D(|V(G)| - |V(H)|) - \operatorname{rank}(M(\tilde{G})/E(\tilde{H})) = k,$$

and every base of $M(\tilde{G})/E(\tilde{H})$ meets $\tilde{e}$ for every edge e of G that is not an edge of H.

Proof. As H is proper, it has at least two vertices, and $\operatorname{def}(\tilde{G}) = k$ as G is a k-dof-graph. So the identity is Lemma 21.6, and the second claim is Lemma 21.7. □

Katoh and Tanigawa state their Lemma 3.5 for the graph $G/E(H)$. It follows from Lemma 21.8 and KT eq. (2.4) for $G/E(H)$ if the matroid $M(\widetilde{G/E(H)})$ is $M(\tilde{G})/E(\tilde{H})$. The two matroids have the same ground set, and they are equal because H is rigid. The reason is Katoh and Tanigawa's (3.1) [KT11, p. 658]. Let a base of $M(\tilde{G})$ contain a base of $M(\tilde{H})$, and let it be the union of D disjoint forests. Then each of these forests meets $E(\tilde{H})$ in a spanning tree on $V(H)$. The algebraic induction applies its induction hypothesis to $G/E(H)$. For $D \geq 2$ we prove the identity, and with it Katoh and Tanigawa's form of their Lemma 3.5.

Lemma 21.9 (The contraction is a minimal k-dof-graph; KT Lemma 3.5)**.** *Let $D \geq 2$, let G be a minimal k-dof-graph, let H be a proper rigid subgraph of G, and let $G/E(H)$ be the contraction of H to a vertex of H (Definition 21.3). Then $G/E(H)$ is a minimal k-dof-graph with fewer vertices than G (Lemma 21.51).*

Proof. By Lemma 21.8,

$$D(|V(G)| - |V(H)|) - \operatorname{rank}(M(\tilde{G})/E(\tilde{H})) \;=\; k,$$

and every base of $M(\tilde{G})/E(\tilde{H})$ meets $\tilde{e}$ for every edge e of G not in H. These edges are the edges of $G/E(H)$, and their copies form the common ground set of $M(\tilde{G})/E(\tilde{H})$ and $M(\widetilde{G/E(H)})$. We show below that

$$M(\widetilde{G/E(H)}) \;=\; M(\tilde{G})/E(\tilde{H}).$$

Then every base of $M(\widetilde{G/E(H)})$ meets the copies of every edge of $G/E(H)$. The graph $G/E(H)$ has $|V(G)| - |V(H)| + 1$ vertices (Lemma 21.51), so KT eq. (2.4) (Theorem 20.12) gives

$$\operatorname{def}(\widetilde{G/E(H)}) \;=\; D(|V(G)| - |V(H)|) - \operatorname{rank} M(\widetilde{G/E(H)}) \;=\; k.$$

Hence $G/E(H)$ is a minimal k-dof-graph.

The identity is not a renaming of edges. Let M_c be the cycle matroid of $\tilde{G}$, so that $M(\tilde{G})$ is the union of D copies of M_c (Definition 20.1). As H is rigid, it is connected (Lemma 20.18), and so is $\tilde{H}$, since $D \geq 2$ gives every edge of H a copy in $\tilde{H}$. So contracting the edges of $\tilde{H}$ in $\tilde{G}$ identifies all the vertices of H,

and gives $G\widetilde{/E(H)}$. The cycle matroid of $G\widetilde{/E(H)}$ is therefore $M_c/E(\tilde{H})$ [Oxl11, Proposition 3.2.1]. So $M(G\widetilde{/E(H)})$ is the union of D copies of $M_c/E(\tilde{H})$, while $M(\tilde{G})/E(\tilde{H})$ is the contraction of the union of D copies of M_c. In general a union of matroids does not commute with contraction. Let M be the matroid of rank one on two elements a and c, neither of which is a loop. Then a is independent in the contraction by c of the union of two copies of M, but a is a loop of M/c, and so of the union of two copies of M/c.

The two matroids coincide here because the union attains D times the cycle rank on $E(\tilde{H})$. Let r_c be the rank function of M_c. The rank of $E(\tilde{H})$ in $M(\tilde{G})$ is the rank of $M(\tilde{H})$ (Lemma 20.24), which is $D(|V(H)|-1)$ as H is rigid (Lemma 21.4). The connected graph $\tilde{H}$ has $r_c(E(\tilde{H})) = |V(H)|-1$. So

$$\operatorname{rank}_{M(\tilde{G})} E(\tilde{H}) \;=\; D\, r_c(E(\tilde{H})).$$

Hence a base of $E(\tilde{H})$ in $M(\tilde{G})$ has $D(|V(H)|-1)$ elements. It is the union of D disjoint forests in $\tilde{H}$ (Theorem 16.18), each with at most $|V(H)|-1$ edges, and so each of them is a spanning tree on $V(H)$. This is Katoh and Tanigawa's (3.1) [KT11, p. 658], on which their proof of Lemma 3.5 rests.

We derive the identity from this equality. A set X is independent in the union of D copies of a matroid with rank function ρ if and only if $|Y| \leq D\,\rho(Y)$ for every $Y \subseteq X$ (Theorem 15.11). Let $C = E(\tilde{H})$, let J be a base of C in $M(\tilde{G})$, so that $|J| = D\,r_c(C)$, and let I be a set of copies of edges not in H. Then I is independent in $M(\tilde{G})/C$ if and only if $I \cup J$ is independent in $M(\tilde{G})$ [Oxl11, Proposition 3.1.7], that is, if and only if $|Z| \leq D\,r_c(Z)$ for every $Z \subseteq I \cup J$. The rank function of M_c/C is $Y \mapsto r_c(Y \cup C) - r_c(C)$ [Oxl11, Proposition 3.1.6]. So I is independent in the union of D copies of M_c/C if and only if $|Y| \leq D\,(r_c(Y \cup C) - r_c(C))$ for every $Y \subseteq I$.

Suppose this second condition, and let $Z \subseteq I \cup J$. Write $Z = Y \cup W$ with $Y \subseteq I$ and $W \subseteq J$. Since $W \subseteq C$, submodularity gives $r_c(Y \cup C) + r_c(W) \leq r_c(Y \cup W) + r_c(C)$. Also $|W| \leq D\,r_c(W)$, as J is independent in $M(\tilde{G})$. Adding these two bounds to the bound for Y gives $|Z| \leq D\,r_c(Z)$. Conversely, suppose the first condition, and let $Y \subseteq I$. Then

$$|Y| + D\,r_c(C) \;=\; |Y \cup J| \;\leq\; D\,r_c(Y \cup J) \;\leq\; D\,r_c(Y \cup C).$$

So the two matroids have the same independent sets. The converse used $|J| = D\,r_c(C)$, and it fails without it, as in the example above. □

21.2. **Removal, splitting off and edge-splitting.** The other operation of the induction acts at a vertex of degree two. Katoh and Tanigawa define splitting off at a vertex v as the removal of v followed by the insertion of new edges between neighbours of v, and they use it only when v has degree two [KT11, §4.1, p. 660]. They also use its inverse, edge-splitting (p. 660), and the removal of v alone (p. 662).

Definition 21.10 (Removal, splitting off and edge-splitting)**.** Let G be a multigraph, and let v be a vertex of G of degree two, whose two edges join v to vertices a and b.

- The *removal* of v gives $G_v = G - v$, obtained from G by deleting v and its two edges.
- The *splitting off* at v gives G_v^{ab}, obtained from G by deleting v and its two edges and adding a new edge ab.

Let H be a multigraph with an edge ab. The *edge-splitting* along ab gives H_{ab}^v, obtained from H by deleting ab and adding a new vertex v and two new edges va and vb.

Formalization note 21.11. In the formalization the edges of a multigraph are elements of a fixed set of labels. An operation that adds an edge takes the label of the new edge as an argument: the label e_0 of the edge ab of G_v^{ab}, and the labels of va and vb in H_{ab}^v. The formal splitting off deletes every edge with the label e_0 before it adds ab. So it gives G_v^{ab} when no edge of $G-v$ has the label e_0, for instance when $e_0 \notin E(G)$. A label outside $E(G)$ exists when there are enough labels (Remark 23.40). The formalization also fixes, for each edge of G, an ordered pair of its ends (`endsOf`), the *endpoint choice* of G. The realizations of Cases I and II of the algebraic induction take the ends of the edges of G from it (Formalization note 23.6).

Splitting off the new vertex v of H_{ab}^v gives back H.

Several proofs below, and the three cases of the algebraic induction, pass between G and G_v^{ab} through the graph G_v.

Lemma 21.12 (The common subgraph of G and G_v^{ab})**.** *Let G be a multigraph, let v be a vertex of G of degree two, whose edges e_a and e_b join it to vertices a and b, and let G_v and G_v^{ab} be obtained by removing v and by splitting off v (Definition 21.10). Then G_v is a subgraph of G and of G_v^{ab}, while G_v^{ab}, whose edge ab is new, is not a subgraph of G. Every edge of G that is not an edge of G_v has v as an end.*

Proof. The graphs G_v and G_v^{ab} both have the vertices of G other than v. An edge of G_v is an edge of G whose ends differ from v, with the same ends. It is not the new edge ab, which is not an edge of G, so it is an edge of G_v^{ab} with the same ends. Hence G_v is a subgraph of both graphs, and the new edge ab is an edge of G_v^{ab} but not of G. An edge of G whose ends both differ from v is an edge of G_v, which gives the last assertion. □

Formalization note 21.13. The four pinned theorems state the two inclusions and the last assertion, for every vertex v. The second and third state the inclusion in G_v^{ab} under a condition on the label of the new edge ab (Formalization note 21.11): the third assumes that the label lies outside $E(G-v)$, which is the condition under which the formal splitting off is G_v^{ab}, and the second, the case that its callers meet, assumes the stronger condition that it lies outside $E(G)$. No pinned theorem states that G_v^{ab} is not a subgraph of G.

21.3. **Forest surgery: KT Lemmas 4.1 and 4.2.** Let v be a vertex of G of degree two, whose edges e_a and e_b join it to vertices $a, b \neq v$. For an edge e of G, let $\tilde{e}$ be the set of its $D-1$ copies in $\tilde{G}$ (Definition 20.8). The edges of $\tilde{G}$ incident with v are the $2(D-1)$ edges of $\tilde{e}_a \cup \tilde{e}_b$. We write $\widetilde{ab}$ for the set of the $D-1$ copies of the new edge ab of G_v^{ab}. Katoh and Tanigawa [KT11, p. 660] compare the independent sets

of $M(\tilde{G})$ and of $M(\widetilde{G_v^{ab}})$ in their Lemma 4.1. They prove it by a *forest surgery*: they partition an independent set of $M(\tilde{G})$ into D forests of $\tilde{G}$, and change each forest at v. Their proof assumes that every forest of the partition has an edge incident with v.

We discuss the lemma and this assumption in Remark 21.15, and then prove the lemma for bases in three steps. The independent sets of $M(\tilde{G})$ are the unions of D forests (Lemma 21.17). A base has a partition into D forests each of which has an edge incident with v (Lemma 21.21). When $a \neq b$, the surgery on such a partition gives the two conclusions of the lemma (Lemma 21.24). The second step uses KT Lemma 4.4 (Lemma 21.38). We prove that lemma in Section 21.4 by comparing the values of partitions of $V(G)$, without forests. KT Lemma 4.2 (Lemma 21.29), the converse for edge-splitting, ends the subsection. The next remark (Remark 21.14) says which lemmas of this subsection the formal proof of KT Theorem 5.5 does not use, and what it uses instead.

Remark 21.14 (Katoh and Tanigawa's forests and the formal proof)**.** Katoh and Tanigawa prove KT Lemmas 4.1 and 4.2 with forests, and they derive KT Lemmas 4.3 and 4.4 from them (KT pp. 660–662). The formal proof of KT Theorem 5.5 (Theorem 23.38) uses the forests where KT Lemmas 4.3 and 4.4 concern bases, but not where they bound a deficiency. We keep the lemmas of this subsection that it does not take beside the argument they belong to, and say here what the formal proof does instead.

The deficiency bounds. Katoh and Tanigawa derive the upper bound of KT Lemma 4.3(i), their eq. (4.1), from KT Lemma 4.1. They derive the lower bound from KT Lemma 4.2(ii) (KT p. 661). They derive the first assertion of KT Lemma 4.4 from the case of KT Lemma 4.2(i) in which the independent set contains no copy of ab (KT p. 662). The formal proof compares the values of partitions of $V(G)$ instead, for splitting off (Lemma 21.33). It compares them for removal as well (Lemma 21.38). These comparisons need neither forests nor KT eq. (2.4) (Theorem 20.12). So the formal proof does not use Lemma 21.25, which derives the upper bound from the surgery on a base, as Katoh and Tanigawa derive eq. (4.1). Corollary 21.26 restates that lemma as the upper bound of KT Lemma 4.3(i).

The bases. KT Lemma 4.3(ii) and the second assertion of KT Lemma 4.4 concern bases, and so does the minimality of G_v^{ab} in KT Lemma 4.3(i). For these the formal proof keeps the forests. One direction of KT Lemma 4.3(ii) (Lemma 21.41) follows from the surgery on a base, as in Katoh and Tanigawa's proof of eq. (4.1). The surgery takes a partition of the base into D forests each of which has an edge incident with v (Lemma 21.21). The other direction follows from case (i) of KT Lemma 4.2 (Lemma 21.29) and the upper bound of KT Lemma 4.3(i). The second assertion of KT Lemma 4.4 follows from the case of KT Lemma 4.2(i) for an independent set with no copy of ab, as Katoh and Tanigawa derive it (Lemma 21.45). Lemma 21.31 states this case on its own. It is a case of Lemma 21.29, and the formal proof applies that lemma. The formal proof shows the minimality of G_v^{ab} in the proof of KT Lemma 4.8(ii) (Lemma 21.53). For an edge $e \neq ab$ of G_v^{ab}, it applies case (ii) of KT Lemma 4.2 there, as Katoh and Tanigawa do (KT p. 661).

Remark 21.15 (On KT Lemma 4.1)**.** KT Lemma 4.1 [KT11, p. 660] states that for any independent set I of $M(\tilde{G})$ there is an independent set I' of $M(\widetilde{G_v^{ab}})$ with $|I'| = |I| - D$ and $|\widetilde{ab} \cap I'| < D-1$. The arXiv version of the paper (arXiv:0902.0236, 2009, Lemma 5.1, p. 11) has essentially the same statement and proof. We separate three points.

(1) *The statement holds only for bases.* If $|I| < D$, for instance if $I = \emptyset$, then no set I' has $|I'| = |I| - D$ (Example 21.16). Katoh and Tanigawa apply the lemma to a base, in the proof of their Lemma 4.3, and we read it for bases.

(2) *The proof assumes a balanced partition.* Let $F_1, \dots, F_D$ be a partition of I into forests of $\tilde{G}$ (Lemma 21.17). A forest has at most two edges incident with v, at most one in $\tilde{e}_a$ and one in $\tilde{e}_b$, since two copies of an edge form a cycle. Let $h = |(\tilde{e}_a \cup \tilde{e}_b) \cap I|$, and let h' be the number of forests with two edges incident with v. Katoh and Tanigawa note that $2h' + (D - h') = h$, and they turn each F_i into a forest F_i' of $\widetilde{G_v^{ab}}$ with $|F_i'| = |F_i| - 1$. Both steps assume that every F_i has an edge incident with v. We call such a partition *balanced at* v. The proof does not show that I has a balanced partition, and if $h < D$, it has none.

We show that a base B has one. First, B contains at least D edges incident with v (Lemma 21.18). Now let F_j be a forest of a partition of B with no edge incident with v. The other $D-1$ forests contain at least D such edges, so one of them contains two. Moving one of these two to F_j gives a new partition of B into forests, as the moved edge is a pendant edge at v of the new F_j (Lemma 21.19). Each move lowers the number of forests with no edge incident with v, so repeated moves give a balanced partition (Lemma 21.21). So, for bases, this step is a gap in the proof and not an error.

On a balanced partition the surgery goes as Katoh and Tanigawa describe it. A forest with two edges $p_a \in \tilde{e}_a$ and $p_b \in \tilde{e}_b$ replaces them by one copy of ab (Lemma 21.23). A forest with one edge incident with v loses it and gains nothing (Lemma 21.22). Each forest loses one edge, so $|I'| = |I| - D$: the h edges incident with v leave, and h' copies of ab enter. This count is the equation $2h' + (D - h') = h$. As $h \le 2(D-1)$, it gives $h' \le D - 2$. So the $D - 1$ copies of ab suffice for distinct forests to take distinct copies, and $|\widetilde{ab} \cap I'| = h' < D - 1$ (Lemma 21.24). The count and KT eq. (2.4) (Theorem 20.12) give $\operatorname{def}(\widetilde{G_v^{ab}}) \le \operatorname{def}(\tilde{G})$, which is Lemma 21.25.

(3) *What the induction uses.* The proof of KT Theorem 4.9 needs KT Lemma 4.1 only through its consequence $\operatorname{def}(\widetilde{G_v^{ab}}) \le \operatorname{def}(\tilde{G})$, the upper bound of KT Lemma 4.3(i). We prove the deficiency bounds of KT Lemmas 4.3(i) and 4.4 in Section 21.4, by comparing the values of partitions of $V(G)$, without forests. These bounds (Lemma 21.40) are what our proof of KT Theorem 4.9 uses. The surgery also gives the second conclusion of KT Lemma 4.1, $|\widetilde{ab} \cap I'| < D-1$. From it we derive one half of KT Lemma 4.3(ii) (Lemma 21.41), which the algebraic induction uses in Cases II and III. The other half uses KT Lemma 4.2, the converse for edge-splitting, which adds edges at the new vertex and needs no balanced partition (Lemma 21.29).

Example 21.16 (KT Lemma 4.1 fails for $I = \emptyset$)**.** Let $D \geq 1$. The empty set is independent in $M(\tilde{G})$, and no set I' has $|I'| + D = |\emptyset| = 0$. So no independent set I' of $M(\widetilde{G_v^{ab}})$ has $|I'| = |\emptyset| - D$, and KT Lemma 4.1, stated for every independent set, fails at $I = \emptyset$.

Lemma 21.17 (The independent sets of $M(\tilde{G})$ are the unions of D forests)**.** *Let G be a multigraph. A set I of edges of $\tilde{G}$ is independent in $M(\tilde{G})$ if and only if $I = F_1 \cup \cdots \cup F_D$ for some forests $F_1, \dots, F_D$ of $\tilde{G}$.*

Proof. By Definition 20.1, $M(\tilde{G})$ is the union of D copies of the cycle matroid of $\tilde{G}$. The independent sets of the cycle matroid are the forests (Section 16.1). The independent sets of a union of matroids are the unions of independent sets of the matroids (Lemma 15.8). □

A subset of a forest is a forest. So removing from each F_i the edges of $F_1, \dots, F_{i-1}$ gives pairwise disjoint forests with the same union. Hence every independent set of $M(\tilde{G})$ has a partition into D forests. Katoh and Tanigawa begin their proofs of Lemmas 4.1 and 4.2 with such a partition.

Lemma 21.18 (A base contains at least D edges at a vertex of degree two)**.** *Let $D \geq 2$, and let G be a multigraph with a vertex v of degree two, whose edges e_a and e_b join v to vertices $a, b \neq v$. Then every base B of $M(\tilde{G})$ contains at least D of the $2(D-1)$ edges of $\tilde{G}$ incident with v:*

$$\big|\, B \cap (\tilde{e}_a \cup \tilde{e}_b) \,\big| \;\geq\; D,$$

where $\tilde{e}$ is the set of the $D-1$ copies of e in $\tilde{G}$.

Proof. Let $G_v = G - v$ (Definition 21.10), and let $B_0 = B \setminus (\tilde{e}_a \cup \tilde{e}_b)$. An edge of B_0 is a copy of an edge of G other than e_a and e_b, so it is not incident with v. Hence B_0 is a set of edges of $\widetilde{G_v}$. It is independent in $M(\tilde{G})$, and hence in the restriction $M(\widetilde{G_v})$ of $M(\tilde{G})$ (Lemma 20.24). So $|B_0| \leq \operatorname{rank} M(\widetilde{G_v})$.

The multigraph G_v has the vertex a, so KT eq. (2.4) (Theorem 20.12) applies to G and to G_v:

$$|B| = D\,(|V(G)| - 1) - \operatorname{def}(\tilde{G}), \qquad \operatorname{rank} M(\widetilde{G_v}) = D\,(|V(G)| - 2) - \operatorname{def}(\widetilde{G_v}).$$

Hence

$$\big|\, B \cap (\tilde{e}_a \cup \tilde{e}_b) \,\big| \;=\; |B| - |B_0| \;\geq\; D + \operatorname{def}(\widetilde{G_v}) - \operatorname{def}(\tilde{G}) \;\geq\; D.$$

The last step is KT Lemma 4.4 (Lemma 21.38), which uses $D \geq 2$: it gives $\operatorname{def}(\tilde{G}) \leq \operatorname{def}(\widetilde{G_v})$. Remark 21.15 uses this count to find a balanced partition of a base. □

Lemma 21.19 (A forest with no edge at v accepts an edge at v)**.** *Let G be a multigraph with a vertex v of degree two. Let F be a forest of $\tilde{G}$ none of whose edges is incident with v, and let x be an edge of $\tilde{G}$ that joins v to a vertex $w \neq v$. Then $F \cup \{x\}$ is a forest of $\tilde{G}$.*

Proof. In $\tilde{G} \restriction (F \cup \{x\})$ the only edge incident with v is x, and x is not a loop. So deleting x leaves v isolated, and in particular separates v from w: the edge x is a bridge. A bridge lies on no cycle, and F contains no cycle, so $F \cup \{x\}$ contains none. The argument uses only that no edge of F is incident with v, not the degree of v. Remark 21.15 uses the lemma to move an edge at v between the forests of a partition of a base. □

Lemma 21.20 (Moving an edge at v into a forest with none)**.** *Let G be a multigraph with a vertex v of degree two, and let $F_1, \dots, F_D$ be forests of $\tilde{G}$ with union I. Suppose that no edge of F_j is incident with v, and let $x \in I$ be an edge of $\tilde{G}$ that joins v to a vertex $w \neq v$. Then the sets $F'_j = F_j \cup \{x\}$ and $F'_i = F_i \setminus \{x\}$ for $i \neq j$ are forests of $\tilde{G}$ with union I.*

Proof. The set F'_j is a forest by Lemma 21.19, and each F'_i with $i \neq j$ is a forest as a subset of F_i. The union is still I, since $x \in I$ moves from the other forests to F'_j. □

Lemma 21.21 (A base has a balanced partition into forests)**.** *Let $D \geq 2$, and let G be a multigraph with a vertex v of degree two, whose edges join v to vertices $a, b \neq v$. Then every base B of $M(\tilde{G})$ is the union of D pairwise disjoint forests of $\tilde{G}$, each of which has an edge incident with v.*

Proof. By Lemma 21.17, B is the union of D forests, which we make pairwise disjoint as above. We induct on the number of these forests with no edge incident with v. If there is none, we are done. Otherwise let F_j be one. By Lemma 21.18, B contains at least D edges incident with v, and they lie in the $D-1$ forests other than F_j. So one of these, F_i, contains two of them. Let x be one of the two. It joins v to a or to b.

Move x from F_i to F_j (Lemma 21.20): the new F_j is a forest by Lemma 21.19, and the forests still have union B. They are still pairwise disjoint, since x lay in F_i only. Now F_j has an edge incident with v, and F_i keeps its other one. So the number of forests with no edge incident with v drops by one, and the induction hypothesis gives the partition. Katoh and Tanigawa's proof of their Lemma 4.1 assumes such a partition (Remark 21.15). □

We now perform the surgery on a balanced partition of a base.

Lemma 21.22 (Removing the edges at v from a forest)**.** *Let G be a multigraph with a vertex v of degree two, whose edges join v to vertices $a, b \neq v$, and let F be a forest of $\tilde{G}$. Then the edges of F that are not incident with v form a forest of $\widetilde{G_v^{ab}}$.*

Proof. The edges of $\tilde{G}$ not incident with v are the edges of $\widetilde{G_v}$, where $G_v = G - v$ (Definition 21.10). As G_v is a subgraph of G_v^{ab}, they are edges of $\widetilde{G_v^{ab}}$ with the same ends. Those of them in F contain no cycle, since F contains none. □

Lemma 21.23 (Replacing the two edges at v of a forest by a copy of ab)**.** *Let G be a multigraph with a vertex v of degree two, whose edges e_a and e_b join v to vertices $a, b \neq v$. Let F be a forest of $\tilde{G}$ whose edges incident with v are $p_a \in \tilde{e}_a$ and $p_b \in \tilde{e}_b$, and let $r \in \widetilde{ab}$. Then $(F \setminus \{p_a, p_b\}) \cup \{r\}$ is a forest of $\widetilde{G_v^{ab}}$.*

Proof. Let $F_0 = F \setminus \{p_a, p_b\}$, the set of edges of F not incident with v. By Lemma 21.22, F_0 is a forest of $\widetilde{G_v^{ab}}$, so a cycle C of $F_0 \cup \{r\}$ passes through r. Then $C - r$ is a path from a to b in F_0. Its edges have the same ends in $\tilde{G}$, and it does not pass through v. So together with p_a and p_b it forms a cycle of $\tilde{G}$ in F, which is impossible. Hence $F_0 \cup \{r\}$ contains no cycle. □

Lemma 21.24 (The surgery on a balanced partition)**.** *Let $D \geq 2$, and let G be a multigraph with a vertex v of degree two, whose edges e_a and e_b join v to distinct vertices $a, b \neq v$. Let $F_1, \dots, F_D$ be pairwise disjoint forests of $\tilde{G}$, each of which has an edge incident with v, and let I be their union. Then there are pairwise disjoint forests $F'_1, \dots, F'_D$ of $\widetilde{G_v^{ab}}$ whose union I' is independent in $M(\widetilde{G_v^{ab}})$ and satisfies*

$$|I'| = |I| - D \qquad \text{and} \qquad |\widetilde{ab} \cap I'| < D - 1.$$

Proof. Two copies of an edge form a cycle, so each F_i has at most one edge in $\tilde{e}_a$ and at most one in $\tilde{e}_b$. These are its only edges incident with v, and it has at least one, so it has one or two. Let h' be the number of forests with two, and let $h = |(\tilde{e}_a \cup \tilde{e}_b) \cap I|$. As the F_i are disjoint, $h = 2h' + (D - h') = D + h'$. As $|\tilde{e}_a \cup \tilde{e}_b| = 2(D-1)$, also $h \leq 2(D-1)$, so $h' \leq D - 2$.

Since $\widetilde{ab}$ has $D - 1 > h'$ elements, we can choose a different copy $r_i \in \widetilde{ab}$ for each forest F_i with two edges incident with v. For such an i, let F'_i be obtained from F_i by replacing these two edges by r_i. It is a forest of $\widetilde{G_v^{ab}}$ by Lemma 21.23. For every other i, let F'_i be F_i without its edge incident with v, a forest of $\widetilde{G_v^{ab}}$ by Lemma 21.22. In both cases $|F'_i| = |F_i| - 1$.

The edges of the F'_i other than the r_i lie in the disjoint sets F_i, and none of them is a copy of ab. So the F'_i are pairwise disjoint, and their union I' has $|I'| = \sum_i |F_i| - D = |I| - D$. Its copies of ab are the h' copies r_i, so $|\widetilde{ab} \cap I'| = h' < D - 1$. By Lemma 21.17, I' is independent in $M(\widetilde{G_v^{ab}})$, as the union of D forests. □

Lemma 21.25 (Splitting off does not increase the deficiency; KT Lemma 4.1)**.** *Let $D \geq 2$, and let G be a multigraph with a vertex v of degree two, whose edges join v to distinct vertices $a, b \neq v$. Then*

$$\operatorname{def}(\widetilde{G_v^{ab}}) \;\leq\; \operatorname{def}(\tilde{G}).$$

Proof. Let B be a base of $M(\tilde{G})$. By Lemma 21.21, B is the union of D pairwise disjoint forests of $\tilde{G}$, each of which has an edge incident with v. By Lemma 21.24, $M(\widetilde{G_v^{ab}})$ has an independent set I' with $|I'| = |B| - D$. Hence

$$\operatorname{rank} M(\widetilde{G_v^{ab}}) \;\geq\; |B| - D.$$

The vertices of G_v^{ab} are the $|V(G)| - 1$ vertices of G other than v, among them a. So KT eq. (2.4) (Theorem 20.12) applies to G_v^{ab} and to G, and gives

$$\operatorname{def}(\widetilde{G_v^{ab}}) \;=\; D\,(|V(G)| - 2) - \operatorname{rank} M(\widetilde{G_v^{ab}}) \;\leq\; D\,(|V(G)| - 1) - |B| \;=\; \operatorname{def}(\tilde{G}).$$

Katoh and Tanigawa [KT11, p. 661] derive their eq. (4.1), $\operatorname{def}(\widetilde{G_v^{ab}}) \leq k$ for a minimal k-dof-graph G, from their Lemma 4.1 in the same way. □

The bound of Lemma 21.25 is the upper bound of KT Lemma 4.3(i). We prove the same bound by comparing the values of partitions (Lemma 21.33), for every $D \geq 1$ and without assuming $a \neq b$. The next corollary restates Lemma 21.25, with the same hypotheses, as this bound of KT Lemma 4.3(i).

Corollary 21.26 (The upper bound of KT Lemma 4.3(i) by forest surgery)**.** *Let $D \geq 2$, and let G be a multigraph with a vertex v of degree two, whose edges join v to distinct vertices $a, b \neq v$. Then*

$$\operatorname{def}(\widetilde{G_v^{ab}}) \;\leq\; \operatorname{def}(\tilde{G}).$$

Proof. This is Lemma 21.25. □

We turn to KT Lemma 4.2. Let H be a multigraph with an edge ab, and let H_{ab}^v be obtained from H by edge-splitting along ab (Definition 21.10). Katoh and Tanigawa [KT11, pp. 660–661] partition an independent set I' of $M(\tilde{H})$ into D forests and change each forest at the new vertex v. When no forest contains a copy of ab, each forest gains one edge at v.

We state KT Lemma 4.2 for a vertex v of G of degree two, whose edges e_a and e_b join v to distinct vertices $a, b \neq v$. Then G is obtained from G_v^{ab} by edge-splitting along ab, so the lemma applies to $H = G_v^{ab}$. In its proof, a forest of $\widetilde{G_v^{ab}}$ that contains a copy of ab loses it and gains a copy of e_a and a copy of e_b, and some of the other forests gain a copy of e_a or of e_b. The next two lemmas show that each change gives a forest of $\tilde{G}$.

Lemma 21.27 (Replacing a copy of ab in a forest by the two edges at v)**.** *Let G be a multigraph with a vertex v of degree two, whose edges e_a and e_b join v to distinct vertices $a, b \neq v$. Let F' be a forest of $\widetilde{G_v^{ab}}$ that contains a copy r of ab, and let $p_a \in \tilde{e}_a$ and $p_b \in \tilde{e}_b$. Then $(F' \setminus \{r\}) \cup \{p_a, p_b\}$ is a forest of $\tilde{G}$.*

Proof. We reverse the argument of Lemma 21.23. Two copies of ab form a cycle, so r is the only copy of ab in F'. Hence the edges of $F_0 = F' \setminus \{r\}$ are edges of $\widetilde{G_v}$. They have the same ends in $\tilde{G}$ as in $\widetilde{G_v^{ab}}$, and none of them is incident with v. Let C be a cycle of $\tilde{G}$ in $F_0 \cup \{p_a, p_b\}$. If C does not pass through v, then C lies in F_0, and it is a cycle of $\widetilde{G_v^{ab}}$ in F'. Otherwise C uses both p_a and p_b, the only edges of the set at v. Then the rest of C is a path from b to a in F_0, and with r it forms a cycle of $\widetilde{G_v^{ab}}$ in F'. Both cases are impossible, as F' is a forest. □

Lemma 21.28 (Adding an edge at v to a forest with no copy of ab)**.** *Let G be a multigraph with a vertex v of degree two, whose edges join v to vertices $a, b \neq v$. Let F be a forest of $\widetilde{G_v^{ab}}$ that contains no copy of ab, and let x be an edge of $\tilde{G}$ that joins v to a vertex $w \neq v$. Then $F \cup \{x\}$ is a forest of $\tilde{G}$.*

Proof. As F contains no copy of ab, its edges are edges of $\widetilde{G_v}$. As in the proof of Lemma 21.22, they have the same ends in $\tilde{G}$ as in $\widetilde{G_v^{ab}}$. So F is a forest of $\tilde{G}$, and none of its edges is incident with v. By Lemma 21.19, $F \cup \{x\}$ is a forest of $\tilde{G}$. □

We add to Katoh and Tanigawa's statement the equation $I \setminus (\tilde{e}_a \cup \tilde{e}_b) = I' \setminus \widetilde{ab}$, which their proof gives. Their proof of Lemma 4.3 relies on it: a base of $M(\widetilde{G_v^{ab}})$ with no copy of an edge $e \neq ab$ extends to a base of $M(\tilde{G})$ with no copy of e [KT11, p. 661]. As $|\widetilde{ab}| = D - 1$, the two cases of the lemma cover every I'.

Lemma 21.29 (Independent sets under edge-splitting; KT Lemma 4.2)**.** *Let $D \geq 2$, and let G be a multigraph with a vertex v of degree two, whose edges e_a and e_b join v to distinct vertices $a, b \neq v$. Let I' be an independent set of $M(\widetilde{G_v^{ab}})$, and let $h' = |\widetilde{ab} \cap I'|$. Then $M(\tilde{G})$ has an independent set I with*

$$I \setminus (\tilde{e}_a \cup \tilde{e}_b) \;=\; I' \setminus \widetilde{ab}$$

such that (i) if $h' < D - 1$, then $|I| = |I'| + D$ and $|\tilde{e}_b \cap I| = h' + 1$, and (ii) if $h' = D - 1$, then $|I| = |I'| + D - 1$.

Proof. We follow Katoh and Tanigawa's proof. By Lemma 21.17, I' is the union of D forests of $\widetilde{G_v^{ab}}$, which we make pairwise disjoint as after that lemma. Two copies of ab form a cycle, so exactly h' of the forests contain a copy of ab, one each. Renumbering the forests and the copies of ab, we call them $F'_1, \dots, F'_D$, where F'_i contains (ab, i) for $i \leq h'$ and no copy of ab for $i > h'$.

For $i \leq h'$, let F_i be F'_i with (ab, i) replaced by (e_a, i) and (e_b, i), a forest of $\tilde{G}$ by Lemma 21.27. In case (i), let $F_i = F'_i \cup \{(e_a, i)\}$ for $h' < i \leq D - 1$, and $F_D = F'_D \cup \{(e_b, h' + 1)\}$. These are forests of $\tilde{G}$ by Lemma 21.28. In case (ii), let $F_D = F'_D$. It contains no copy of ab, so it is a forest of $\tilde{G}$, as in the proof of Lemma 21.28.

The added edges are different copies of e_a and e_b, and they are not edges of $\widetilde{G_v^{ab}}$. So $F_1, \dots, F_D$ are pairwise disjoint forests of $\tilde{G}$, and their union I is independent in $M(\tilde{G})$ (Lemma 21.17). Deleting the copies of e_a and e_b from I leaves the union of the F'_i without their copies of ab, which is $I' \setminus \widetilde{ab}$. In case (i) every forest gains one edge, so $|I| = |I'| + D$, and the copies of e_b in I are $(e_b, 1), \dots, (e_b, h' + 1)$. In case (ii) the forests $F_1, \dots, F_{D-1}$ gain one edge each and F_D none, so $|I| = |I'| + D - 1$. □

Formalization note 21.30. The first pin is case (ii), with $h' = D - 1$ stated as $\widetilde{ab} \subseteq I'$, and the second is case (i). The formal proof does not renumber the copies of ab. A forest that contains the copy (ab, j) receives (e_a, j) and (e_b, j), and the copies added to the other forests take the indices j for which (ab, j) is not in I'.

The next lemma restates the case $h' = 0$ of case (i) of Lemma 21.29. In this case no forest contains a copy of ab, and each forest gains one edge at v.

Lemma 21.31 (Edge-splitting and independent sets avoiding ab; KT Lemma 4.2(i))**.** *Let $D \geq 2$, and let G be a multigraph with a vertex v of degree two, whose edges e_a and e_b join v to distinct vertices $a, b \neq v$. Let I' be an independent set of $M(\widetilde{G_v^{ab}})$ that contains no copy of ab. Then $M(\tilde{G})$ has an independent set I with $I \setminus (\tilde{e}_a \cup \tilde{e}_b) = I'$, $|I| = |I'| + D$ and $|\tilde{e}_b \cap I| = 1$.*

Proof. By Lemma 21.17, I' is the union of D forests $F'_1, \dots, F'_D$ of $\widetilde{G_v^{ab}}$, which we make pairwise disjoint as after that lemma. None of them contains a copy of ab. Let

$F_i = F'_i \cup \{(e_a, i)\}$ for $i \le D-1$, and $F_D = F'_D \cup \{(e_b, 1)\}$; the copy $(e_b, 1)$ exists as $D \ge 2$. These are forests of $\tilde{G}$ by Lemma 21.28. The added edges are distinct copies of e_a and e_b, and they are not edges of $\widetilde{G_v^{ab}}$. So $F_1, \dots, F_D$ are pairwise disjoint, and their union I is independent in $M(\tilde{G})$ (Lemma 21.17). Each forest gains one edge, so $|I| = |I'| + D$. The edges of I outside $\tilde{e}_a \cup \tilde{e}_b$ are those of I', and $(e_b, 1)$ is the only copy of e_b in I. This is Katoh and Tanigawa's construction when no forest contains a copy of ab. □

Formalization note 21.32. The pin is the second pin of Lemma 21.29, its case (i) for every $h' < D-1$. The statement is the case $h' = 0$, as $0 < D-1$ when $D \ge 2$. The proof of the pin also replaces the copies of ab in a forest (Lemma 21.27); here there are none.

Katoh and Tanigawa [KT11, p. 662] use this case to prove their Lemma 4.4, which we prove without forests (Lemma 21.38); Remark 21.39 compares the two proofs. The case needs $D \ge 2$: for $D = 1$ the multigraphs $\tilde{G}$ and $\widetilde{G_v^{ab}}$ have no edges, so $I' = \emptyset$, and no set I of edges of $\tilde{G}$ has $|I| = 1$.

21.4. **Splitting off and removal: KT Lemmas 4.3–4.8.** This subsection proves KT Lemmas 4.3 to 4.8 [KT11, pp. 661–666]. Let v be a vertex of G of degree two, whose edges e_a and e_b join it to vertices $a, b \ne v$. KT Lemmas 4.3(i) and 4.4 compare the deficiency of G with the deficiencies of the splitting off G_v^{ab} and of the removal G_v (Definition 21.10). Splitting off does not raise the deficiency and lowers it by at most one. Removal does not lower it when $D \ge 2$. Katoh and Tanigawa prove these bounds with forests and KT eq. (2.4). We compare the values of partitions of $V(G)$ instead (Definition 20.3), and we compare the two proofs of KT Lemma 4.4 in Remark 21.39. Two variants of the upper bound for splitting off follow the first lemma; the pencil chapter uses them.

When G is a k-dof-graph, KT Lemma 4.3(ii) tells whether G_v^{ab} is a k-dof-graph or a $(k-1)$-dof-graph, in terms of the bases of $M(\widetilde{G_v^{ab}})$. Its proof uses the forest surgery and edge-splitting (Section 21.3). KT Lemmas 4.5 to 4.7 concern a minimal k-dof-graph with no proper rigid subgraph. They bound its number of edges, find a vertex of degree two in it when it is 2-edge-connected, and show that removing such a vertex raises its deficiency. KT Lemma 4.8 ends the subsection: splitting off at such a vertex gives a minimal 0-dof-graph if $k = 0$, and a minimal $(k-1)$-dof-graph if $k > 0$. This is the step of the induction in KT Theorem 4.9, together with the contraction of a proper rigid subgraph (Lemma 21.8).

Lemma 21.33 (Splitting off lowers the deficiency by zero or one; KT Lemma 4.3(i))**.** *Let $D \ge 1$, and let G be a multigraph with a vertex v of degree two, whose edges e_a and e_b join v to vertices $a, b \ne v$. Then*

$$\operatorname{def}(\tilde{G}) - 1 \;\le\; \operatorname{def}(\widetilde{G_v^{ab}}) \;\le\; \operatorname{def}(\tilde{G}).$$

In particular, if G is a k-dof-graph, then G_v^{ab} is a k-dof-graph or a $(k-1)$-dof-graph.

Proof. We compare the values $\operatorname{def}_{\tilde{G}}(P)$ of partitions (Definition 20.3). The vertices of G_v^{ab} are those of G other than v. As $D \ge 1$, the value $D\,(|P|-1) - (D-1)\,d_G(P)$ does not increase when $d_G(P)$ increases.

For the upper bound, let P' be a partition of $V(G)-v$, and let P be the partition of $V(G)$ obtained by adding v to the part of P' that contains a. Then $|P| = |P'|$. The edge e_a joins two vertices of one part of P. The edge e_b joins different parts of P exactly when a and b lie in different parts of P', that is, when the new edge ab of G_v^{ab} joins different parts of P'. Every other edge has the same ends in G and in G_v^{ab}. Hence $d_G(P) = d_{G_v^{ab}}(P')$, and the value of P' for $\widetilde{G_v^{ab}}$ is the value of P for $\tilde{G}$, which is at most $\mathrm{def}(\tilde{G})$.

For the lower bound, let P be a partition of $V(G)$ of value $\mathrm{def}(\tilde{G})$, and let P' be the partition of $V(G)-v$ into the nonempty sets $X-v$ with $X \in P$. The edges of G_v^{ab} are the edges of G other than e_a and e_b, with the same ends, and the new edge ab.

Suppose first that v is not alone in its part of P. Then $|P'| = |P|$. If ab joins different parts of P', then v lies in a different part from a or from b, so e_a or e_b joins different parts of P. Hence $d_{G_v^{ab}}(P') \le d_G(P)$, and the value of P' is at least $\mathrm{def}(\tilde{G})$.

Suppose now that $\{v\}$ is a part of P. Then $|P'| = |P| - 1$, and both e_a and e_b join different parts of P, so $d_{G_v^{ab}}(P') \le d_G(P) - 1$. The value of P' is then at least

$$D\,(|P|-2) - (D-1)\,(d_G(P)-1) \;=\; \mathrm{def}(\tilde{G}) - 1.$$

In both cases $\mathrm{def}(\widetilde{G_v^{ab}}) \ge \mathrm{def}(\tilde{G}) - 1$. □

KT Lemma 4.3 assumes that G is a minimal k-dof-graph, and its part (i) adds that G_v^{ab} is minimal when it is a $(k-1)$-dof-graph. Katoh and Tanigawa use this clause only in the proof of their Lemma 4.8(ii). We prove the minimality there (Lemma 21.53).

The ear steps of the pencil chapter (Section 30.7) suppress a vertex of degree two, and keep one of its two edges as the new edge that joins its neighbours.

Lemma 21.34 (Suppressing a vertex of degree two)**.** *Let $D \ge 1$, and let G be a multigraph with a vertex v of degree two, whose edges e_a and e_b join v to vertices $a, b \ne v$. Let G'' be obtained from G by replacing the end v of e_a by b, and deleting v and e_b. Then*

$$\mathrm{def}(\tilde{G''}) \;\le\; \mathrm{def}(\tilde{G}).$$

Proof. The graph G'' is G_v^{ab} with e_a as its new edge ab, and we argue as for the upper bound of Lemma 21.33. Let P' be a partition of $V(G)-v$, and let P be the partition of $V(G)$ obtained by adding v to the part of a. Then $|P| = |P'|$. In G the edge e_a joins two vertices of one part of P, and e_b joins different parts of P exactly when e_a joins different parts of P' in G''. Every other edge has the same ends in G and in G''. So P' has the same value for $\tilde{G''}$ as P for $\tilde{G}$. □

Formalization note 21.35. In the formalization G'' is the splitting off at v with $e_0 = e_a$ as the label of the new edge (Formalization note 21.11). It is the case $e_0 = e_a$ of the upper bound of Lemma 21.33, which assumes $e_0 \notin E(G-v)$, a condition that e_a meets. The two bounds are one theorem.

The ear steps of the pencil chapter whose ends are not adjacent (Section 30.8) compare a graph with the graph obtained from it by adding an edge.

Lemma 21.36 (Splitting off with a gap in the merged deficiency)**.** *Let $D \geq 1$. Let v, a and b be vertices of a multigraph G with $a, b \neq v$. Let G' be obtained from G by deleting v, and let H be obtained from G' by adding a new edge ab. Suppose that*

$$\operatorname{def}(\tilde{G'}; a, b) \;\leq\; \operatorname{def}(\tilde{G'}) - (D-1)$$

(Definition 20.33). Then $\operatorname{def}(\tilde{H}) \leq \operatorname{def}(\tilde{G'}) - (D-1)$.

Proof. The graphs H and G' have the same vertices, and H has one more edge, ab. Let P be a partition of $V(G')$. If P separates a and b, then $d_H(P) = d_{G'}(P) + 1$. So the value of P for $\tilde{H}$ is its value for $\tilde{G'}$ minus $D-1$, which is at most $\operatorname{def}(\tilde{G'}) - (D-1)$. If a and b lie in one part of P, then $d_H(P) = d_{G'}(P)$. So the value of P for $\tilde{H}$ is at most $\operatorname{def}(\tilde{G'}; a, b)$, which the hypothesis bounds. □

Formalization note 21.37. The pin builds H with the formal splitting off at v (Formalization note 21.11), which is defined at a vertex of any degree, and a label e_0 for the new edge. It asks only that every edge of G with the label e_0 have v as an end. So e_0 is not the label of an edge of G', and it may be the label of an edge at v, as in Lemma 21.34.

Lemma 21.38 (Removal does not lower the deficiency; KT Lemma 4.4)**.** *Let $D \geq 2$, and let G be a multigraph with a vertex v of degree two, whose edges e_a and e_b join v to vertices $a, b \neq v$. Then*

$$\operatorname{def}(\tilde{G}) \;\leq\; \operatorname{def}(\widetilde{G_v}).$$

In particular, if G is a k-dof-graph, then $\operatorname{def}(\widetilde{G_v}) \geq k$.

Proof. We compare the values of partitions, as for the lower bound of Lemma 21.33. Let P be a partition of $V(G)$ of value $\operatorname{def}(\tilde{G})$, and let P' be the partition of $V(G) - v$ into the nonempty sets $X - v$ with $X \in P$. The edges of G_v are the edges of G other than e_a and e_b, the only edges at v, with the same ends. So

$$d_{G_v}(P') \;=\; d_G(P) - c,$$

where $c \in \{0, 1, 2\}$ is the number of the edges e_a and e_b that join different parts of P.

Suppose first that v is not alone in its part of P. Then $|P'| = |P|$, and the value of P' for $\widetilde{G_v}$ is $\operatorname{def}(\tilde{G}) + (D-1)\,c \geq \operatorname{def}(\tilde{G})$.

Suppose now that $\{v\}$ is a part of P. Then $|P'| = |P| - 1$. The neighbours a and b of v lie in other parts, so both e_a and e_b join different parts of P, and $c = 2$. The value of P' is

$$D\,(|P| - 2) - (D-1)\,(d_G(P) - 2) \;=\; \operatorname{def}(\tilde{G}) + D - 2,$$

which is at least $\operatorname{def}(\tilde{G})$ as $D \geq 2$. In both cases $\operatorname{def}(\widetilde{G_v}) \geq \operatorname{def}(\tilde{G})$. □

Remark 21.39 (On KT Lemma 4.4)**.** Katoh and Tanigawa [KT11, p. 662] prove KT Lemma 4.4 with forests. Let B' be a base of $M(\widetilde{G_v})$. As G_v is a subgraph of G_v^{ab}, the set B' is independent in $M(\widetilde{G_v^{ab}})$, and it contains no copy of ab. The

graph G is obtained from G_v^{ab} by edge-splitting along ab, so KT Lemma 4.2(i) with $h' = 0$ (Lemma 21.31) gives an independent set I of $M(\tilde{G})$ with $|I| = |B'| + D$ and $|\tilde{e}_b \cap I| = 1$: each of the D forests whose union is B' gains one edge at v.

By KT eq. (2.4) (Theorem 20.12), $|B'| = D\,(|V(G)| - 2) - \operatorname{def}(\widetilde{G_v})$ and $|I| \leq D\,(|V(G)| - 1) - \operatorname{def}(\tilde{G})$. Hence $\operatorname{def}(\tilde{G}) \leq \operatorname{def}(\widetilde{G_v})$. If equality holds, then I is a base of $M(\tilde{G})$ with $|\tilde{e}_b \cap I| = 1$. This is the second assertion of KT Lemma 4.4, and we prove it in this way (Lemma 21.45).

Katoh and Tanigawa's argument is correct. It also needs $D \geq 2$, since KT Lemma 4.2(i) at $h' = 0$ asks that $0 < D - 1$. Our proof of Lemma 21.38 compares the values of partitions instead, and needs neither forests nor KT eq. (2.4). The comparison works because of the case where $\{v\}$ is a part of P. There P' has one part fewer than P, which lowers the value by D. But the neighbours a and b of v lie in other parts, so both e_a and e_b join different parts of P. Hence $d_{G_v}(P') = d_G(P) - 2$, which raises the value by $2(D - 1)$. The value changes by $2(D - 1) - D = D - 2 \geq 0$. When v is not alone in its part, the number of parts stays the same, and fewer edges joining different parts can only raise the value.

Without the forced crossing of e_a and e_b the count fails. If we used only that the number of edges joining different parts does not increase, as in the other case, we would get only $\operatorname{def}(\widetilde{G_v}) \geq \operatorname{def}(\tilde{G}) - D$. Rank monotonicity gives the same bound. As $M(\widetilde{G_v})$ is a restriction of $M(\tilde{G})$ (Lemma 20.24), its rank is at most that of $M(\tilde{G})$, and KT eq. (2.4) turns this into $\operatorname{def}(\widetilde{G_v}) \geq \operatorname{def}(\tilde{G}) - D$.

The step of the induction uses the two bounds together (Lemma 21.52).

Lemma 21.40 (The deficiency after splitting off and after removal; KT Lemmas 4.3(i) and 4.4)**.** *Let $D \geq 2$, and let G be a k-dof-graph with a vertex v of degree two, whose edges e_a and e_b join v to vertices $a, b \neq v$. Then*

$$k - 1 \;\leq\; \operatorname{def}(\widetilde{G_v^{ab}}) \;\leq\; k \qquad \text{and} \qquad \operatorname{def}(\widetilde{G_v}) \;\geq\; k.$$

Proof. Combine Lemma 21.33, which holds as $D \geq 1$, with Lemma 21.38. □

Lemma 21.41 (When splitting off keeps the deficiency; KT Lemma 4.3(ii))**.** *Let $D \geq 2$, and let G be a k-dof-graph with a vertex v of degree two, whose edges e_a and e_b join v to distinct vertices $a, b \neq v$. Then G_v^{ab} is a k-dof-graph if and only if $M(\widetilde{G_v^{ab}})$ has a base B' with $|\widetilde{ab} \cap B'| < D - 1$.*

Proof. The vertices of G_v^{ab} are the $|V(G)| - 1$ vertices of G other than v. So KT eq. (2.4) (Theorem 20.12) gives

$$\operatorname{rank} M(\tilde{G}) \;=\; D\,(|V(G)| - 1) - k, \qquad \operatorname{rank} M(\widetilde{G_v^{ab}}) \;=\; D\,(|V(G)| - 2) - \operatorname{def}(\widetilde{G_v^{ab}}).$$

Suppose that G_v^{ab} is a k-dof-graph, and let B be a base of $M(\tilde{G})$. By Lemma 21.21, B is the union of D pairwise disjoint forests of $\tilde{G}$, each of which has an edge incident with v. The surgery on these forests (Lemma 21.24) gives an independent set I' of $M(\widetilde{G_v^{ab}})$ with $|I'| = |B| - D$ and $|\widetilde{ab} \cap I'| < D - 1$. Then $|I'| = D(|V(G)| - 2) - k$, which is the rank of $M(\widetilde{G_v^{ab}})$, so I' is a base. This is Katoh and Tanigawa's argument [KT11, p. 661], with the balanced partition that they assume (Remark 21.15).

Conversely, let B' be a base of $M(\widetilde{G_v^{ab}})$ with $|\widetilde{ab} \cap B'| < D-1$. By case (i) of KT Lemma 4.2 (Lemma 21.29), $M(\tilde{G})$ has an independent set I with $|I| = |B'| + D$. As $|I| \le \operatorname{rank} M(\tilde{G})$, we get $|B'| \le D(|V(G)|-2) - k$, that is, $\operatorname{def}(\widetilde{G_v^{ab}}) \ge k$. The upper bound of KT Lemma 4.3(i) (Lemma 21.33) gives $\operatorname{def}(\widetilde{G_v^{ab}}) \le k$. □

Formalization note 21.42. Both pinned theorems assume that the label of the new edge ab lies outside $E(G)$ (Formalization note 21.11).

The rest of the subsection follows Katoh and Tanigawa's §4.2 [KT11, pp. 663–666], on minimal k-dof-graphs with no proper rigid subgraph (Definition 20.17). Katoh and Tanigawa's multigraphs have no loops [KT11, §2.5], and ours may have them. A minimal k-dof-graph G has none. Indeed, the copies of a loop e of G are loops of $\tilde{G}$. A loop is a cycle, so it lies in no forest, and hence in no independent set of $M(\tilde{G})$ (Lemma 21.17). So no base of $M(\tilde{G})$ meets $\tilde{e}$, against minimality (Definition 20.8). Hence every circuit X of $M(\tilde{G})$ has $|V(X)| \ge 2$. If G also has no proper rigid subgraph, then $V(X) = V(G)$, as we noted after Lemma 21.2.

Lemma 21.43 (An edge bound with no proper rigid subgraph; KT Lemma 4.5(i))**.** *Let $D \ge 2$, and let G be a minimal k-dof-graph with at least one vertex and no proper rigid subgraph. Then*

$$(D-1)\,|E(G)| \;<\; D\,(|V(G)|-1) - k + (D-1).$$

Proof. We follow Katoh and Tanigawa's proof [KT11, p. 663], which they give for $k = 0$. If G has no edge, then the partition of $V(G)$ into single vertices has the largest value, so $k = D(|V(G)|-1)$, and the bound reads $0 < D-1$. Otherwise, fix an edge e of G. Let h^* be the least value of $|\tilde{e} \cap B|$ over the bases B of $M(\tilde{G})$, and let B^* be a base with $|\tilde{e} \cap B^*| = h^*$. As G is minimal, $h^* \ge 1$. By KT eq. (2.4) (Theorem 20.12), every base of $M(\tilde{G})$ has $D(|V(G)|-1) - k$ elements.

We show that every edge of $\tilde{G}$ outside $\tilde{e}$ lies in B^* (KT eq. (4.3)). Suppose that an edge $f \notin \tilde{e}$ of $\tilde{G}$ is not in B^*, and let X be the fundamental circuit of f with respect to B^*, the only circuit contained in $B^* + f$. Then $V(X) = V(G)$, as G has no loops and no proper rigid subgraph. Suppose first that X does not meet $\tilde{e}$, and let $x \in X$. By Lemma 21.2, $X - x$ is an independent set with $D(|V(G)|-1)$ elements. It lies in a base, which has $D(|V(G)|-1) - k$ elements, so $k \le 0$. As $V(G)$ is not empty, $k \ge 0$. So $k = 0$, and $X - x$ is a base of $M(\tilde{G})$ that does not meet $\tilde{e}$, against the minimality of G. Hence X contains a copy $e_j \in \tilde{e}$. Then $B^* + f - e_j$ is a base of $M(\tilde{G})$ that meets $\tilde{e}$ in $h^* - 1$ elements, against the choice of B^*.

So the edges of $\tilde{G}$ outside B^* are the $D - 1 - h^*$ copies of e outside B^*. As $\tilde{G}$ has $(D-1)\,|E(G)|$ edges,

$$(D-1)\,|E(G)| \;=\; |B^*| + (D-1-h^*) \;\le\; D\,(|V(G)|-1) - k + (D-2).$$

□

Lemma 21.44 (The edges of $\tilde{G}$ are independent when $k > 0$; KT Lemma 4.5(ii))**.** *Let $D \ge 2$, and let G be a minimal k-dof-graph with $k > 0$ and no proper rigid*

subgraph. Then $E(\tilde{G})$ is independent in $M(\tilde{G})$. In particular, $E(\tilde{G})$ is the only base of $M(\tilde{G})$.

Proof. Suppose that $E(\tilde{G})$ contains a circuit X of $M(\tilde{G})$. As G has no loops and no proper rigid subgraph, $V(X) = V(G)$. So $H = G[V(X)]$ is a rigid subgraph of G with the vertices of G (Lemma 21.2). By Lemma 21.4, $\operatorname{rank} M(\tilde{H}) = D(|V(G)|-1)$, and $M(\tilde{H})$ is a restriction of $M(\tilde{G})$ (Lemma 20.24). So $\operatorname{rank} M(\tilde{G}) \geq D(|V(G)|-1)$, and KT eq. (2.4) (Theorem 20.12) gives $k = \operatorname{def}(\tilde{G}) \leq 0$, a contradiction. Hence $E(\tilde{G})$ is independent. A base of $M(\tilde{G})$ is a maximal independent subset of $E(\tilde{G})$, so it is $E(\tilde{G})$. □

Katoh and Tanigawa state part (ii) of their Lemma 4.5 for every k-dof-graph with $k > 0$ and no proper rigid subgraph. They use that a circuit X has $|V(X)| \geq 2$, which holds as their multigraphs have no loops. We assume that G is minimal, which excludes loops. They add that $(D-1)\,|E(G)| = D(|V(G)|-1) - k$, which follows by KT eq. (2.4).

For $k > 0$, Katoh and Tanigawa derive KT Lemma 4.7 from KT Lemma 4.5(ii) and the second assertion of KT Lemma 4.4, which they prove with KT Lemma 4.2 (Remark 21.39). The next two lemmas prove that assertion and KT Lemma 4.7.

Lemma 21.45 (When removal keeps the deficiency; the second assertion of KT Lemma 4.4)**.** *Let $D \geq 2$, and let G be a k-dof-graph with a vertex v of degree two, whose edges e_a and e_b join v to distinct vertices $a, b \neq v$. If $\operatorname{def}(\widetilde{G_v}) = k$, then $M(\tilde{G})$ has a base B with $|\tilde{e}_b \cap B| = 1$.*

Proof. We argue as Katoh and Tanigawa do (Remark 21.39). Let B' be a base of $M(\widetilde{G_v})$. As G_v is a subgraph of G_v^{ab}, the set B' is independent in $M(\widetilde{G_v^{ab}})$ (Lemma 20.24), and it contains no copy of ab. By case (i) of KT Lemma 4.2 with $h' = 0$ (Lemma 21.29), $M(\tilde{G})$ has an independent set I with $|I| = |B'| + D$ and $|\tilde{e}_b \cap I| = 1$. The vertices of G_v are the $|V(G)| - 1$ vertices of G other than v. So KT eq. (2.4) (Theorem 20.12), for G_v and for G, gives

$$|I| \;=\; D\,(|V(G)|-2) - k + D \;=\; D\,(|V(G)|-1) - k \;=\; \operatorname{rank} M(\tilde{G}),$$

and I is a base. □

Formalization note 21.46. The pin also takes a label $e_0 \notin E(G)$ for the new edge of G_v^{ab}, which the proof uses (Formalization note 21.11).

Lemma 21.47 (Removal raises the deficiency; KT Lemma 4.7)**.** *Let $D \geq 3$, and let G be a minimal k-dof-graph with no proper rigid subgraph. Let v be a vertex of G of degree two, whose edges e_a and e_b join v to distinct vertices $a, b \neq v$. Then $\operatorname{def}(\widetilde{G_v}) > k$.*

Proof. KT Lemma 4.4 (Lemma 21.38) gives $\operatorname{def}(\widetilde{G_v}) \geq k$, and we show that equality fails. Suppose that $\operatorname{def}(\widetilde{G_v}) = k$. If $k = 0$, then G_v is a rigid subgraph of G whose vertices, those of G other than v, include a and b: at least two, and not all of those of G. This is impossible. If $k > 0$, then by Lemma 21.45 some base B of $M(\tilde{G})$ has

$|\tilde{e}_b \cap B| = 1$. But $B = E(\tilde{G})$ (Lemma 21.44), which contains all $D - 1 \geq 2$ copies of e_b. This is Katoh and Tanigawa's proof [KT11, p. 665]. □

Formalization note 21.48. The pin also takes a label $e_0 \notin E(G)$ for the new edge of G_v^{ab}, which the case $k > 0$ of the proof uses (Formalization note 21.11).

KT Lemma 4.6 shows that a 2-edge-connected minimal k-dof-graph with no proper rigid subgraph is a cycle on at most d vertices, or contains a chain $v_0 v_1 \ldots v_d$ of length d whose interior vertices have degree two. We prove this for $k = 0$ (Lemma 21.66). Katoh and Tanigawa's proof begins with a vertex of degree two, which the next two lemmas give.

Lemma 21.49 (A vertex of degree at most two; from the proof of KT Lemma 4.6)**.** *Let $D \geq 3$, and let G be a minimal k-dof-graph with at least one vertex and no proper rigid subgraph. Then G has a vertex of degree at most two.*

Proof. As $V(G)$ is not empty, $k \geq 0$, so Lemma 21.43 gives

$$(D-1)\,|E(G)| \;<\; D\,(|V(G)|-1) + D - 1 \;<\; D\,|V(G)|.$$

The degrees of the vertices of G sum to $2|E(G)|$, so their average is less than $2D/(D-1) = 2 + 2/(D-1)$, which is at most 3 as $D \geq 3$. Hence some vertex has degree at most two. This is KT eq. (4.5) [KT11, p. 664]. □

Lemma 21.50 (A vertex of degree two; from the proof of KT Lemma 4.6)**.** *Let $D \geq 3$, and let G be a 2-edge-connected minimal k-dof-graph with at least two vertices and no proper rigid subgraph. Then G has a vertex of degree two.*

Proof. By Lemma 21.49, G has a vertex of degree at most two. As G is 2-edge-connected and has at least two vertices, every vertex of G has degree at least two (Lemma 20.21). □

A 0-dof-graph is 2-edge-connected (KT Lemma 3.1, Lemma 20.18). So for $k = 0$ the hypothesis of 2-edge-connectivity holds automatically.

Katoh and Tanigawa prove their Theorem 4.9 by induction on the number of vertices [KT11, pp. 666–667].

Lemma 21.51 (The reductions lower the number of vertices; from the proof of KT Theorem 4.9)**.** *Let G be a multigraph. If v is a vertex of G of degree two, whose edges join v to vertices a and b, then $|V(G_v^{ab})| < |V(G)|$. If H is a subgraph of G, and $G/E(H)$ is the contraction of H to one of its vertices, then $|V(G/E(H))| = |V(G)| - |V(H)| + 1$, which is less than $|V(G)|$ if H has at least two vertices.*

Proof. The vertices of G_v^{ab} are those of G other than v (Definition 21.10). The vertices of $G/E(H)$ are the chosen vertex of H and the vertices of G not in H (Definition 21.3), which gives the count. □

Lemma 21.52 (Splitting off and contraction keep a minimal 0-dof-graph minimal; KT Lemmas 4.8(i) and 3.5)**.** *Let $D \geq 2$, and let G be a minimal 0-dof-graph.*

(1) Suppose that G has at least three vertices and no proper rigid subgraph, and let v be a vertex of G of degree two, whose edges join v to vertices $a, b \neq v$. Then G_v^{ab} is a minimal 0-dof-graph.

(2) Let H be a proper rigid subgraph of G. Then

$$\operatorname{rank}(M(\tilde{G})/E(\tilde{H})) \;=\; D\,(|V(G)| - |V(H)|),$$

and every base of $M(\tilde{G})/E(\tilde{H})$ meets $\tilde{e}$ for every edge e of G that is not an edge of H.

Proof. Part (ii) is the case $k = 0$ of Lemma 21.8.

For (i) we follow Katoh and Tanigawa's proof [KT11, pp. 665–666] up to its last step. Let $G' = G_v^{ab}$, and let $G_v = G - v$. Both have the $|V(G)| - 1$ vertices of G other than v, and the edges of $\widetilde{G'}$ are those of $\widetilde{G_v}$ and the $D - 1$ copies of ab. By the upper bound of Lemma 21.33, $\operatorname{def}(\widetilde{G'}) \le \operatorname{def}(\tilde{G}) = 0$, and $\operatorname{def}(\widetilde{G'}) \ge 0$ as G' has a vertex. So G' is a 0-dof-graph. The subgraph G_v of G has $|V(G)| - 1 \ge 2$ vertices, but not all of those of G. As G has no proper rigid subgraph, G_v is not rigid, and so $\operatorname{def}(\widetilde{G_v}) > 0$. This is the case $k = 0$ of KT Lemma 4.7.

Every circuit X of $M(\widetilde{G'})$ meets $\widetilde{ab}$ (KT eq. (4.10)). Otherwise X is a set of edges of $\widetilde{G_v}$. As G_v is a subgraph of both G' and G, X is then a circuit of $M(\widetilde{G_v})$ and of $M(\tilde{G})$ (Lemma 20.24). As G has no loops, $|V(X)| \ge 2$, and no edge of X is incident with v. So $G[V(X)]$ is a proper rigid subgraph of G (Lemma 21.2), which is impossible. Hence $E(\widetilde{G_v})$ contains no circuit of $M(\widetilde{G'})$, and it is independent in $M(\widetilde{G'})$ and in its restriction $M(\widetilde{G_v})$, whose ground set it is. Thus $E(\widetilde{G_v})$ is the only base of $M(\widetilde{G_v})$, and KT eq. (2.4) (Theorem 20.12) gives

$$|E(\widetilde{G_v})| \;=\; D\,(|V(G)| - 2) - \operatorname{def}(\widetilde{G_v}) \;<\; D\,(|V(G)| - 2).$$

Now let B' be a base of $M(\widetilde{G'})$, and let e be an edge of G'. By KT eq. (2.4), $|B'| = D(|V(G)|-2)$. Suppose that B' does not meet $\tilde{e}$. If e is the new edge ab, then B' is a set of edges of $\widetilde{G_v}$. Otherwise e is an edge of G_v, and B' consists of at most $D-1$ copies of ab and of edges of $\widetilde{G_v}$ outside $\tilde{e}$, of which there are $|E(\widetilde{G_v})| - (D-1)$. In both cases $|B'| \le |E(\widetilde{G_v})| < D(|V(G)| - 2)$, a contradiction. So every base of $M(\widetilde{G'})$ meets $\tilde{e}$ for every edge e of G', and G' is minimal.

Katoh and Tanigawa end differently. They start from a base of $M(\widetilde{G'})$ that misses $\tilde{e}$ for an edge $e \ne ab$, and use KT eq. (4.10) to exchange its copies of ab, one at a time, for copies of e. This gives a base of $M(\widetilde{G'})$ with no copy of ab, which is a base of $M(\widetilde{G_v})$ with $D(|V(G)| - 2)$ elements, against KT Lemma 4.7. We replace this exchange by the count above. □

In part (i) of Lemma 21.52, the bound $|V(G)| \ge 3$ is needed. The two-vertex double edge is a minimal 0-dof-graph with no proper rigid subgraph, and both of its vertices have degree two. Splitting off either of them leaves one vertex with a loop. This graph is a 0-dof-graph, as it has one vertex, but it is not minimal, as the only base of its matroid is empty.

By Lemma 21.51, the graphs G_v^{ab} and $G/E(H)$ of the lemma have fewer vertices than G. For $D \ge 2$, its part (ii) gives Katoh and Tanigawa's form of their Lemma 3.5, as we explained after Lemma 21.8: the graph $G/E(H)$ is again a minimal 0-dof-graph (Lemma 21.9).

Lemma 21.53 (Splitting off when $k > 0$; KT Lemma 4.8(ii))**.** *Let $D \geq 3$, and let G be a minimal k-dof-graph with $k > 0$ and no proper rigid subgraph. Let v be a vertex of G of degree two, whose edges e_a and e_b join v to distinct vertices $a, b \neq v$. Then G_v^{ab} is a minimal $(k-1)$-dof-graph.*

Proof. Let $G' = G_v^{ab}$. Its vertices are the $|V(G)| - 1$ vertices of G other than v, and by Lemma 21.40, $\operatorname{def}(\widetilde{G'})$ is $k-1$ or k.

We show, as Katoh and Tanigawa do [KT11, p. 666], that $\operatorname{def}(\widetilde{G'}) \neq k$. Suppose that $\operatorname{def}(\widetilde{G'}) = k$. By KT Lemma 4.3(ii) (Lemma 21.41), $M(\widetilde{G'})$ has a base B' that misses a copy p of ab. Let X be the fundamental circuit of p with respect to B', and let $H = G'[V(X)]$, a rigid subgraph of G' (Lemma 21.2). If $V(H) = V(G')$, then $\operatorname{rank} M(\widetilde{G'}) \geq \operatorname{rank} M(\tilde{H}) = D(|V(G')| - 1)$ (Lemma 21.4), and KT eq. (2.4) gives $\operatorname{def}(\widetilde{G'}) \leq 0 < k$. So $V(H)$ is a proper subset of $V(G')$. It contains the ends a and b of p.

Let $K = G[V(H) \cup \{v\}]$. Then v has degree two in K, and $K_v^{ab} = H$. The set $I = X - p$ is independent in $M(\tilde{H})$, with $|I| = D(|V(H)| - 1)$ (Lemma 21.2). As $I \subseteq B'$ and $p \notin B'$, also $|\widetilde{ab} \cap I| < D - 1$. By case (i) of KT Lemma 4.2 (Lemma 21.29) for K, $M(\tilde{K})$ has an independent set with $|I| + D = D(|V(K)| - 1)$ elements. So $\operatorname{def}(\tilde{K}) = 0$ by KT eq. (2.4), and K is a rigid subgraph of G. Its vertex set $V(H) \cup \{v\}$ has at least two elements, and it is a proper subset of $V(G)$. This contradicts the hypothesis, and so $\operatorname{def}(\widetilde{G'}) = k - 1$.

It remains to show that G' is minimal: every base B' of $M(\widetilde{G'})$ meets $\tilde{e}$ for every edge e of G'. This is the minimality clause of KT Lemma 4.3(i), and we follow Katoh and Tanigawa's proof of it [KT11, p. 661]. As G' is not a k-dof-graph, B' contains all $D-1$ copies of ab (Lemma 21.41), and so it meets $\widetilde{ab}$. Suppose that B' misses $\tilde{e}$. Then $e \neq ab$, and e is an edge of G other than e_a and e_b. By KT eq. (2.4), $|B'| = D(|V(G)| - 2) - (k-1)$. By case (ii) of KT Lemma 4.2 (Lemma 21.29), $M(\tilde{G})$ has an independent set J with $|J| = |B'| + D - 1 = D(|V(G)| - 1) - k$ and $J \setminus (\tilde{e}_a \cup \tilde{e}_b) = B' \setminus \widetilde{ab}$. By KT eq. (2.4), J is a base of $M(\tilde{G})$, and it does not meet $\tilde{e}$, against the minimality of G. □

21.5. **KT Theorem 4.9 and the induction for KT Theorem 5.5.** Katoh and Tanigawa combine their Lemmas 3.5, 4.6 and 4.8 into their Theorem 4.9: a minimal 0-dof-graph with at least two vertices reduces to the two-vertex double edge by splitting off at vertices of degree two and contracting proper rigid subgraphs [KT11, pp. 666–667]. We state it with an induction principle on the number of vertices, from which it follows (Theorem 21.54). Katoh and Tanigawa prove their Theorem 5.5 by a second induction on the number of vertices, over minimal k-dof-graphs of every deficiency k, with four cases [KT11, p. 671]. This induction is Theorem 21.56. In its first case the graph is not 2-edge-connected. Katoh and Tanigawa realize such a graph in their Lemma 6.1, which we prove with the algebraic induction (Section 23.4).

Theorem 21.54 (Reduction of minimal 0-dof-graphs; KT Theorem 4.9)**.** *Let $D \geq 3$.*

(i) *Let G be a minimal 0-dof-graph with at least two vertices. Then there are minimal 0-dof-graphs $G = G_1, G_2, \dots, G_m$ such that G_m has two vertices and two*

edges, each joining them, and for $1 \leq i < m$, G_{i+1} *is obtained from* G_i *by splitting off at a vertex of degree two or by contracting a proper rigid subgraph.*

(ii) *Let* P *be a property of multigraphs such that for every minimal* 0*-dof-graph* G*: (a)* $P(G)$ *holds if* G *has two vertices; (b) if* G *has no proper rigid subgraph, and* v *is a vertex of* G *of degree two whose edges join it to vertices* $a, b \neq v$*, then* $P(G_v^{ab})$ *implies* $P(G)$*; (c) if* G *has at least three vertices and a proper rigid subgraph, and* P *holds for every minimal* 0*-dof-graph with at least two and fewer than* $|V(G)|$ *vertices, then* $P(G)$ *holds. Then* P *holds for every minimal* 0*-dof-graph with at least two vertices.*

Proof. We prove (ii) by induction on $|V(G)|$. If G has two vertices, then $P(G)$ holds by (a). Let G have at least three vertices, and suppose that P holds for every minimal 0-dof-graph with at least two and fewer than $|V(G)|$ vertices. If G has a proper rigid subgraph, then $P(G)$ holds by (c).

Otherwise, G is 2-edge-connected, as it is a 0-dof-graph (Lemma 20.21). So G has a vertex v of degree two (Lemma 21.50). As we showed before Lemma 21.43, G has no loops. So the two edges at v join it to vertices $a, b \neq v$. By part (i) of Lemma 21.52, G_v^{ab} is a minimal 0-dof-graph. Its vertices are the $|V(G)| - 1 \geq 2$ vertices of G other than v (Lemma 21.51). So $P(G_v^{ab})$ holds by the induction hypothesis, and $P(G)$ holds by (b).

Part (i) is Katoh and Tanigawa's statement. As in their proof, it follows from (ii), with $P(G)$ the existence of such a sequence for G. A minimal 0-dof-graph with two vertices has two edges, each joining them (Lemma 20.39), which gives (a). A sequence for G_v^{ab} extends to one for G, which gives (b).

For (c), let H be a proper rigid subgraph of G, and let $G/E(H)$ be its contraction to a vertex of H. It has $|V(G)| - |V(H)| + 1$ vertices, at least two and fewer than $|V(G)|$ (Lemma 21.51). By KT Lemma 3.5, $G/E(H)$ is a minimal 0-dof-graph. We prove this lemma for the matroid $M(\tilde{G})/E(\tilde{H})$ (Lemma 21.8), and for the graph $G/E(H)$ (Lemma 21.9). So $G/E(H)$ has a sequence, which extends to one for G. □

Formalization note 21.55. The formalization proves both parts. Part (i) states the steps as a relation on graphs and the end as a double edge. In (b) the declaration derives $P(G)$ from $P(G_v^{ab})$ at every label $e_0 \notin E(G-v)$ of the new edge, the labels at which the formal splitting off is G_v^{ab} (Formalization note 21.11). It assumes nothing else of the labels.

In (c), P is assumed for every smaller graph, and not only for $G/E(H)$. In Katoh and Tanigawa's proof of their Theorem 5.5, the case of a proper rigid subgraph H needs it for both H and $G/E(H)$ [KT11, p. 673]. That proof is the following induction over all deficiencies. In each case it assumes the induction hypothesis KT eq. (6.1), which we state for a property P.

Theorem 21.56 (Induction over all deficiencies; KT p. 671)**.** *Let* P *be a property of multigraphs. Suppose that* $P(G)$ *holds for every minimal* k*-dof-graph* G *with one or two vertices, for every* k*. Suppose also that* $P(G)$ *holds for every minimal* k*-dof-graph* G *with at least three vertices, in each of the following cases, provided that* P *holds for every minimal* k'*-dof-graph with at least one and fewer than* $|V(G)|$ *vertices,*

for every k': (i) *G is not 2-edge-connected;* (ii) *G has a proper rigid subgraph;* (iii) *G is 2-edge-connected, has no proper rigid subgraph, and $k > 0$;* (iv) *G is 2-edge-connected, has no proper rigid subgraph, and $k = 0$. Then P holds for every minimal k-dof-graph with at least one vertex, for every k.*

Proof. We argue by induction on $|V(G)|$, for all k at once. Let G be a minimal k-dof-graph with at least one vertex, and suppose that P holds for every minimal k'-dof-graph with at least one and fewer than $|V(G)|$ vertices. If G has one or two vertices, then $P(G)$ holds by hypothesis. Otherwise one of the four cases applies. Indeed, if G is 2-edge-connected and has no proper rigid subgraph, then $k \geq 0$, as the partition of $V(G)$ into one part has value 0 (Definition 20.3). In each case $P(G)$ holds by hypothesis. □

21.6. **Simplicity, and two adjacent vertices of degree two.** A multigraph is *simple* if it has no loops and no two of its edges join the same pair of vertices [KT11, p. 668]. Katoh and Tanigawa remark that a minimal k-dof-graph with at least three vertices and no proper rigid subgraph is simple, as two edges that join the same pair of vertices form a proper rigid subgraph [KT11, p. 677]. We prove this remark (Lemma 21.57) from the deficiency of the two-vertex double edge (Lemma 20.36). With it, KT Lemma 6.7 shows that splitting off a vertex of degree two keeps such a graph simple when it has at least four vertices (Lemma 21.58). For $d = 3$, Katoh and Tanigawa derive from KT Lemma 4.6 that such a graph with $k = 0$ has two adjacent vertices v and a of degree two, and they write $N_G(v) = \{a, b\}$ and $N_G(a) = \{v, c\}$ [KT11, p. 680]. We prove the first fact for $d \geq 3$ by a count of degrees (Lemma 21.60), and derive the chain $b\,v\,a\,c$ from it (Lemma 21.61). Case III of the algebraic induction uses neither: it extracts a chain of length d at every d (Lemma 21.68).

Lemma 21.57 (A minimal k-dof-graph with no proper rigid subgraph is simple; KT p. 677)**.** *Let $D \geq 2$, and let G be a minimal k-dof-graph with at least three vertices and no proper rigid subgraph. Then G is simple.*

Proof. As we showed before Lemma 21.43, G has no loops. Suppose that two edges $e \neq f$ join vertices x and y, so that $x \neq y$. The subgraph H of G with the vertices x, y and the edges e, f is a 0-dof-graph (Lemma 20.36), and $1 < |V(H)| = 2 < |V(G)|$. So H is a proper rigid subgraph of G (Definition 20.17), which is impossible. □

Lemma 21.58 (Splitting off keeps the graph simple; KT Lemma 6.7)**.** *Let $D \geq 3$. A triangle, a multigraph with three vertices and three edges that join them in pairs, is a 0-dof-graph (Definition 20.8). Let G be a simple multigraph with $|V(G)| \geq 4$, and let $e_a \neq e_b$ be edges of G that join a vertex v to vertices a and b. If an edge of G joins a and b, then G has a proper rigid subgraph (Definition 20.17). So if G has none, then G_v^{ab} (Definition 21.10) is simple.*

Proof. We follow Katoh and Tanigawa's proof [KT11, p. 677]. A triangle with vertices x, y, z and edges joining x and y, y and z, z and x is a cycle on three vertices. As $3 \leq D$, it is a 0-dof-graph (Lemma 20.37).

As G is simple, e_a and e_b do not join v to the same vertex, so $a \neq b$. Let f be an edge of G that joins a and b. The subgraph of G induced on $\{v, a, b\}$ has no edge

other than e_a, e_b and f, as G is simple, so it is a triangle, and a 0-dof-graph. As $1 < 3 < 4 \leq |V(G)|$, it is a proper rigid subgraph of G.

Now let G have no proper rigid subgraph. Then no edge of G joins a and b. The new edge ab of G_v^{ab} is not a loop, as $a \neq b$, and its other edges are edges of the simple graph G with the same ends (Definition 21.10). None of them joins a and b, so none is parallel to ab, and G_v^{ab} is simple. □

Formalization note 21.59. The first pinned theorem is the first assertion. The second is the next, with the triangle on v, a and b as the proper rigid subgraph. The third is the last assertion, which is KT Lemma 6.7(ii). It combines the second with `splitOff_simple_of_noRigid`, by which G_v^{ab} is simple when G has no proper rigid subgraph and no edge joining a and b. Katoh and Tanigawa assume that G is a 2-edge-connected minimal k-dof-graph with no proper rigid subgraph, which is simple (Lemma 21.57), and that v has degree two. Their part (i) says that if $|V(G)| = 3$, then $k = 0$ and G has a realization of rank $D(|V(G)| - 1)$. Such a G is a triangle, as it is simple and 2-edge-connected, so $k = 0$ by the first assertion; Lemma 23.198 realizes it.

Lemma 21.60 (Two adjacent vertices of degree two; from KT Lemma 4.6, for $d \geq 3$ (superseded))**.** *Let $D \geq 6$, that is, $d \geq 3$, and let G be a minimal 0-dof-graph with at least three vertices and no proper rigid subgraph. Then G has two adjacent vertices of degree two.*

Proof. Suppose not. As we showed before Lemma 21.43, G has no loops. As G is a 0-dof-graph with at least two vertices, every vertex of G has degree at least two (Lemma 20.23). Let X_2 be the set of vertices of degree two, and let X_3 be the set of the other vertices, which have degree at least three. No edge joins two vertices of X_2, so every edge at a vertex of X_2 has its other end in X_3. Hence

$$\sum_{v \in X_2} \deg v \;\leq\; \sum_{w \in X_3} \deg w.$$

The degrees of the vertices of G sum to $2|E(G)|$. So $2|E(G)| \geq 2\sum_{v \in X_2} \deg v = 4|X_2|$, and $2|E(G)| \geq 2|X_2| + 3|X_3|$. Adding the first inequality to four times the second gives $10|E(G)| \geq 12(|X_2| + |X_3|)$, that is, $5|E(G)| \geq 6|V(G)|$.

On the other hand, KT Lemma 4.5(i) (Lemma 21.43) at $k = 0$ gives $(D - 1)\,|E(G)| < D\,|V(G)| - 1$. So

$$5|E(G)| \;<\; \frac{5D}{D-1}\,|V(G)| \;\leq\; 6|V(G)|,$$

as $D \geq 6$. This is a contradiction. Katoh and Tanigawa prove their Lemma 4.6 with maximal chains [KT11, pp. 664–665]; for two adjacent vertices of degree two, the count above suffices. □

Lemma 21.61 (A chain $b\,v\,a\,c$ with inner vertices of degree two; KT p. 680 (superseded))**.** *Let $D \geq 6$, and let G be a minimal 0-dof-graph with at least four vertices and no proper rigid subgraph. Then G has distinct vertices v, a, b, c and edges $e_a = va$, $e_b = vb$ and $e_c = ac$, such that e_a and e_b are the only edges at v, and e_a and e_c are the only edges at a.*

Proof. By Lemma 21.60, G has adjacent vertices v and a of degree two. Let e_a be an edge that joins them. By Lemma 21.57, G is simple. So the other edge e_b at v joins v to a vertex $b \notin \{v, a\}$, and the other edge e_c at a joins a to a vertex $c \notin \{v, a\}$.

Suppose that $b = c$. Then the subgraph T of G with the vertices v, a, b and the edges e_a, e_b, e_c is a triangle. It is a 0-dof-graph. Indeed, two edges of T join the parts of a partition of $V(T)$ into two parts, of value $D-2(D-1) \leq 0$, and three join those of the partition into three parts, of value $2D-3(D-1) \leq 0$ (Definition 20.3). As $1 < |V(T)| = 3 < |V(G)|$, T is a proper rigid subgraph of G (Definition 20.17), which is impossible. Hence $b \neq c$. □

21.7. **A chain or a short cycle: KT Lemma 4.6.** We now prove KT Lemma 4.6 for $k = 0$, which we stated before Lemma 21.49. Katoh and Tanigawa use it in the proof of their Lemma 6.13 [KT11, p. 692], which is Case III of the algebraic induction at general d. If G is a cycle on at most d vertices, they realize it by their Lemma 5.4 (Lemma 23.80). Otherwise they split off at the interior vertices of a chain $v_0 v_1 \dots v_d$, beginning with $G_{v_1}^{v_0 v_2}$, which is a minimal 0-dof-graph by KT Lemma 4.8. We define chains and cycles, and prove the dichotomy (Lemma 21.66), using that a short cycle is a 0-dof-graph (Lemma 20.37). Then we show that $G_{v_1}^{v_0 v_2}$ is also simple, by KT Lemma 6.7 (Lemma 21.58), and smaller than G when G has at least four vertices (Lemma 21.68), as Case III needs.

Definition 21.62 (A chain of length d; KT Lemma 4.6)**.** Let G be a multigraph, and let $d \geq 1$. A *chain of length* d in G, written $v_0 v_1 \dots v_d$, consists of distinct vertices $v_0, \dots, v_d$ of G and, for $0 \leq i \leq d-1$, an edge f_i of G joining v_i and v_{i+1}, such that for $1 \leq i \leq d-1$ the only edges of G at the *interior* vertex v_i are f_{i-1} and f_i.

Formalization note 21.63. The formal chain also records a label $e_0 \notin E(G)$, which Lemma 21.68 uses for the new edge of $G_{v_1}^{v_0 v_2}$ (Formalization note 21.11). So the declarations that give a chain, pinned on Lemma 21.66 and Lemma 21.68, assume that a label outside $E(G)$ exists.

Definition 21.64 (Cycle graphs; KT Lemma 4.6)**.** Let G be a multigraph, and let $m \geq 3$. Then G is a *cycle* on m vertices, or a cycle graph, if it has distinct vertices $v_0, \dots, v_{m-1}$ and, for $0 \leq i \leq m-1$, an edge f_i joining v_i and v_{i+1}, with indices modulo m, such that every vertex and every edge of G is among them.

Formalization note 21.65. The second pin says that $V(G) = \{v_0, \dots, v_{m-1}\}$. So $|V(G)| = m$, as the v_i are distinct.

Lemma 21.66 (A short cycle or a chain; KT Lemma 4.6)**.** *Let $D \geq 3$, and let G be a minimal 0-dof-graph with at least three vertices and no proper rigid subgraph. Then G is a cycle on at most d vertices (Definition 21.64), or G has a chain of length d (Definition 21.62).*

Proof. We follow Katoh and Tanigawa's proof [KT11, pp. 664–665], and give its count in full. As $D \geq 3$, we have $d \geq 2$, and $d \leq D$. By KT Lemma 3.1 (Lemma 20.18), G is 2-edge-connected. So it is connected, and each of its vertices has degree at least two (Lemma 20.23). By Lemma 21.57, G is simple.

Let u_0 be a vertex of G, and let g_1 be an edge at u_0, joining it to u_1. We extend the path u_0u_1 one edge at a time. While its last vertex u_j has degree two and $j < d$, let g_{j+1} be the edge at u_j other than g_j, and let u_{j+1} be its other end. The vertices $u_1, \dots, u_{j-1}$ have degree two and both of their edges are on the path, so u_{j+1} is not one of them. As G is simple, $u_{j+1} \neq u_j$, and $u_2 \neq u_0$. So u_{j+1} is a new vertex, or $u_{j+1} = u_0$ with $j \geq 2$. We call the path so obtained the *walk* from u_0 along g_1. It ends in one of three ways.

If the walk reaches length d, then $u_0u_1 \dots u_d$ is a chain of length d (Definition 21.62).

If the walk closes up, with $u_{j+1} = u_0$, then $u_0, \dots, u_j$ and $g_1, \dots, g_{j+1}$ form a cycle C on $m = j+1$ vertices, where $3 \leq m \leq d$. If u_0 has degree two, then no edge of G leaves C. As G is connected, $G = C$ is a cycle on at most d vertices (Definition 21.64). If u_0 has degree at least three, then C is a 0-dof-graph, as $m \leq d \leq D$ (Lemma 20.37). A third edge at u_0 has its other end outside C, since the other vertices of C have degree two. So C is a proper rigid subgraph of G (Definition 20.17), which is impossible. Katoh and Tanigawa's maximal chains may begin and end at the same vertex. A closed one of length d has only d distinct vertices, and this case excludes it.

Otherwise the walk stops at a vertex u_j of degree at least three, with $1 \leq j \leq d-1$.

Suppose now that G is not a cycle on at most d vertices and has no chain of length d. Then every walk stops at a vertex of degree at least three, after at most $d-1$ edges. So a walk that starts at a vertex of degree at least three has at most $d-2$ interior vertices, all of degree two. For $i \geq 2$, let X_i be the set of vertices of G of degree i. Let $v \in X_2$, and let e be an edge at v. The walk from v along e stops at a vertex u of degree at least three. Read backwards, it is a path that leaves u along an edge g and passes through vertices of degree two up to v. At a vertex of degree two, a walk leaves by the edge by which it did not arrive. So this path is the beginning of the walk from u along g, and v, of degree two, is an interior vertex of that walk. The pair (u, g) and the position of v on its walk determine v. They also determine e, as the edge by which that walk arrives at v. There are $\sum_{i\geq 3} i\,|X_i|$ pairs (u, g) with u of degree at least three, and so

$$2\,|X_2| \;\leq\; (d-2)\sum_{i\geq 3} i\,|X_i|.$$

Katoh and Tanigawa obtain this inequality from their eqs. (4.6) and (4.7), by counting maximal chains.

As every vertex of G has degree at least two, $|V(G)| = \sum_{i\geq 2} |X_i|$. Adding $2|X_i|$ for each $i \geq 3$ to both sides of the inequality gives KT eq. (4.8),

$$2\,|V(G)| \;\leq\; \sum_{i\geq 3} (i(d-2)+2)|X_i|.$$

For $i \geq 3$ we have $i(d-2)+2 \leq (D-1)(i-2)$, the inequality before KT eq. (4.9). Indeed, at $i = 3$ twice the right side minus twice the left side is $(d-2)(d-3) \geq 0$, as $D = d(d+1)/2$. Each step from i to $i+1$ adds $d-2$ to the left side and $D-1 \geq d-2$ to the right side. The degrees of G sum to $2|E(G)|$, so $\sum_{i\geq 3}(i-2)|X_i| =$

$2|E(G)| - 2|V(G)|$, and KT eq. (4.9) reads

$$2\,|V(G)| \;\leq\; (D-1)\sum_{i\geq 3}(i-2)|X_i| \;=\; (D-1)(2|E(G)| - 2|V(G)|).$$

Hence $D\,|V(G)| \leq (D-1)\,|E(G)|$. But KT Lemma 4.5(i) (Lemma 21.43, at $k = 0$) gives $(D-1)\,|E(G)| < D(|V(G)| - 1) + D - 1 < D\,|V(G)|$, a contradiction. □

Formalization note 21.67. The first pin is the lemma. The second is the case of a walk that closes up at a vertex of degree at least three, in a simple multigraph: the cycle is a proper rigid subgraph.

The cycle alternative cannot be dropped. Let $3 \leq m \leq d$, and let C be a cycle on m vertices. It is a 0-dof-graph (Lemma 20.37). A subgraph of C that misses an edge or a vertex has no cycle, so its partition into single vertices has positive value when it has at least two vertices. So removing an edge of C raises the deficiency, and C is minimal, as we noted after Definition 20.8. Also, C has no proper rigid subgraph. A chain of length d has $d + 1 > m$ vertices, so C has none. Hence the cycle alternative occurs for every $d \geq 3$. Under the bound $|V(G)| \geq 4$ of the next lemma, it occurs exactly when $d \geq 4$.

Lemma 21.68 (The chain and its splitting off; KT Lemmas 4.6, 4.8 and 6.7)**.** *Let $D \geq 3$, and let G be a minimal 0-dof-graph with at least four vertices and no proper rigid subgraph. Then G is a cycle on at most d vertices, or G has a chain $v_0v_1 \dots v_d$ of length d such that $G_{v_1}^{v_0v_2}$ is a simple minimal 0-dof-graph with at least two and fewer than $|V(G)|$ vertices.*

Proof. By Lemma 21.66, G is a cycle on at most d vertices, or it has a chain $v_0v_1 \dots v_d$ of length d. Suppose the second. As $D \geq 3$, we have $d \geq 2$, so v_1 is an interior vertex of the chain. Its only edges are f_0 and f_1, which join it to v_0 and to v_2, and v_0, v_1, v_2 are distinct. As G has at least three vertices and no proper rigid subgraph, $G_{v_1}^{v_0v_2}$ is a minimal 0-dof-graph by KT Lemma 4.8(i) (part (i) of Lemma 21.52). Its vertices are the $|V(G)| - 1 \geq 3$ vertices of G other than v_1 (Lemma 21.51).

It remains to show that $G_{v_1}^{v_0v_2}$ is simple. The graph G is simple by Lemma 21.57, as it has at least three vertices and no proper rigid subgraph, and it has at least four vertices. So KT Lemma 6.7(ii) (Lemma 21.58), applied to v_1 and its edges $f_0 \neq f_1$, shows that $G_{v_1}^{v_0v_2}$ is simple. □

Formalization note 21.69. The pinned theorem is the lemma. It takes the simplicity of $G_{v_1}^{v_0v_2}$ from the third pinned theorem of Lemma 21.58.

22. Grassmann–Cayley meet and projective duality

This chapter defines the meet, the operation of the Grassmann–Cayley algebra dual to the join, and proves that for $d = 3$ the meet of two planes is a scalar multiple of the join of two points of their line of intersection. Katoh and Tanigawa use the join and not the meet. For the Grassmann–Cayley algebra they refer to Barnabei, Brini and Rota [BBR85] and to White [Whi94], among others [KT11, §2.1]. They define the extensor of an affine subspace through points that span it, and in a nonparallel panel-hinge framework the hinge of an edge uv is the intersection $\Pi(u) \cap \Pi(v)$ of

the panels of its ends [KT11, §5.1]. The formalization records such a hinge by the meet of the two panels (Definition 23.1). For $d = 3$, Lemma 22.16 shows that this meet agrees with the join of two points of the hinge up to a scalar. This is what lets the proof of KT Lemma 6.10, which spans $\bigwedge^2 \mathbb{R}^4$ by joins of points in KT eq. (6.45) [KT11, p. 691], apply to hinges given by panels.

The chapter builds only on Section 18.2: the exterior algebra of $V = K^{d+1}$, its extensors, and the join. As there, K is an arbitrary field. The constructions use the standard basis of V, through a volume form and a coordinate pairing, and no order on K or hypothesis on its characteristic. We write $\vee$ for the product of an exterior algebra, as in Definition 18.4, and $\wedge$ for the meet, as Barnabei, Brini and Rota and White do.

The first subsection defines the volume form (Definition 22.1). It then defines the duality between the exterior powers of V and of its dual space (Definition 22.3). The pairing $(X, Y) \mapsto \mathrm{vol}(X \vee Y)$ is nondegenerate, and it gives the complement isomorphism $*$ (Definition 22.5), whose defining property is Lemma 22.6. The meet is then $A \wedge B = *(*A \vee *B)$ (Definition 22.7), and the meet of two hyperplanes is an extensor of their intersection (Lemma 22.9). The second subsection takes $d = 3$ and proves the agreement of meet and join (Lemma 22.16). It ends with the converse fact that a nonzero 2-extensor determines its plane (Lemma 22.18). The remark at the start of the second subsection (Remark 22.11) says which of these results the formal proof of KT Theorem 5.5 does not use, and what it uses instead.

22.1. **The complement isomorphism and the meet.** Let $V = K^{d+1}$, with standard basis $e_1, \dots, e_{d+1}$. For a set $S = \{s_1 < \cdots < s_i\} \subseteq \{1, \dots, d+1\}$ we write $e_S = e_{s_1} \vee \cdots \vee e_{s_i}$. The products e_S over the i-element sets S form a basis of $\bigwedge^i V$ (Lemma 18.6), and X_S denotes the coordinate of $X \in \bigwedge^i V$ at e_S. For $i = d+1$ there is one such product.

Definition 22.1 (The volume form $\bigwedge^{d+1} V \cong K$)**.** The *volume form* $\mathrm{vol} \colon \bigwedge^{d+1} V \to K$ is the linear isomorphism with $\mathrm{vol}(e_1 \vee \cdots \vee e_{d+1}) = 1$.

Proof. The space $\bigwedge^{d+1} V$ has dimension $\binom{d+1}{d+1} = 1$, with basis $e_1 \vee \cdots \vee e_{d+1}$. The volume form takes an element to its coordinate in this basis. □

Formalization note 22.2. The second pin is the same isomorphism for R^n over a commutative ring R, and the first is its case $R = K$, $n = d+1$. As in Section 19, the formalization writes $d = k+1$, so that $V = K^{k+2}$.

By Lemma 18.6, $\mathrm{vol}(v_1 \vee \cdots \vee v_{d+1})$ is the determinant of the matrix with rows $v_1, \dots, v_{d+1}$.

The exterior powers of V and of its dual space V^* are dual to each other through determinants.

Definition 22.3 (Duality of exterior powers $\bigwedge^j V^* \cong (\bigwedge^j V)^*$)**.** Let $j \geq 0$. The linear map $\bigwedge^j V^* \to (\bigwedge^j V)^*$ that sends $\varphi_1 \vee \cdots \vee \varphi_j$ to the linear form

$$v_1 \vee \cdots \vee v_j \;\longmapsto\; \det(\varphi_a(v_b))_{a,b=1}^{j}$$

is an isomorphism.

Proof. Let $\varphi_1, \dots, \varphi_{d+1}$ be the basis of V^* dual to $e_1, \dots, e_{d+1}$, and let φ_S be the products of its elements, as for e_S. For j-element sets S and T, the matrix $(\varphi_{s_a}(e_{t_b}))$ is the identity if $S = T$, and it has a zero row otherwise. So the map sends φ_S to the linear form $X \mapsto X_S$. It takes the basis (φ_S) of $\bigwedge^j V^*$ to the basis of $(\bigwedge^j V)^*$ dual to (e_S), and is therefore an isomorphism. □

Formalization note 22.4. The second pin is the isomorphism for a finite free module with an ordered basis over a commutative ring, built from that basis as in the proof, and the first is its case $V = K^{d+1}$ with the standard basis.

The complement isomorphism compares two pairings. For each i, the *coordinate pairing* on $\bigwedge^i V$ is the bilinear form $\langle X, Y\rangle = \sum_S X_S Y_S$, for which the basis (e_S) is orthonormal. On $V = \bigwedge^1 V$ it is the dot product. It identifies $\bigwedge^i V$ with its dual, as $Z \mapsto \langle Z, \cdot\rangle$ is an isomorphism $\bigwedge^i V \to (\bigwedge^i V)^*$. The second pairing uses the volume form. For $0 \le j \le d+1$, the join of $X \in \bigwedge^j V$ and $Y \in \bigwedge^{d+1-j} V$ lies in $\bigwedge^{d+1} V$, and the *wedge pairing* of X and Y is $\mathrm{vol}(X \vee Y)$.

Definition 22.5 (The complement isomorphism $\bigwedge^j V \cong \bigwedge^{d+1-j} V$)**.** Let $0 \le j \le d+1$. The wedge pairing $\bigwedge^j V \times \bigwedge^{d+1-j} V \to K$ is nondegenerate, so $X \mapsto \mathrm{vol}(X \vee \cdot)$ is an isomorphism $\bigwedge^j V \to (\bigwedge^{d+1-j} V)^*$. The *complement isomorphism* $*\colon \bigwedge^j V \to \bigwedge^{d+1-j} V$ is this isomorphism followed by the inverse of the coordinate identification of $\bigwedge^{d+1-j} V$ with its dual.

Proof. Let S be a j-element set and T a $(d+1-j)$-element set in $\{1, \dots, d+1\}$. The join $e_S \vee e_T$ is the extensor of the family that lists e_s for $s \in S$ and then e_t for $t \in T$ (Definition 18.4). If S and T meet, this family repeats a vector, and $e_S \vee e_T = 0$ (the remark after Definition 18.3). Since $|S| + |T| = d+1$, the sets S and T are disjoint exactly when T is the complement S^c of S. In that case the family lists each standard basis vector once. It is linearly independent, so $e_S \vee e_{S^c} \neq 0$, and $\varepsilon_S = \mathrm{vol}(e_S \vee e_{S^c}) \neq 0$, as the volume form is injective. In the bases (e_S) and (e_T), the matrix of the wedge pairing therefore has exactly one nonzero entry in each row and each column: ε_S, in row S and column S^c.

Hence $X \mapsto \mathrm{vol}(X \vee \cdot)$ is injective. If $\mathrm{vol}(X \vee Y) = 0$ for every Y, then $Y = e_{S^c}$ gives $\varepsilon_S X_S = 0$, so $X_S = 0$ for every S, and $X = 0$. The spaces $\bigwedge^j V$ and $(\bigwedge^{d+1-j} V)^*$ both have dimension $\binom{d+1}{j} = \binom{d+1}{d+1-j}$, so this injective linear map is an isomorphism. The coordinate identification of $\bigwedge^{d+1-j} V$ with its dual is an isomorphism because the basis (e_T) is orthonormal, and $*$ is the first isomorphism followed by the inverse of the second.

In fact $\varepsilon_S = \pm 1$ is the sign of the permutation that lists the elements of S and then those of S^c, each in increasing order. The argument needs only $\varepsilon_S \neq 0$. □

Lemma 22.6 (The defining property of the complement isomorphism)**.** *Let $0 \le j \le d+1$, $X \in \bigwedge^j V$ and $Y \in \bigwedge^{d+1-j} V$. Then*

$$\langle *X, Y\rangle = \mathrm{vol}(X \vee Y).$$

Proof. The coordinate identification sends $*X$ to the linear form $\langle *X, \cdot \rangle$, which is $\mathrm{vol}(X \vee \cdot)$ by Definition 22.5. □

Taking $X = e_S$ and $Y = e_T$ shows that $\langle *e_S, e_T \rangle$ is ε_S if $T = S^c$ and 0 otherwise. So $*e_S = \varepsilon_S \, e_{S^c}$ (`complementIso_exteriorPower_basis_eq_smul_compl`), and $*$ is the Hodge star operator of the standard basis, which Barnabei, Brini and Rota use to state the duality between join and meet [BBR85, p. 146].

The meet is the join transported by the complement isomorphism.

Definition 22.7 (The meet (regressive product))**.** Let $a, b \geq 0$ with $a + b \leq d + 1$, and let $A \in \bigwedge^{d+1-a} V$ and $B \in \bigwedge^{d+1-b} V$. The *meet* of A and B is

$$A \wedge B \;=\; *(*A \vee *B) \;\in\; \bigwedge^{d+1-a-b} V,$$

where $*A \in \bigwedge^{a} V$ and $*B \in \bigwedge^{b} V$, so that their join lies in $\bigwedge^{a+b} V$.

Formalization note 22.8. The formalization proves no property of the meet itself. It proves the formula for $*e_S$ and the transformation law below, and the case of two hyperplanes in the form $*(n_A \vee n_B)$ (Lemma 22.9), which is how the later chapters use the meet (Definition 23.1). The statements below for a general subspace U and for general subspaces X and Y are not formalized.

The meet of extensors of two subspaces is an extensor of their intersection. Let U be a subspace of V, and let $U^\perp$ be its orthogonal complement for the dot product. The complement isomorphism sends a nonzero extensor of U to a nonzero extensor of $U^\perp$. For $U = U_S$, the span of the e_i with $i \in S$, this is $*e_S = \varepsilon_S \, e_{S^c}$. In general, write $U = g(U_S)$ for a linear automorphism g of V, and let h be its inverse transpose, so that $h(x) \cdot g(y) = x \cdot y$. The extensors of U are the multiples of $g(e_{s_1}) \vee \cdots \vee g(e_{s_i})$, and

$$*(g(e_{s_1}) \vee \cdots \vee g(e_{s_i})) \;=\; \det(g)\, \varepsilon_S \; h(e_{t_1}) \vee \cdots \vee h(e_{t_{d+1-i}}),$$

where $S^c = \{t_1 < \cdots < t_{d+1-i}\}$ (`complementIso_map_contragredient_eq`). The right side is a nonzero extensor of $h(U_{S^c})$. This is $U^\perp$, since $h(e_t) \cdot g(e_s) = e_t \cdot e_s = 0$ for $t \in S^c$ and $s \in S$, and the two spaces have the same dimension.

Now let A and B be nonzero extensors of subspaces X and Y of V with $X + Y = V$. Then $*A$ and $*B$ are nonzero extensors of $X^\perp$ and $Y^\perp$, and $X^\perp \cap Y^\perp = (X + Y)^\perp = 0$. So $*A \vee *B$ is a nonzero extensor of $X^\perp + Y^\perp = (X \cap Y)^\perp$ (the remark after Definition 18.4), and $A \wedge B$ is a nonzero extensor of $X \cap Y$. Barnabei, Brini and Rota define the meet by a sum over the splits of A and prove this property for it [BBR85, §4, Proposition 4.3]. White gives the same definition as a sum over shuffles [Whi94, §4].

The later chapters use the case of two distinct hyperplanes H_A and H_B, where $a = b = 1$. Then $n_A = *A$ spans $H_A^\perp$, so it is a normal of H_A, and likewise $n_B = *B$ for H_B. The meet $A \wedge B = *(n_A \vee n_B)$ is a nonzero extensor of $H_A \cap H_B$. The next lemma states this for every d, in the form the formalization proves, and Lemma 22.16 below proves it for $d = 3$ by a count of dimensions.

Lemma 22.9 (The meet of two hyperplanes is an extensor of their intersection)**.** *Let $d \geq 1$, let $n_1, n_2 \in V$, and let W be a $(d-1)$-dimensional subspace of V whose*

vectors are orthogonal to n_1 and n_2. Then $*(n_1 \vee n_2)$ lies in the image of the map $\bigwedge^{d-1} W \to \bigwedge^{d-1} V$ induced by the inclusion of W in V. If moreover n_1 and n_2 are linearly independent, and $p_1, \dots, p_{d-1} \in V$ are linearly independent and orthogonal to n_1 and n_2, then $p_1 \vee \cdots \vee p_{d-1} = c\,*(n_1 \vee n_2)$ for some $c \in K$.

Proof. If n_1 and n_2 are linearly dependent, then $n_1 \vee n_2 = 0$ (the remark after Definition 18.3), and the first statement holds as $*0 = 0$. Otherwise there is a linear automorphism g of V with $g(e_1) = n_1$ and $g(e_2) = n_2$. Let h be its inverse transpose. By the formula displayed after Definition 22.7, with $S = \{1, 2\}$, which follows from Lemma 22.6,

$$*(n_1 \vee n_2) \;=\; \det(g)\, \varepsilon_S\, h(e_3) \vee \cdots \vee h(e_{d+1}).$$

For $t \geq 3$ and $i = 1, 2$ we have $h(e_t) \cdot n_i = h(e_t) \cdot g(e_i) = e_t \cdot e_i = 0$. So the vectors $h(e_t)$ lie in the orthogonal complement of n_1 and n_2. This complement has dimension $d - 1$ and contains W, so it is W, and $*(n_1 \vee n_2)$ is a multiple of a join of $d - 1$ vectors of W. Hence it lies in the image of $\bigwedge^{d-1} W$.

For the second statement, let W be the orthogonal complement of n_1 and n_2, of dimension $d-1$. The space $\bigwedge^{d-1} W$ has dimension $\binom{d-1}{d-1} = 1$, so its image in $\bigwedge^{d-1} V$ has dimension at most 1. The vectors p_i lie in W, so their join lies in this image, and it is nonzero, as they are linearly independent. The meet $*(n_1 \vee n_2)$ lies in the image by the first statement. So it is a multiple $\mu\,(p_1 \vee \cdots \vee p_{d-1})$, and $\mu \neq 0$, as $n_1 \vee n_2 \neq 0$ and $*$ is an isomorphism. Then $c = \mu^{-1}$ will do. □

Formalization note 22.10. The first pin is the first statement, and the second pin the second.

22.2. **Lines in projective three-space.** In this subsection $d = 3$ and $V = K^4$, except in part (1) of Lemma 22.12 and in Lemma 22.18, which hold for every d. A point $p \in K^3$ has the homogeneous coordinate vector $\bar{p} = (p, 1) \in K^4$ (Definition 18.1). The plane $\{x \in K^3 : a \cdot x + a_0 = 0\}$ has the normal $n = (a, a_0) \in K^4$, and a point p lies on it exactly when $\bar{p} \cdot n = 0$. Let L be the line in which two nonparallel planes meet, and let n_1 and n_2 be their normals, which are linearly independent. Let

$$W = \{x \in K^4 : x \cdot n_1 = x \cdot n_2 = 0\},$$

a 2-dimensional subspace that contains $\bar{p}$ for every $p \in L$. The extensor of L is $\bar{p} \vee \bar{q}$ for distinct points p and q of L (Definition 18.8). The meet of the two planes (two panels, in a panel-hinge framework) is a scalar multiple of $*(n_1 \vee n_2)$, a nonzero extensor of W by the remarks after Definition 22.7 and by Lemma 22.9. Lemma 22.16 proves that the two extensors agree up to a scalar, by a count of dimensions in $\bigwedge^2 K^4$. It states the agreement in the form that the algebraic induction applies: a linear form on $\bigwedge^2 K^4$ that vanishes on the meet vanishes on the join.

Remark 22.11 (Katoh and Tanigawa's extensors of lines and the formal proof)**.** Katoh and Tanigawa write the 2-extensor $C(L)$ supporting a line L as the join of two points of L (KT §2.1, p. 651). In the proof of KT Claim 6.12, they use that the join of two points of a line in a panel is such an extensor (KT eq. (6.45), p. 691). The formalization records a hinge by the meet of two panels (Definition 23.1), so

it compares the meet with the join. Some definitions and lemmas of this chapter concern this comparison in a form that the formal proof of KT Theorem 5.5 (Theorem 23.38) does not use. We keep them, and say here what the formal proof does instead.

The line $\bigwedge^2 W$. After the proof of Lemma 22.16, we identify the line of $\bigwedge^2 K^4$ that the proof finds with the image of $\bigwedge^2 W$. So the meet of the two planes is an extensor of their line of intersection. The identification uses two facts. The space $\bigwedge^2 W$ is a line (Lemma 22.14). It embeds in $\bigwedge^2 K^4$ (Lemma 22.15). The formal proof of Lemma 22.16 makes no such identification. There the line is the orthogonal complement of the span of the 2-extensors $n_i \vee v$, a span of dimension 5. The join lies on it, as a determinant of dot products vanishes.

The formal proof of KT Theorem 5.5 uses neither Lemma 22.16 nor Lemma 22.12. It takes the case $d = 3$ as a case of KT Lemma 6.13 (Remark 23.137). It compares the meet and the join by Lemma 22.9, which holds for every d. In that lemma, W is a subspace of K^{d+1} of dimension $d-1$ whose vectors are orthogonal to n_1 and n_2. The first statement places the meet in the image of $\bigwedge^{d-1} W$, by the transformation law of the complement isomorphism (the passage after Definition 22.7). The second compares the meet and the join in this image. Its formal proof uses the facts of Lemma 22.14 and Lemma 22.15 in degree $d-1$. The two lemmas state their case $d = 3$.

The meet and the duality of exterior powers. Katoh and Tanigawa give each panel of a nonparallel framework by a vector $c(v)$, and each hinge as the intersection of two panels (KT §5.1, p. 668). The formal proof gives each panel by a normal. It gives a hinge by $*(n_A \vee n_B)$, from the normals n_A and n_B of its two panels. It does not form the meet $A \wedge B$ of Definition 22.7, which is $*(n_A \vee n_B)$ when $n_A = *A$ and $n_B = *B$ (the passage before Lemma 22.9). Of Definition 22.3, the formal proof uses the map and its determinant formula, but not that the map is an isomorphism. With V identified with V^* by the dot product, the formula gives the coordinate pairing of two j-extensors as a determinant of dot products (the proof of Lemma 22.16). The formal proof of the transformation law compares coordinate pairings in this form.

The proof of Lemma 22.16 finds a line of $\bigwedge^2 K^4$ that contains both extensors, cut out by the coordinate pairing. The first lemma shows that the meet lies on it.

Lemma 22.12 (The complement of a 2-extensor is orthogonal to the 2-extensors sharing a vector with it)**.**

(1) Let $d \geq 1$ *and* $0 \leq j \leq d+1$*, and let* $X \in \bigwedge^j K^{d+1}$ *and* $Y \in \bigwedge^{d+1-j} K^{d+1}$ *with* $X \vee Y = 0$*. Then* $\langle *X, Y \rangle = 0$*.*

(2) Let $n_1, n_2, c \in K^4$*, and let* $Y = n_1 \vee c$*. Then* $(n_1 \vee n_2) \vee Y = 0$ *and* $\langle *(n_1 \vee n_2), Y \rangle = 0$*.*

Proof. For part (1), Lemma 22.6 gives $\langle *X, Y \rangle = \mathrm{vol}(X \vee Y) = \mathrm{vol}(0) = 0$. For part (2), $(n_1 \vee n_2) \vee (n_1 \vee c)$ is the extensor of the family n_1, n_2, n_1, c (Definition 18.4). This family repeats n_1, so its extensor is zero (the remark after Definition 18.3). The second claim follows by part (1), with $X = n_1 \vee n_2$ and $j = 2$.

The same argument applies to $Y = n_2 \vee c$, as the family n_1, n_2, n_2, c repeats n_2. So $*(n_1 \vee n_2)$ is orthogonal to every 2-extensor $n \vee c$ with $n \in \{n_1, n_2\}$. □

Formalization note 22.13. The first pin is part (1), and the second and third pins are the two claims of part (2). No pin states the case $Y = n_2 \vee c$. The formal proof of Lemma 22.16 obtains it from the first pin, as at the end of the proof above.

Geometrically, the line in the proof below is the image of $\bigwedge^2 W$ in $\bigwedge^2 K^4$. The next two lemmas give the facts that identify it, which we use after the proof.

Lemma 22.14 (The exterior square of a 2-dimensional space is a line)**.** *Let W be a 2-dimensional vector space over K. Then $\dim \bigwedge^2 W = \binom{2}{2} = 1$. Consequently, every element of $\bigwedge^2 W$ is a scalar multiple of any nonzero one.*

Proof. The exterior power $\bigwedge^n W$ has dimension $\binom{\dim W}{n}$, which is 1 for $n = \dim W = 2$. A space of dimension 1 is spanned by each of its nonzero elements. □

Lemma 22.15 (The exterior square of a subspace embeds in $\bigwedge^2 K^4$)**.** *Let W be a subspace of K^4. The linear map $\bigwedge^2 W \to \bigwedge^2 K^4$ induced by the inclusion of W in K^4 is injective.*

Proof. Extend a basis $w_1, \dots, w_m$ of W to a basis of K^4. The map sends the basis of $\bigwedge^2 W$ formed by the $w_a \vee w_b$ with $a < b$ to part of the corresponding basis of $\bigwedge^2 K^4$, so it is injective. □

Lemma 22.16 (Point-join $\leftrightarrow$ panel-meet duality; KT eq. (6.45))**.** *Let $n_1, n_2 \in K^4$ be linearly independent, and let $x, y \in K^4$ satisfy $x \cdot n_i = y \cdot n_i = 0$ for $i = 1, 2$. Then $x \vee y$ is a scalar multiple of $*(n_1 \vee n_2)$. In particular, every linear form r on $\bigwedge^2 K^4$ with $r(*(n_1 \vee n_2)) = 0$ satisfies $r(x \vee y) = 0$.*

Proof. Let Φ be the subspace of $\bigwedge^2 K^4$ spanned by the 2-extensors $n_1 \vee v$ and $n_2 \vee v$ for $v \in K^4$, and let

$$\Omega = \{Z \in \textstyle\bigwedge^2 K^4 : \langle Z, F \rangle = 0 \text{ for every } F \in \Phi\}.$$

We show that Ω has dimension 1 and contains both $x \vee y$ and $*(n_1 \vee n_2)$.

First, Φ has dimension 5. For $n \neq 0$, the map $v \mapsto n \vee v$ has kernel $\operatorname{span}\{n\}$, since $n \vee v = 0$ exactly when n and v are linearly dependent (the remark after Definition 18.3). So its image $n \vee K^4$ has dimension 3. The images for n_1 and n_2 meet in the line spanned by $n_1 \vee n_2$. This line lies in both, as $n_1 \vee n_2 = n_2 \vee (-n_1)$. Conversely, let $n_1 \vee v = n_2 \vee w$. The join with n_2 on the left gives $n_2 \vee n_1 \vee v = n_2 \vee n_2 \vee w = 0$, so n_2, n_1 and v are linearly dependent. As n_1 and n_2 are independent, $v = \alpha n_1 + \beta n_2$ for some $\alpha, \beta \in K$, and $n_1 \vee v = \beta\,(n_1 \vee n_2)$. Hence $\dim \Phi = 3 + 3 - 1 = 5$ (`finrank_sup_range_wedgeFixedLeft`). The coordinate pairing is nondegenerate on the 6-dimensional space $\bigwedge^2 K^4$, so $\dim \Omega = 6 - 5 = 1$.

Next, $x \vee y \in \Omega$. It suffices to show that $\langle x \vee y, n_i \vee v \rangle = 0$ for $i = 1, 2$ and $v \in K^4$. Identify V with V^* by the dot product. Then the coordinate pairing of two

2-extensors is the determinant of Definition 22.3 (`exteriorPower_basis_toDual_eq_pairingDual_comp_map`):

$$\langle x \vee y, n_i \vee v \rangle = \det \begin{pmatrix} x \cdot n_i & x \cdot v \\ y \cdot n_i & y \cdot v \end{pmatrix}.$$

Its first column is zero by hypothesis, so it vanishes (`extensor_toDual_extensor_eq_zero_of_perp`).

The meet lies in Ω as well. By part (2) of Lemma 22.12, the pairing $\langle *(n_1 \vee n_2), n_1 \vee v \rangle$ vanishes for every $v \in K^4$. The end of the proof of that lemma gives the same for $n_2 \vee v$.

Finally, $n_1 \vee n_2 \neq 0$, as n_1 and n_2 are linearly independent, and so $*(n_1 \vee n_2) \neq 0$, as $*$ is an isomorphism. So $*(n_1 \vee n_2)$ spans Ω, and $x \vee y = \lambda *(n_1 \vee n_2)$ for some $\lambda \in K$. For a linear form r with $r(*(n_1 \vee n_2)) = 0$, it follows that $r(x \vee y) = \lambda\, r(*(n_1 \vee n_2)) = 0$. □

Formalization note 22.17. The first pin is the first statement, and the second and third are the second statement, with the orthogonality written through the coordinate pairing of K^4 and through the dot product. The formal proof of Case III at $d = 3$ (Lemma 23.195) applies the third, in the step from a join to a line of the proof of Lemma 23.191. The general case applies Lemma 22.9 instead.

The line Ω is the image of $\bigwedge^2 W$. The inclusion of W in K^4 induces an injective map $\bigwedge^2 W \to \bigwedge^2 K^4$ (Lemma 22.15). Its image is therefore a line, as $\bigwedge^2 W$ is one (Lemma 22.14). It is spanned by $x \vee y$ for any basis x, y of W, and $x \vee y$ lies in Ω by the proof. So $*(n_1 \vee n_2)$ is a nonzero extensor of W, and the meet of the two planes is an extensor of their line of intersection. For lines in 3-space this is White's theorem that the Plücker coordinates of a subspace and its dual Plücker coordinates, computed from hyperplanes that contain it, agree up to a common scalar after a reordering with signs [Whi94, §2]. By the formula for $*e_S$ after Lemma 22.6, the complement isomorphism performs the reordering.

Conversely, a nonzero 2-extensor determines the plane spanned by its vectors. This is the case $j = 2$ of the proposition of Barnabei, Brini and Rota recalled after Definition 18.4, here for every d.

Lemma 22.18 (Equal nonzero 2-extensors span the same plane)**.** *Let* $p_0, p_1, p_0', p_1' \in K^{d+1}$ *with* $p_0 \vee p_1 = p_0' \vee p_1' \neq 0$. *Then* $\operatorname{span}\{p_0, p_1\} = \operatorname{span}\{p_0', p_1'\}$.

Proof. Let $x \in K^{d+1}$. The join $x \vee p_0 \vee p_1$ depends on p_0 and p_1 only through $p_0 \vee p_1$ (Definition 18.4), so $x \vee p_0 \vee p_1 = x \vee p_0' \vee p_1'$. The vectors p_0 and p_1 are linearly independent, as their join is nonzero, so $x \vee p_0 \vee p_1 = 0$ exactly when $x \in \operatorname{span}\{p_0, p_1\}$ (the remark after Definition 18.3). The same holds for p_0' and p_1'. So the two spans have the same elements. □

23. The algebraic induction: KT Theorem 5.5 and its three cases

This chapter proves Katoh and Tanigawa's Theorem 5.5 and the results that they derive from it [KT11, §§5–6]. Let $D = \binom{d+1}{2}$, where d is the dimension. KT

Theorem 5.5 states that every minimal k-dof-graph $G = (V, E)$ (Definition 20.8) with $|V| \geq 2$ has a panel-hinge realization (G, p) in $\mathbb{R}^d$ with

$$\operatorname{rank} R(G, p) \;=\; D(|V| - 1) - k,$$

where $R(G, p)$ is the rigidity matrix (Definition 19.7). If G is simple, the realization is nonparallel: the panels of any two bodies meet in a $(d-2)$-dimensional affine subspace (KT §5.1). We prove the theorem for every $d \geq 2$, in two forms (Theorem 23.38). Every such G has a realization of this rank whose hinges lie in the panels of the bodies they join and are nondegenerate, that is, have nonzero extensors. If G is simple, it also has a panel-hinge realization of this rank in which any two panels have linearly independent normals.

Katoh and Tanigawa derive their Theorem 5.6 from Theorem 5.5: every multigraph $G = (V, E)$ has a panel-hinge realization (G, p) with $\operatorname{rank} R(G, p) = D(|V| - 1) - \operatorname{def}(\tilde{G})$, where $\operatorname{def}(\tilde{G})$ is the deficiency of the multigraph $\tilde{G} = (D-1) \cdot G$ (Definition 20.3). With their Proposition 2.3, Theorem 5.6 proves the molecular conjecture of Tay and Whiteley [TW84] (KT Conjecture 1.2; KT p. 670). We prove KT Theorem 5.6 and the molecular conjecture for simple graphs (Theorem 23.50, Theorem 23.56). For multigraphs we prove them with nondegenerate hinges that need only lie in the panels of the bodies they join (Theorem 23.66, Theorem 23.70). KT Proposition 2.3 contains the rank formula of Jackson and Jordán [JJ10, Theorem 6.1] at a generic body-hinge framework on a multigraph. We prove it in a conditional form: at a body-hinge framework, a bound on the space of motions that a generic framework meets gives the formula (Proposition 23.19). A later chapter proves the formula at every generic hinge-point assignment of a multigraph (Theorem 28.40), and KT Proposition 2.3 itself there (Proposition 28.42).

Field generality. We work over an arbitrary infinite field K, of any characteristic. Katoh and Tanigawa work over $\mathbb{R}$, which is the case $K = \mathbb{R}$. The extensor algebra (Section 18) and the rigidity matrix (Section 19) need no hypothesis on the field. The arguments by genericity use only that a nonzero polynomial over an infinite field does not vanish at every point. For KT Theorems 5.5 and 5.6 and the molecular conjecture, the statement over an arbitrary infinite field appears to be new. For body-bar frameworks there is a precedent. In his proof of Tay's theorem, Whiteley chooses a point of an irreducible variety off the zeros of finitely many polynomials, and he states this step over any infinite field [Whi88, Proposition 6 and Lemma 7].

We follow Katoh and Tanigawa's proof of Theorem 5.5, an induction on $|V|$ over minimal k-dof-graphs of every deficiency k at once (Theorem 21.56). A graph with at least three vertices falls into one of four cases (KT p. 671). If G is not 2-edge-connected, the realization comes from KT Lemma 6.1. We call the other three cases Cases I, II and III, after KT §§6.2–6.4. In Case I, G has a proper rigid subgraph (Definition 20.17). In Cases II and III, G is 2-edge-connected and has no proper rigid subgraph, with $k > 0$ in Case II and $k = 0$ in Case III.

The first five subsections set up the frameworks and derive the main results from the three cases. Section 23.1 builds panel-hinge frameworks from the normals of their panels, in homogeneous coordinates. The hinge of an edge is the meet of the panels of its ends. Section 23.2 defines two operations on frameworks, and Section 23.3 proves

the rank formula. Section 23.4 proves KT Lemma 6.1 and the base of the induction, states KT Theorem 5.5 and gives its induction. It then derives KT Theorem 5.6 and the molecular conjecture for simple graphs, and Section 23.5 proves the results for multigraphs. The remaining subsections prove the three cases in order: Case I in Section 23.6, Case II in Section 23.9, and Case III in Section 23.10. Between Cases I and II, Section 23.7 treats the rank at generic normals, and Section 23.8 counts rows and motions. The realization lemmas of all three cases use results of the second.

23.1. **Panel-hinge frameworks.** As recalled in Section 17.2, Katoh and Tanigawa call a body-hinge framework a panel-hinge framework if the hinges at each body lie in a common hyperplane, the panel of the body [KT11, p. 648]. Their realizations of simple graphs are built from the panels: each body v has a panel $\Pi(v)$, and the hinge of an edge uv is the intersection $\Pi(u) \cap \Pi(v)$ [KT11, §5.1]. We build panel-hinge frameworks in the same way, in homogeneous coordinates. A panel is given by a normal vector in K^{d+1}. The hinge of uv is recorded by the meet of the two panels (Definition 22.7), which we compute from their normals. The hinges at a body then lie in its panel by construction, and the extensor of a hinge is nonzero exactly when the normals at its ends are linearly independent.

This subsection defines these frameworks (Definition 23.5). It then gives a condition on the normals under which every hinge between two distinct bodies has a nonzero extensor (Definition 23.9). Katoh and Tanigawa's nonparallel panels meet a stronger condition (Definition 23.10). Normals on the moment curve satisfy both, the second for $d \geq 2$ (Lemma 23.12).

Definition 23.1 (Panel support extensor)**.** Let $n_1, n_2 \in K^{d+1}$. The *panel support extensor* of n_1 and n_2 is

$$C(n_1, n_2) \;=\; *(n_1 \vee n_2) \;\in\; \bigwedge^{d-1} K^{d+1},$$

the image of the join $n_1 \vee n_2$ (Definition 18.4) under the complement isomorphism $*$ (Definition 22.5).

Formalization note 23.2. As in Section 19 and Section 22, the formalization writes $d = k+1$, so that normals lie in K^{k+2} and $C(n_1, n_2)$ lies in $\bigwedge^{k} K^{k+2}$. Several pinned theorems of this chapter also take the dimension that determines D in the deficiency $\mathrm{def}(\tilde{G})$ as a separate parameter, with a hypothesis that forces it to equal $k+1$.

The extensor $C(n_1, n_2)$ lies in the screw space $\bigwedge^{d-1} K^{d+1}$ of Definition 19.3, and $C(n_2, n_1) = -C(n_1, n_2)$. It is nonzero if and only if n_1 and n_2 are linearly independent, since $*$ is an isomorphism and $n_1 \vee n_2 \neq 0$ exactly in that case (the remark after Definition 18.3). If they are, then $C(n_1, n_2)$ is the meet of the hyperplanes $n_1^{\perp}$ and $n_2^{\perp}$ of K^{d+1} (the remarks after Definition 22.7). Their intersection has dimension $d-1$, and $C(n_1, n_2)$ is a nonzero extensor of it.

Lemma 23.3 (The panel support extensor as a join)**.** *Let $d \geq 2$, and let $n_1, n_2 \in K^{d+1}$ be linearly independent. Then there are $x_1, \ldots, x_{d-1} \in n_1^{\perp} \cap n_2^{\perp}$ with*

$$C(n_1, n_2) \;=\; x_1 \vee \cdots \vee x_{d-1}.$$

Proof. The subspace $n_1^\perp \cap n_2^\perp$ has dimension $d-1$, as n_1 and n_2 are linearly independent. Let $q_1, \dots, q_{d-1}$ be a basis of it. By Lemma 22.9, $q_1 \vee \cdots \vee q_{d-1} = c\, C(n_1, n_2)$ for some $c \in K$. The vectors q_i are linearly independent, so their join is nonzero (the remark after Definition 18.3), and $c \neq 0$. Take $x_1 = c^{-1} q_1$ and $x_i = q_i$ for $2 \leq i \leq d-1$. □

As $C(n_1, n_2) \neq 0$, the vectors $x_1, \dots, x_{d-1}$ of the lemma are linearly independent, so they form a basis of $n_1^\perp \cap n_2^\perp$.

We reduce the linear independence of panel support extensors to that of joins. The proofs of Lemma 23.61 and Lemma 23.62 use this reduction.

Lemma 23.4 (Independence of panel support extensors)**.** *Let $a_i, b_i \in K^{d+1}$ for $1 \leq i \leq m$. Then the extensors $C(a_i, b_i)$ with $1 \leq i \leq m$ are linearly independent if and only if the joins $a_i \vee b_i$ are linearly independent in $\bigwedge^2 K^{d+1}$.*

Proof. By definition, $C(a_i, b_i) = *(a_i \vee b_i)$ (Definition 23.1), and the complement isomorphism $*$ (Definition 22.5) is a linear isomorphism. A linear isomorphism preserves and reflects linear independence. □

The panel support extensor describes the hinge of two panels. As in Section 22 for $d = 3$, the hyperplane $\{x \in K^d : a \cdot x + a_0 = 0\}$ has the normal $n = (a, a_0) \in K^{d+1}$, and a point x lies on it exactly when $\bar{x} \cdot n = 0$, where $\bar{x} = (x, 1)$ (Definition 18.1). Katoh and Tanigawa's panel $\{x : x \cdot c(v) + 1 = 0\}$ has the normal $(c(v), 1)$ [KT11, §5.1]. Suppose that the hyperplanes with normals n_1 and n_2 meet in a $(d-2)$-dimensional affine subspace H, spanned by affinely independent points $p_1, \dots, p_{d-1}$. The two hyperplanes are distinct, so n_1 and n_2 are linearly independent. The vectors $\bar{p}_1, \dots, \bar{p}_{d-1}$ are linearly independent (Lemma 18.2) and orthogonal to n_1 and n_2. Let $C(H) = \bar{p}_1 \vee \cdots \vee \bar{p}_{d-1}$ be the extensor of H (Definition 18.8). It is a nonzero multiple of $C(n_1, n_2)$ (`extensor_join_proportional_complementIso_meet`), so the two give the same hinge constraint.

A panel-hinge framework takes the normals as its data.

Definition 23.5 (Panel-hinge framework)**.** A *panel-hinge framework* (G, n) is a multigraph $G = (V, E)$ together with a vector $n_v \in K^{d+1}$ for each body v, the normal of its panel $\Pi(v)$. Its *body-hinge interpretation* is the body-hinge framework (G, p) (Definition 19.3) whose hinge at an edge e with ends u and v has the supporting extensor

$$C(p(e)) \;=\; C(n_u, n_v)$$

(Definition 23.1).

Since $C(n_v, n_u) = -C(n_u, n_v)$, the hinge constraint at e does not depend on the order of its ends. We identify a panel-hinge framework with its body-hinge interpretation, and write $R(G, p)$ and $Z(G, p)$ for its rigidity matrix and its space of infinitesimal motions (Definition 19.7). The hinge at an edge e of a body-hinge framework is *nondegenerate* if $C(p(e)) \neq 0$. In a panel-hinge framework this means that the normals at the ends of e are linearly independent. For Katoh and Tanigawa a hinge is a $(d-2)$-dimensional affine subspace, and its extensor is nonzero (the remark after Definition 19.3). Here a hinge may be degenerate. If $C(p(e)) = 0$, the hinge constraint at e is $S(u) = S(v)$, and it welds the two bodies together.

Formalization note 23.6. In the formalization a multigraph records which bodies an edge joins as a relation. A panel-hinge framework therefore also records, for every edge label e, an ordered pair of bodies, and the hinge at e is the panel support extensor of their normals. We call this record the *endpoint selector*. The definition asks that it assign each edge of G its two ends, in either order, so the hinge at an edge with ends u and v is $C(n_u, n_v)$ or $C(n_v, n_u)$, as in Definition 23.5. A label outside $E(G)$ also gets a pair, on which the definition places no condition and which does not affect $R(G, p)$. The endpoint choice of G (Formalization note 21.11) is such a selector. The definition does not ask for nondegenerate hinges: a result that needs them assumes them. The molecular conjecture (Theorem 23.56) assumes them at the edges of G, on both of its sides.

A nondegenerate hinge at a body v lies in the panel of v, since its extensor $C(n_v, n_w)$ is an extensor of $n_v^\perp \cap n_w^\perp$ (Lemma 23.3). So the body-hinge interpretation of a panel-hinge framework meets Katoh and Tanigawa's condition [KT11, p. 648], and we use their name for the body-hinge frameworks that arise in this way.

Definition 23.7 (Hinge-coplanar body-hinge framework)**.** A body-hinge framework is *hinge-coplanar* if some panel-hinge framework on its multigraph has, at every edge, the hinge of the framework.

Formalization note 23.8. The formal predicate inherits the endpoint selector of Formalization note 23.6. It holds for a body-hinge framework whose hinge at each edge is the panel support extensor of the normals of its two ends. It asks nothing of the hinge at a label outside $E(G)$.

Katoh and Tanigawa's condition asks only that the hinges at each body lie in a common hyperplane. It also admits two bodies with the same panel, joined by any $(d-2)$-dimensional hinge in that panel. The meet of two equal panels is zero, so under Definition 23.7 the hinge between two bodies with the same panel is degenerate. The containment form of Definition 23.63, later in this chapter, admits such a hinge.

A condition on the normals alone makes every hinge between two distinct bodies nondegenerate.

Definition 23.9 (Panel normals in general position)**.** The normals of a panel-hinge framework (G, n) are in *general position* if n_u and n_v are linearly independent for any two distinct bodies u and v.

Katoh and Tanigawa ask more of the panels of a simple graph. Two panels are nonparallel if they meet in a $(d-2)$-dimensional affine subspace, and a panel-hinge framework on a simple graph is nonparallel if the panels of any two distinct bodies are [KT11, §5.1, p. 668]. The notion is essentially Jackson and Jordán's, who call such a framework non-degenerate [JJ10, §7]. As above, the panel with normal $n = (a, a_0)$, where $a \in K^d$, is the hyperplane $\{x \in K^d : a \cdot x + a_0 = 0\}$. We write $\hat{n} = a$ for the first d coordinates of n. Let $d \geq 2$. Two such hyperplanes meet in a $(d-2)$-dimensional affine subspace if and only if the vectors $\hat{n}$ of their normals are linearly independent. Otherwise the two are parallel or equal, or one of them is empty or all of K^d.

Definition 23.10 (Nonparallel panels)**.** The panels of a panel-hinge framework (G, n) are *nonparallel* if $\hat{n}_u$ and $\hat{n}_v$ are linearly independent for any two distinct bodies u and v, where $\hat{n}_v \in K^d$ consists of the first d coordinates of n_v.

For $d \geq 2$ this is Katoh and Tanigawa's condition, asked of any two distinct bodies. The definition reads only the normals, so G need not be simple. The panel $\{x : x \cdot c(v) + 1 = 0\}$ of Katoh and Tanigawa has the normal $(c(v), 1)$, so their panels are nonparallel exactly when the vectors $c(v)$ are pairwise linearly independent. They build their nonparallel realizations from vectors $c(v)$ with this property [KT11, p. 668]. For $d = 1$ the vectors $\hat{n}_v$ lie in K^1, so no two panels are nonparallel.

Nonparallel panels have normals in general position (Definition 23.9), since a linear relation between n_u and n_v restricts to one between $\hat{n}_u$ and $\hat{n}_v$. The converse fails for two parallel panels, whose normals are linearly independent and which meet only at infinity. In the proof of KT Theorem 5.6, Katoh and Tanigawa make any two panels meet by a projective transformation, under which infinitesimal rigidity is invariant [KT11, p. 670].

Lemma 23.11 (General position gives nondegenerate hinges)**.** *Let the normals of a panel-hinge framework (G, n) be in general position. Then every edge of G with two distinct ends has a nondegenerate hinge.*

Proof. Let e have ends $u \neq v$. Then n_u and n_v are linearly independent, so $C(p(e)) = C(n_u, n_v) \neq 0$ (the remark after Definition 23.1). □

Lemma 23.12 (Moment-curve normals are in general position and nonparallel)**.** *Let $(t_v)_{v \in V}$ be distinct elements of K, and let $n_v = (1, t_v, t_v^2, \dots, t_v^d)$ be the point of the moment curve at t_v. Then the normals n_v are in general position, and if $d \geq 2$, their panels are nonparallel. In particular, if K is infinite and V is finite, then there are normals in general position, and for $d \geq 2$ there are nonparallel panels.*

Proof. Let $u \neq v$. The first two coordinates of n_u and n_v form the matrix with rows $(1, t_u)$ and $(1, t_v)$, whose determinant $t_v - t_u$ is not zero. So n_u and n_v are linearly independent. If $d \geq 2$, then $\hat{n}_v = (1, t_v, \dots, t_v^{d-1})$ is the point of the moment curve in K^d at t_v, and the same argument shows that $\hat{n}_u$ and $\hat{n}_v$ are linearly independent. An infinite field has $|V|$ distinct elements for every finite V. □

23.2. **Two operations on frameworks.** This subsection defines two operations, each of which changes one part of a framework and keeps the rest. The first replaces the multigraph of a framework and keeps its hinges (Definition 23.13). The second replaces the normal of one body of a panel-hinge framework (Definition 23.15). If that body lies on no edge, the space of infinitesimal motions does not change (Lemma 23.16). Case II uses this to choose the panel of the vertex of degree two that it puts back (Lemma 23.135, in Section 23.9). Passing to a subgraph can only enlarge the space of motions (Lemma 23.18). The lower bound on the rank in the rank formula (KT Proposition 2.3) and the proofs of KT Theorem 5.6 use this to compare a realization of G with one of a spanning subgraph.

Definition 23.13 (Framework on a new graph)**.** Let (G, p) be a body-hinge framework, and let G' be a multigraph on the same set of bodies. The framework (G', p)

has the multigraph G' and the hinge $p(e)$ at each edge e of G'. If G' is a subgraph of G, then (G', p) is the restriction of (G, p) to G'. The same operation on a panel-hinge framework, for a subgraph G' of G, replaces its multigraph by G' and keeps its normals and the hinge at each edge of G'.

Formalization note 23.14. In the formalization a body-hinge framework has a hinge at every edge label, not only at the edges of its multigraph, so (G', p) is defined for every multigraph G' on its bodies. On a panel-hinge framework the operation keeps the endpoint selector (Formalization note 23.6), and with it the hinge at every label. It takes G' with a proof that G' is a subgraph of G, so that the selector assigns each edge of G' its two ends. The third pinned theorem states that the operation commutes with the body-hinge interpretation.

If G' is a subgraph of G, each edge of G' has the same ends in G' as in G. So the restriction of a panel-hinge framework (G, n) to G' is the panel-hinge framework (G', n) (Definition 23.5).

Definition 23.15 (Replacing one body's panel normal)**.** Let (G, n) be a panel-hinge framework, let v be a body, and let $m \in K^{d+1}$. Replacing the normal of v by m gives the panel-hinge framework (G, n') with $n'_v = m$ and $n'_w = n_w$ for every body $w \neq v$.

Lemma 23.16 (Replacing the normal of a body that lies on no edge)**.** *Let (G, n) be a panel-hinge framework, let v be a body, and let (G, n') be obtained by replacing the normal of v (Definition 23.15). Then each edge of G whose ends differ from v has the same hinge in both frameworks. If v lies on no edge of G, then the two frameworks have the same space of infinitesimal motions, and the same subspace* pinned(w) *for every body w (Lemma 19.20).*

Proof. The hinge at an edge with ends u and w is $C(n_u, n_w)$ (Definition 23.5). It depends only on the normals at u and w, so it does not change when $u, w \neq v$. If v lies on no edge of G, then no hinge changes, and so no hinge constraint changes (Definition 19.3). So the space of infinitesimal motions does not change, nor does its subspace pinned(w) of the motions S with $S(w) = 0$. □

Formalization note 23.17. The pinned theorems read the ends of an edge from the endpoint selector (Formalization note 23.6). The first assumes that neither body of the pair that the selector assigns to the edge is v, and the other two assume this for every edge of G.

Lemma 23.18 (Deleting edges enlarges the space of motions)**.** *Let (G, p) be a body-hinge framework, and let G' be a subgraph of G. Then $Z(G, p) \subseteq Z(G', p)$. In particular,* $\dim Z(G, p) \leq \dim Z(G', p)$*, that is,* $\operatorname{rank} R(G', p) \leq \operatorname{rank} R(G, p)$*.*

Proof. Each edge of G' is an edge of G with the same ends, and it has the same hinge in (G', p) as in (G, p) (Definition 23.13). So every infinitesimal motion of (G, p) satisfies the hinge constraint at every edge of G' (Definition 19.3), and $Z(G, p) \subseteq Z(G', p)$. The rank form follows, as $\operatorname{rank} R(G, p) = D|V| - \dim Z(G, p)$ (Definition 19.7). □

23.3. **The rank formula from a bound on the motions.** Katoh and Tanigawa's Proposition 2.3, which they credit to Jackson and Jordán [JJ10], relates the rank of the rigidity matrix of a generic body-hinge framework (G, p) on a multigraph G (Definition 19.16) to the deficiency $\mathrm{def}(\tilde{G})$ and to the rank of the matroid $M(\tilde{G})$ [KT11, p. 655]. It has two halves of different kinds. The matroidal half is KT eq. (2.4), $\operatorname{rank} M(\tilde{G}) = D(|V| - 1) - \mathrm{def}(\tilde{G})$, which Theorem 20.12 proves. The analytic half is Jackson and Jordán's rank formula [JJ10, Theorem 6.1]

$$\operatorname{rank} R(G, p) \;=\; D(|V| - 1) - \mathrm{def}(\tilde{G})$$

at a generic body-hinge framework (G, p).

The analytic half is in turn two inequalities of different kinds. The upper bound on the rank holds at every body-hinge framework whose hinges are nondegenerate (Lemma 20.15). The lower bound on the rank, in the form

$$\dim Z(G, p) \;\leq\; D + \mathrm{def}(\tilde{G}),$$

is where genericity enters. A generic framework has the largest rank over all realizations of G, so it meets the bound as soon as one realization does. Jackson and Jordán build such a realization from D edge-disjoint forests in $\tilde{G}$ with $D(|V| - 1) - \mathrm{def}(\tilde{G})$ edges in all [JJ10, Theorem 6.1]. Katoh and Tanigawa's Theorem 5.6, which they derive from their Theorem 5.5, gives one for every multigraph [KT11, p. 670] (Theorem 23.66 below).

We state the analytic half in a conditional form, with the bound as its hypothesis (Proposition 23.19), and prove the bound where we use the formula. The proofs of KT Theorem 5.6 below prove it for the realizations that they construct (Theorem 23.48, Theorem 23.66). A later chapter proves the formula, and so the bound, at every generic hinge-point assignment of a multigraph (Theorem 28.40), which is Jackson and Jordán's Theorem 6.1 for the frameworks whose hinges are spanned by points. Katoh and Tanigawa's unconditional statement of Proposition 2.3, the three-way equivalence at those assignments, is Proposition 28.42 there.

Proposition 23.19 (The rank formula from a bound on the motions)**.** *Let $G = (V, E)$ be a multigraph with $V \neq \emptyset$, and let (G, p) be a body-hinge framework with $C(p(e)) \neq 0$ at every edge e of G whose two ends differ. Suppose that*

$$\dim Z(G, p) \;\leq\; D + \mathrm{def}(\tilde{G}).$$

Then

$$\operatorname{rank} R(G, p) \;=\; D(|V| - 1) - \mathrm{def}(\tilde{G}).$$

Proof. Let $k = \mathrm{def}(\tilde{G})$. Since $\operatorname{rank} R(G, p) = D|V| - \dim Z(G, p)$ (Definition 19.7), the claim is that $\dim Z(G, p) = D + k$. The hypothesis is the inequality $\dim Z(G, p) \leq D + k$. The reverse inequality holds as the hinges at the edges whose ends differ are nondegenerate (Lemma 20.15). □

Formalization note 23.20. The pinned theorem asks for a nonzero supporting extensor only at the edges of G whose two ends differ; the hinge at a loop, which constrains no motion, and at a label outside $E(G)$ is unconstrained. Its screw assignments are defined on the whole body set, so its conclusion $\dim Z(G, p) = D + \mathrm{def}(\tilde{G})$ is the rank

formula when G spans the body set. The formula at every generic framework in the sense of Definition 19.16, the hinge-point frameworks of maximum rank, is Corollary 28.41. The theorems that apply the pinned theorem, such as Theorem 23.48, take G spanning and prove the bound for the realization they construct. The formula at every generic realization is formalized, with genericity as the independence of rows, for generic normals of a simple spanning graph (Theorem 28.12) and for generic hinge points of a spanning multigraph (Theorem 28.40), where KT Proposition 2.3 itself, the three-way equivalence, is also formalized (Proposition 28.42).

At deficiency zero the formula reads $\operatorname{rank} R(G,p) = D(|V|-1)$. So a generic body-hinge framework is infinitesimally rigid if and only if $\operatorname{def}(\tilde{G}) = 0$, by Lemma 19.18. By KT Proposition 2.2 (Lemma 20.14), this holds if and only if $\tilde{G}$ contains D edge-disjoint spanning trees. This is Tay and Whiteley's theorem in the form of KT Proposition 1.1 (Section 17.2), for the rigidity matrix of Section 19.

23.4. **KT Theorems 5.5 and 5.6 and the molecular conjecture.** This subsection proves KT Theorem 5.5 and derives from it KT Theorem 5.6 and the molecular conjecture for simple graphs. Katoh and Tanigawa prove Theorem 5.5 by induction on the number of vertices. Its induction hypothesis, KT eq. (6.1), is the theorem for graphs with fewer vertices. We first define two forms of the realization that it asks for (Definition 23.21, Definition 23.24), and prove KT Lemma 6.1, the case of the induction in which the graph is not 2-edge-connected, for each form. We then prove the base of the induction, the graphs with one or two vertices (Lemma 23.36), state the theorem and give the induction (Theorem 23.38). KT Theorem 5.6 and the molecular conjecture follow (Theorem 23.50, Theorem 23.56). The subsection ends with panel normals for the rigid cycles of KT Lemma 5.4, which Case III uses.

Definition 23.21 (Realization hypothesis, rank form; KT eq. (6.1))**.** Let $G = (V, E)$ be a multigraph with $V \neq \emptyset$. A panel-hinge realization (G, p) of G (Definition 23.5) is *at the target rank* if

$$\operatorname{rank} R(G,p) \;=\; D(|V|-1) - \operatorname{def}(\tilde{G}).$$

A *generic full-rank realization* of G is a panel-hinge realization at the target rank whose normals are in general position (Definition 23.9). A *nonparallel full-rank realization* of G is a panel-hinge realization at the target rank whose panels are non-parallel (Definition 23.10). For a minimal k-dof-graph, $\operatorname{def}(\tilde{G}) = k$ (Definition 20.8), and the target rank is the rank $D(|V|-1) - k$ of KT eq. (6.1).

Formalization note 23.22. In the formalization every framework has the same set of bodies, and a graph need not have all of them as vertices. So the hypothesis is stated for the rank (Remark 23.23). The first pinned predicate is the rank form at deficiency zero: a panel-hinge framework on G whose infinitesimal motions are all constant on $V(G)$, that is, of rank $D(|V(G)|-1)$. It asks nothing of the normals, and of the endpoint selector only what every panel-hinge framework meets (Formalization note 23.6). No main result uses this predicate. The second pinned predicate is the generic full-rank realization: a panel-hinge framework on G at the target rank whose normals are in general position. Of the endpoint selector it too asks only what every

panel-hinge framework meets, that it assign each edge of G its two ends. The third pinned predicate is the nonparallel full-rank realization, with the same condition on the endpoint selector. Like general position, it asks the panels of any two distinct bodies to be nonparallel, bodies that are not vertices of G included, so for $d = 1$ it holds only when there is at most one body. It is the realization that KT Theorem 5.5 gives for a simple graph, and it is a generic full-rank realization, as nonparallel panels have normals in general position. For $d \geq 2$ and over an infinite field, Lemma 23.26 obtains one from a generic full-rank realization of a multigraph without loops.

Katoh and Tanigawa call a nonparallel realization generic if the coordinates of the vectors $c(v)$ that define its panels are algebraically independent over $\mathbb{Q}$ [KT11, p. 668]. A generic realization has the largest rank of all nonparallel realizations of its graph, and their proof uses this for the realizations given by the induction hypothesis (KT footnote 6, p. 684). A generic full-rank realization asks only that the normals be in general position. Where the formal proof needs a realization given by the induction hypothesis to attain a rank, it chooses the normals off the zero set of the finitely many polynomials that this step tests, a product of determinants and rank polynomials (Lemma 23.120) fixed before the choice.

Remark 23.23 (Why the target is a rank equality)**.** The rows of $R(G, p)$ involve only the bodies of G. Let W be a set of bodies that contains $V(G)$, and regard (G, p) as a framework on W in which the bodies outside $V(G)$ lie on no edge. Its rank does not change, and each body of $W \setminus V(G)$ adds D dimensions to its space of motions. So the target rank does not depend on W, and it is equivalent to

$$\dim Z(G, p) \;=\; D + \operatorname{def}(\tilde{G}) + D\,|W \setminus V(G)|.$$

For $W = V(G)$ this is $\dim Z(G, p) = D + \operatorname{def}(\tilde{G})$, as in the proof of Proposition 23.19. When $\operatorname{def}(\tilde{G}) = 0$, the target rank says that every infinitesimal motion is constant on $V(G)$.

A panel-hinge realization at the target rank may have degenerate hinges. If all the normals are equal, then every hinge is degenerate and welds its two bodies together, so every connected 0-dof-graph has a panel-hinge realization at the target rank. Katoh and Tanigawa's hinges are nondegenerate (the remark after Definition 19.3). A generic full-rank realization has a nondegenerate hinge at every edge with two distinct ends (Lemma 23.11). The second form of the hypothesis asks for nondegenerate hinges directly, each in the panels of the two ends of its edge.

Definition 23.24 (Realization hypothesis, nondegenerate-hinge form; KT eq. (6.1))**.** Let $G = (V, E)$ be a multigraph with $V \neq \emptyset$. An extensor *lies in the panel* with normal $n \in K^{d+1}$ if it is $x_1 \vee \cdots \vee x_{d-1}$ for some $x_1, \ldots, x_{d-1} \in n^{\perp}$. A *nondegenerate-hinge panel realization* of G is a body-hinge framework (G, p) at the target rank $D(|V| - 1) - \operatorname{def}(\tilde{G})$ (Definition 23.21), together with a nonzero normal $n_v \in K^{d+1}$ for each body v, such that the hinge $C(p(e))$ at each edge e with ends u and v is nonzero and lies in the panels with normals n_u and n_v.

This is the form of the hypothesis in Katoh and Tanigawa's proof. Their panel-hinge frameworks are body-hinge frameworks whose hinges at each body lie in a

common hyperplane, the panel of the body [KT11, p. 668], and their hinges are nondegenerate. Here the hinge at an edge need not be the meet of the two panels. When the two panels coincide, it can be any nonzero extensor in their common panel, as in KT Lemma 5.3 (p. 669). A generic full-rank realization gives a nondegenerate-hinge one.

Lemma 23.25 (A generic full-rank realization has nondegenerate hinges)**.** *Let $d \geq 2$, and let $G = (V, E)$ be a multigraph with $|V| \geq 2$. If G has a generic full-rank realization (Definition 23.21), then G has a nondegenerate-hinge panel realization (Definition 23.24).*

Proof. Let (G, p) be a generic full-rank realization with normals n_v. It is at the target rank. Each normal n_v is nonzero, since $|V| \geq 2$ gives a body $w \neq v$, and n_v and n_w are linearly independent. Let e be an edge with ends $u \neq v$. Its hinge $C(n_u, n_v)$ is nonzero (Lemma 23.11). It is the meet of the hyperplanes $n_u^\perp$ and $n_v^\perp$, so it is $x_1 \vee \cdots \vee x_{d-1}$ for a basis $x_1, \ldots, x_{d-1}$ of $n_u^\perp \cap n_v^\perp$ (Lemma 23.3 and the remark after it). So it lies in both panels. A loop at a body u gives no row of $R(G, p)$, since its constraint involves only $S(u) - S(u) = 0$. So we may replace its hinge by $C(n_u, n_w)$ for a body $w \neq u$, without changing the rank. As n_u and n_w are linearly independent, this hinge is nonzero (the remark after Definition 23.1), and as above it lies in the panel of u. □

A generic full-rank realization also gives a nonparallel one of the same rank. Katoh and Tanigawa make a realization nonparallel by KT Lemma 5.2 [KT11, p. 669], rotating one of two equal panels slightly and arguing by continuity over $\mathbb{R}$ (the passage before Lemma 23.78). For a realization whose normals are in general position, the next lemma chooses new normals off the zeros of two polynomials instead.

Lemma 23.26 (A generic full-rank realization gives a nonparallel one)**.** *Let K be an infinite field, let $d \geq 2$, and let $G = (V, E)$ be a multigraph without loops, with $V \neq \emptyset$. If G has a generic full-rank realization, then G has a nonparallel full-rank realization (Definition 23.21).*

Proof. Let (G, q_0) be a generic full-rank realization. As G has no loops, every edge has two distinct ends, so every hinge of (G, q_0) is nondegenerate (Lemma 23.11). By Lemma 23.120 there is a polynomial Q_1 in the normals with $Q_1(q_0) \neq 0$ such that $\operatorname{rank} R(G, q) \geq \operatorname{rank} R(G, q_0)$ at every q with $Q_1(q) \neq 0$. Let Q_2 be the product, over the ordered pairs (u, v) of distinct bodies, of the 2×2 minor of the first two coordinates of the normals n_u and n_v. As $d \geq 2$, these are coordinates of $\hat{n}_u$ and $\hat{n}_v$, so the panels of (G, q) are nonparallel wherever $Q_2(q) \neq 0$. At moment-curve normals with distinct parameters t_v, each factor is $t_v - t_u \neq 0$ (Lemma 23.12), and an infinite field has distinct parameters for the finitely many bodies, so $Q_2 \neq 0$. As K is infinite, the nonzero polynomial $Q_1 Q_2$ does not vanish at some q.

The panels of (G, q) are nonparallel, so its normals are in general position (Definition 23.10) and its hinges are nondegenerate (Lemma 23.11). Its rank is at least that of (G, q_0), the target rank, and at most the target rank (Lemma 20.16). So (G, q) is a nonparallel full-rank realization. □

Katoh and Tanigawa's Lemma 6.1 [KT11, p. 672] gives the realization in case (i) of their induction (Theorem 21.56), in which G is not 2-edge-connected. Its hypothesis is KT eq. (6.1): every minimal k'-dof-graph H with fewer vertices than G has a panel-hinge realization (H, p_H) with $\operatorname{rank} R(H, p_H) = D(|V(H)| - 1) - k'$, nonparallel if H is simple. We state KT Lemma 6.1 for each of the two forms of this hypothesis.

Lemma 23.27 (Nondegenerate-hinge realization at a cut of at most one edge; KT Lemma 6.1)**.** *Let $D \geq 2$, and let G be a minimal k-dof-graph that is not 2-edge-connected. Suppose that for every k', every minimal k'-dof-graph with at least one and fewer than $|V(G)|$ vertices has a nondegenerate-hinge panel realization. Then G has a nondegenerate-hinge panel realization.*

Proof. Part (2) of Lemma 20.27 (KT Lemmas 3.3 and 3.6) applies, as G is not 2-edge-connected. It gives a set V_1 of vertices with $\emptyset \neq V_1 \neq V(G)$ and the following properties. Let $V_2 = V(G) \setminus V_1$. At most one edge joins V_1 to V_2; let C be the set of such edges. Then $G[V_1]$ and $G[V_2]$ are minimal k_1- and k_2-dof-graphs, with $k = k_1 + k_2 + D - (D-1)\,|C|$. Both have fewer vertices than G. So by hypothesis they have nondegenerate-hinge panel realizations $(G[V_1], p_1)$ and $(G[V_2], p_2)$, with $\operatorname{rank} R(G[V_i], p_i) = D(|V_i| - 1) - k_i$.

Let p have the normals of p_i at the bodies of V_i, and the hinges of p_i at the edges of $G[V_i]$. Suppose that $C = \{e\}$, where e joins $u \in V_1$ to $v \in V_2$. The subspace $n_u^\perp \cap n_v^\perp$ of K^{d+1} has dimension at least $d - 1$, so it contains linearly independent vectors $x_1, \dots, x_{d-1}$. Let p have the hinge $x_1 \vee \cdots \vee x_{d-1} \neq 0$ (Definition 18.3) at e. Katoh and Tanigawa take the intersection of the two panels as this hinge, after they move p_2 so that the panels are not parallel. Here any nonzero extensor of vectors in both panels serves. So (G, p) has nonzero normals, and each of its hinges is nonzero and lies in the panels of the ends of its edge.

The hinges of p at the edges of $G[V_i]$ are those of p_i, so $\operatorname{rank} R(G[V_i], p) = D(|V_i| - 1) - k_i$. By Lemma 19.24,

$$\operatorname{rank} R(G,p) \;\geq\; D(|V_1|-1) - k_1 + (D-1)\,|C| + D(|V_2|-1) - k_2 \;=\; D(|V(G)|-1) - k,$$

by the formula for k and as $|V(G)| = |V_1| + |V_2|$. As every hinge of p is nonzero and $\operatorname{def}(\tilde{G}) = k$, Lemma 20.16 gives $\operatorname{rank} R(G, p) \leq D(|V(G)| - 1) - k$. □

Lemma 23.28 (Generic full-rank realization at a cut of at most one edge; KT Lemma 6.1)**.** *Let $D \geq 2$, and let G be a simple minimal k-dof-graph that is not 2-edge-connected. Suppose that for every k', every simple minimal k'-dof-graph with at least one and fewer than $|V(G)|$ vertices has a generic full-rank realization. Then G has a generic full-rank realization.*

Proof. Let V_1, V_2, C, k_1 and k_2 be as in the proof of Lemma 23.27. As G is simple, so are $G[V_1]$ and $G[V_2]$. By hypothesis they have generic full-rank realizations, with normals q_1 and q_2. For $i = 1, 2$, there is a polynomial Q_i in the normals with $Q_i(q_i) \neq 0$ such that $\operatorname{rank} R(G[V_i], q) \geq D(|V_i| - 1) - k_i$ at all normals q with $Q_i(q) \neq 0$ (Lemma 23.120). Let Q_0 be the product, over the pairs of distinct bodies u and v, of the 2×2 minor of the first two coordinates of n_u and n_v. Where Q_0 does not vanish, the normals are pairwise linearly independent. At normals on the moment curve each of these minors is a nonzero Vandermonde determinant (Lemma 23.12),

so Q_0 is not the zero polynomial. Hence the product $Q_1Q_2Q_0$ of three nonzero polynomials is not the zero polynomial. Since K is infinite, there are normals q_0 at which it does not vanish. This is the argument by which nonempty Zariski-open sets of normals meet in the proof of Lemma 23.112. Katoh and Tanigawa instead apply an isometry to one of the two realizations, which keeps its rank, so that no panel of the one is parallel to a panel of the other (KT p. 672).

Let (G, q_0) be the panel-hinge framework on G with the normals q_0, whose hinge at each edge is the intersection of the panels of its ends (Definition 23.5). Its normals are pairwise linearly independent, so its hinges are nonzero (Lemma 23.11), and $\operatorname{rank} R(G[V_i], q_0) \geq D(|V_i| - 1) - k_i$ for $i = 1, 2$. As in the proof of Lemma 23.27, Lemma 19.24 and Lemma 20.16 give $\operatorname{rank} R(G, q_0) = D(|V(G)| - 1) - k$. □

Formalization note 23.29. The endpoint selector of a panel-hinge framework records the two ends of each edge, in some order (Formalization note 23.6). The framework (G, q_0) takes them from the endpoint choice of G (Formalization note 21.11). The polynomial Q_i is formed at the framework on $G[V_i]$ with the normals q_i and these ends. Its rank is that of the given realization, as reversing the ends of an edge does not change the space of infinitesimal motions.

We now prove the base of the induction in the proof of Theorem 23.38, below. Let G be a minimal k-dof-graph with one or two vertices. By Lemma 20.39, either (i) G has no edge, or (ii) G has two vertices and one edge, which joins them, or (iii) G has two vertices and two edges, each of which joins them. In cases (i) and (ii), every realization of G with nondegenerate hinges is at the target rank. Katoh and Tanigawa note this, and they treat case (iii) by their Lemma 5.3 [KT11, p. 671]. In case (iii), two hinges with linearly independent extensors make the two bodies rigid.

Lemma 23.30 (Two independent hinges make two bodies rigid)**.** *Let (G, p) be a body-hinge framework, and let e_1 and e_2 be edges of G that both join the bodies u and v, with $C(p(e_1))$ and $C(p(e_2))$ linearly independent. Then every infinitesimal motion S of (G, p) has $S(u) = S(v)$. In particular, if u and v are the only bodies, then (G, p) is infinitesimally rigid.*

Proof. An infinitesimal motion satisfies the hinge constraints at e_1 and e_2, so this is KT Lemma 5.3 (Lemma 19.21). □

Katoh and Tanigawa realize the double edge with two coincident panels and two distinct hinges in them (KT Lemma 5.3, p. 669). Distinct hinges have linearly independent extensors (the remark after Lemma 19.21). In homogeneous coordinates, we need two linearly independent extensors in one panel.

Lemma 23.31 (Two independent extensors in one panel)**.** *Let $n \in K^4$. Then there are vectors p_0, p_1, q_0, q_1 in $n^\perp = \{x \in K^4 : x \cdot n = 0\}$ such that the extensors $p_0 \vee p_1$ and $q_0 \vee q_1$ are linearly independent in $\bigwedge^2 K^4$.*

Proof. The subspace $n^\perp$ has dimension at least 3, so it contains linearly independent vectors a, b and c. Extend them by a vector w to a basis of K^4, and take $p_0 = a$, $p_1 = q_0 = b$ and $q_1 = c$. Then $a \vee b$ and $b \vee c$ are two of the extensors of KT Lemma 2.1 for this basis: the first omits c and w, and the second omits a and w. So they are linearly independent (Lemma 18.9 and the remark after it). □

The same argument works in K^{d+1} for every $d \geq 2$. The subspace $n^\perp$ contains linearly independent vectors $v_1, \ldots, v_d$, and the extensors $v_1 \vee \cdots \vee v_{d-1}$ and $v_2 \vee \cdots \vee v_d$ are linearly independent, by KT Lemma 2.1 for a basis $v_1, \ldots, v_d, w$ of K^{d+1} (`exists_linearIndependent_extensor_pair_perp_grade`). The realizations below use this form.

We now realize the three cases. In cases (i) and (ii) we also give a generic full-rank realization. The graph of case (iii) is not simple.

Lemma 23.32 (The base of KT Theorem 5.5: no edge)**.** *Let K be an infinite field, and let G be a minimal k-dof-graph with one or two vertices and no edge. Then G has a nondegenerate-hinge panel realization (Definition 23.24) and a generic full-rank realization (Definition 23.21), both of rank* 0*.*

Proof. By Lemma 20.39, $\mathrm{def}(\tilde{G}) = k = D(|V| - 1)$, so the target rank $D(|V| - 1) - \mathrm{def}(\tilde{G})$ is 0. As G has no edge, $R(G, p)$ has no rows, and every realization of G is at the target rank. For the first realization, give every body the same nonzero normal; there is no hinge. For the second, take normals in general position, which exist as K is infinite (Lemma 23.12). □

Lemma 23.33 (The base of KT Theorem 5.5: one edge)**.** *Let $d \geq 2$, let K be an infinite field, and let G be a minimal* 1*-dof-graph with two vertices and one edge, which joins them. Then G has a nondegenerate-hinge panel realization (Definition 23.24) and a generic full-rank realization (Definition 23.21), both of rank $D - 1$.*

Proof. Let the edge e of G join x and y. As $\mathrm{def}(\tilde{G}) = 1$, the target rank is $D(2 - 1) - 1 = D - 1$. The rows of $R(G, p)$ are the $D - 1$ rows of e, and they are linearly independent when the hinge at e is nondegenerate (Lemma 19.9). So each realization needs only a nondegenerate hinge at e. For the first, give x and y the same nonzero normal n, and give e the extensor $v_1 \vee \cdots \vee v_{d-1}$ of the paragraph after Lemma 23.31. It lies in the panel with normal n, and it is nonzero, as $v_1, \ldots, v_{d-1}$ are linearly independent. For the second, take normals in general position (Lemma 23.12). Then the hinge $C(n_x, n_y)$ at e is nonzero (Lemma 23.11). □

Lemma 23.34 (The base of KT Theorem 5.5: the double edge; KT Lemma 5.3)**.** *Let $d \geq 2$, and let G be a minimal* 0*-dof-graph with two vertices and two edges, each of which joins them. Then G has a nondegenerate-hinge panel realization (Definition 23.24), of rank D.*

Proof. We follow Katoh and Tanigawa's proof of their Lemma 5.3 (KT p. 670). Let the edges e and f of G join x and y. Give x and y the same nonzero normal n, so that their panels coincide. Give e and f two linearly independent extensors that lie in the panel with normal n (the form of Lemma 23.31 for K^{d+1}). Both are nonzero, and each lies in the panels of x and y. By Lemma 23.30, the framework is infinitesimally rigid. So its space of motions is the D-dimensional space of trivial motions (Lemma 19.18), and its rank is $2D - D = D$ (Definition 19.7). As $\mathrm{def}(\tilde{G}) = 0$, this is the target rank. □

Formalization note 23.35. The pinned theorems prove more than the statements of Lemma 23.32, Lemma 23.33 and this lemma, which are the cases that Lemma 23.36

needs. Those of Lemma 23.32 hold for every $d \geq 1$ and any number of vertices, and the first over every field. The first pinned theorem of Lemma 23.33 holds over every field, and the second for every $d \geq 1$. The pinned theorem of this lemma holds over every field, and it assumes $\operatorname{def}(\tilde{G}) = 0$ in place of minimality. The pinned theorems of the last two lemmas take the dimension that determines D in $\operatorname{def}(\tilde{G})$ as a separate parameter (the note after Definition 23.1), with no hypothesis that it equals d.

Lemma 23.36 (The base of KT Theorem 5.5)**.** *Let $d \geq 2$, let K be an infinite field, and let G be a minimal k-dof-graph with one or two vertices. Then G has a nondegenerate-hinge panel realization (Definition 23.24). If G is simple, then G also has a generic full-rank realization (Definition 23.21).*

Proof. By Lemma 20.39, G is a graph of Lemma 23.32, Lemma 23.33 or Lemma 23.34, and that lemma gives the realizations. In the last case the two edges of G join the same two vertices, so G is not simple. □

Formalization note 23.37. The second pinned theorem states that a 0-dof-graph with two vertices is not simple. The first uses it in case (iii).

Katoh and Tanigawa's Theorem 5.5 [KT11, p. 670] gives a minimal k-dof-graph G with $|V| \geq 2$ a panel-hinge realization of rank $D(|V| - 1) - k$, nonparallel if G is simple. We state it as two claims: a nondegenerate-hinge panel realization of every such G, and a nonparallel full-rank realization of a simple one. For simple G the second gives the first. Nonparallel panels have normals in general position (the remark after Definition 23.10), so a nonparallel full-rank realization is a generic one, and a generic one gives a nondegenerate-hinge one (Lemma 23.25).

The second claim needs simplicity. Let G have two vertices u and v, and two edges e and f that join them. Then G is a 0-dof-graph (Lemma 20.36), and its target rank is D. In a panel-hinge realization of G with n_u and n_v linearly independent, the hinges at e and f are both $\pm C(n_u, n_v)$. So the two edges impose the same $D - 1$ conditions, and the rank is $D-1$. Katoh and Tanigawa realize G at rank D with two coincident panels and two distinct hinges in them (KT Lemma 5.3, p. 669). They define nonparallel realizations only for simple graphs (KT §5.1, p. 668), and their induction hypothesis, KT eq. (6.1), asks for a nonparallel realization only of a simple graph.

The proof is by induction on $|V|$, as in KT §6. It uses KT Lemma 6.1, the base cases above, and Cases I, II and III, which Section 23.6, Section 23.9 and Section 23.10 prove.

Theorem 23.38 (Realization at the target rank; KT Theorem 5.5)**.** *Let $d \geq 2$, let K be an infinite field, and let $G = (V, E)$ be a minimal k-dof-graph with $|V| \geq 2$. Then G has a nondegenerate-hinge panel realization (Definition 23.24), of rank $D(|V| - 1) - k$. If G is simple, then G also has a nonparallel full-rank realization (Definition 23.21). In particular, a minimal 0-dof-graph with at least two vertices has an infinitesimally rigid nondegenerate-hinge panel realization.*

Proof. For a multigraph H, let $P(H)$ say that H has a nondegenerate-hinge panel realization, and also a generic full-rank realization if H is simple. We prove $P(G)$ for every minimal k-dof-graph G with at least one vertex, for all k at once, by induction

on $|V(G)|$ (Theorem 21.56). The induction hypothesis is KT eq. (6.1): $P(H)$ holds for every minimal k'-dof-graph H with at least one and fewer than $|V(G)|$ vertices, for every k'.

The base. If G has one or two vertices, then $P(G)$ holds (Lemma 23.36). A minimal k-dof-graph with two vertices has no edge, one edge, or two parallel edges, and the last case is KT Lemma 5.3 (Lemma 23.30).

The four cases. Let $|V(G)| \geq 3$. We follow Katoh and Tanigawa's four cases (KT p. 671). If G is not 2-edge-connected, then KT Lemma 6.1 gives $P(G)$. Lemma 23.27 gives the nondegenerate-hinge realization, and Lemma 23.28 gives the generic one when G is simple. If G has a proper rigid subgraph, then Case I gives $P(G)$ (KT §6.2; Lemma 23.109). It uses KT Lemma 6.2 when G is not simple, KT Lemma 6.3 when G is simple and has a proper rigid subgraph H with $G/E(H)$ simple (Lemma 23.100), and KT Lemma 6.5 otherwise.

In the remaining two cases G is 2-edge-connected and has no proper rigid subgraph, so G is simple (Lemma 21.57). If $k > 0$, then Case II gives a generic full-rank realization of G (KT Lemma 6.8; Lemma 23.135). If $k = 0$, then Case III does (KT Lemmas 6.10 and 6.13; Lemma 23.208). In both cases this realization is also a nondegenerate-hinge one (Lemma 23.25), so $P(G)$ holds.

The nonparallel realization. If G is simple, it has no loops, and $V(G) \neq \emptyset$, so the generic full-rank realization that $P(G)$ gives for it yields a nonparallel full-rank realization (Lemma 23.26). This is the realization that the theorem states.

The deficiencies. The induction hypothesis is used at deficiencies other than k, even when $k = 0$. KT Lemma 6.1 applies it to the two sides of a cut of at most one edge, at deficiencies smaller than k. Case I applies it to a proper rigid subgraph H, a minimal 0-dof-graph, and to the contraction $G/E(H)$, a minimal k-dof-graph. In Case II, splitting off a vertex of degree two gives a minimal $(k-1)$-dof-graph. Case III applies it to a minimal k'-dof-graph with $k' \leq D - 2$ (KT Claim 6.11, eq. (6.22); Lemma 23.140). For general d, Case III also distinguishes whether G is a cycle on at most d vertices or has a chain of length d (KT Lemma 6.13; Lemma 21.68). The case $d = 3$, $k = 0$ of the theorem is Theorem 23.45. □

Formalization note 23.39. The induction gives each edge it adds a label that the graph does not use, so its formal statement, the second pinned theorem, assumes that every minimal k'-dof-graph, for every k', has an edge label outside its edge set (Remark 23.40). The first pinned theorem assumes nothing of the labels. It relabels the edges of G into a set with enough labels that every minimal k'-dof-graph leaves one unused. The relabelled graph is a minimal k-dof-graph with the bodies and the deficiency of G, simple if G is (Lemma 23.43, Lemma 23.42). The first pinned theorem applies the second to it, and reads its realizations on the labels of G: they are realizations of G with the same normals, rows and motions (Lemma 23.44).

Remark 23.40 (The fresh-edge-label convention)**.** The formalization takes the bodies of every framework from one fixed set W, and the edges of every multigraph from one fixed set L of labels (Formalization note 21.11). Katoh and Tanigawa's argument has no analogue of L. An operation of the induction that adds an edge, such as splitting off a vertex, needs a label that the graph does not use. Every minimal k-dof-graph G has one when L is large enough. Indeed, every base of $M(\tilde{G})$ meets

the copies of each edge of G (Definition 20.8), and it has $D(|V(G)|-1)-\mathrm{def}(\tilde{G})$ elements (Theorem 20.12). So G has at most that many edges (`edgeSet_ncard_add_deficiency_le_of_isMinimalKDof`). As $\mathrm{def}(\tilde{G}) \geq 0$, a label outside $E(G)$ exists whenever $D(|W|-1) < |L|$ (`freshEdgeSupply_of_card_lt`). Minimality is needed here: a graph whose edges are loops at one body can use every label.

Some pinned theorems assume one of three forms of this condition, and the statements of this chapter omit it. The supply itself, a label outside $E(H)$ for every minimal k'-dof-graph H and every k', is assumed by the second theorems pinned on Theorem 23.38, Theorem 23.48, Theorem 23.54 and Theorem 23.66, and by Lemma 30.39. The bound $D(|W|-1) < |L|$ is assumed by the second theorem pinned on Theorem 23.50 and the fourth pinned on Theorem 23.66. A label outside $E(G)$ for the graph G of the lemma alone is assumed by Lemma 23.135 and Lemma 23.208.

The first theorems pinned on Theorem 23.38, Theorem 23.48, Theorem 23.54, Theorem 23.50 and Theorem 23.66, and the third pinned on Theorem 23.66, assume nothing of the labels, and so do the results that apply them, such as the third theorem pinned on Theorem 23.54. Each of the six relabels the edges of G into a larger set of labels that has a supply (Definition 23.41, Lemma 23.43), applies the form that assumes one, and reads the realization on the labels of G (Lemma 23.44). Remark 23.59 gives an instance with no label at all.

Definition 23.41 (Relabelling the edges)**.** Let ι be an injective map from the set of edge labels of G into a larger set of labels. The graph G *with its edges relabelled by* ι has the bodies of G, and an edge $\iota(e)$ joining u and v for every edge e of G joining u and v.

Lemma 23.42 (Relabelling the edges keeps simplicity and every deficiency)**.** *Let G' be G with its edges relabelled by an injective map ι (Definition 23.41). Then G' is simple if G is, and $\mathrm{def}_d(G') = \mathrm{def}_d(G)$ for every d.*

Proof. The bodies and the pairs of adjacent bodies are unchanged (`vertexSet_embedEdges`, `embedEdges_isLink`): the edge $\iota(e)$ joins the bodies that e joins. As ι is injective, G' has a loop, or two edges joining one pair of bodies, only if G does. A partition crosses exactly the images of the edges it crossed before (`edgeSet_embedEdges`), so it crosses as many edges and keeps its deficiency, and the largest of these values is unchanged. □

Lemma 23.43 (Relabelling the edges keeps minimality, and the padded label set has a supply)**.** *Let G be a minimal k-dof-graph and G' be G with its edges relabelled injectively (Definition 23.41). Then G' is a minimal k-dof-graph. Moreover, for $D \geq 1$, every minimal k'-dof-graph on the bodies of G with labels in $L \sqcup \{1, \dots, D(|W|-1)+1\}$ has a label outside its edge set, for every k'.*

Proof. Deleting an edge of a minimal k-dof-graph raises the deficiency: a base of $M(\tilde{G})$ of the deleted graph is independent in $M(\tilde{G})$ (Lemma 20.24), it has the size of a base of $M(\tilde{G})$ if the deficiency stayed the same (KT eq. (2.4), Theorem 20.12), so it is a base, and it meets the copies of the deleted edge by minimality, a contradiction. The same count shows that a k-dof-graph whose every edge deletion raises

the deficiency is minimal. Relabelling commutes with deleting an edge and keeps every deficiency (Lemma 23.42), so G' is minimal. The padded label set has more than $D(|W|-1)$ labels, and Remark 23.40 applies. □

Lemma 23.44 (Relabelling the edges of a framework keeps its rows and motions)**.** *Let G' be G with its edges relabelled by an injective map ι (Definition 23.41), let (G', p') be a body-hinge framework, and let (G, p) be the body-hinge framework with $p(e) = p'(\iota(e))$ for every edge label e of G. Then the rigidity matrices of (G, p) and (G', p') have the same rows, and the two frameworks have the same infinitesimal motions. If (G', p') is a panel-hinge framework with normals n_v, then (G, p) is the panel-hinge framework on G with the normals n_v.*

Proof. The edges of G' are the edges $\iota(e)$, for the edges e of G, and $\iota(e)$ joins the bodies that e joins. So the rows of the rigidity matrix at $\iota(e)$ and at e are the rows of one hinge between the same two bodies (Definition 19.7), and a motion satisfies the hinge constraint of $\iota(e)$ exactly when it satisfies that of e (Definition 19.3). In a panel-hinge framework the hinge at an edge is the meet of the panels of its ends, and the ends of $\iota(e)$ are those of e. □

We record the case of the molecular conjecture, $d = 3$ and $k = 0$.

Theorem 23.45 (KT Theorem 5.5 at $d = 3$ and deficiency zero)**.** *Let $d = 3$, so that $D = 6$, let K be an infinite field, and let $G = (V, E)$ be a minimal 0-dof-graph with $|V| \geq 2$. Then G has a nondegenerate-hinge panel realization (Definition 23.24), of rank $6(|V|-1)$. If G is simple, then G also has a nonparallel full-rank realization (Definition 23.21).*

Proof. This is Theorem 23.38 at $d = 3$ and $k = 0$. □

At deficiency zero the realization is infinitesimally rigid. We record this in the form $\dim Z(G, p) = D + \operatorname{def}(\tilde{G})$, as in the proof of Proposition 23.19.

Corollary 23.46 (The rank formula at deficiency zero from KT Theorem 5.5 $(d = 3)$)**.** *Let $d = 3$, so that $D = 6$, let K be an infinite field, and let $G = (V, E)$ be a simple minimal 0-dof-graph with $|V| \geq 2$ that spans the set of bodies. Then G has a panel-hinge realization (G, p) with nonparallel panels (Definition 23.10), a nondegenerate hinge at every edge of G, and*

$$\dim Z(G, p) \;=\; D \;=\; D + \operatorname{def}(\tilde{G})$$

(Definition 23.21).

Proof. By Theorem 23.45, G has a nonparallel full-rank realization (G, p), of rank $D(|V|-1)$. So $\dim Z(G, p) = D|V| - \operatorname{rank} R(G, p) = D$ (Definition 19.7). As $\operatorname{def}(\tilde{G}) = 0$, this is the rank formula at (G, p), in the form of Proposition 23.19. □

Formalization note 23.47. The pinned theorem assumes that G spans the set of bodies, and it takes the realization from Theorem 23.45, whose nonparallel conjunct needs G simple and whose rank needs two bodies. It derives the formula from the pinned theorem of Proposition 23.19, which asks for a nonzero hinge at every link of the realization between two distinct bodies. The realization has one at every link: its

selector assigns each edge its two ends (Formalization note 23.6), which are distinct as G has no loops, and their normals are linearly independent, as nonparallel normals are in general position. Simplicity is needed: at the double edge on two bodies, a minimal 0-dof-graph, the two hinges of the meet model coincide, so no realization has nonparallel panels there. The conclusion records the nonparallel panels and the nonzero hinge at every edge of G. A framework whose hinges are all degenerate also meets the rank formula, since a 0-dof-graph is connected, which is why the hinges are part of the statement.

Katoh and Tanigawa's Theorem 5.6 removes the minimality from KT Theorem 5.5: every multigraph G has a panel-hinge realization of rank $D(|V|-1)-\mathrm{def}(\tilde{G})$ [KT11, p. 670]. Its realization meets the bound on the motions that Proposition 23.19 assumes, so a generic body-hinge framework of G meets it too (the opening of Section 23.3). At deficiency zero the realization is infinitesimally rigid, as Corollary 23.46 records for a simple spanning minimal 0-dof-graph at $d = 3$. Katoh and Tanigawa obtain the molecular conjecture from the theorem in this way (KT p. 670; Theorem 23.56).

We prove the theorem for simple graphs. In the panel-hinge frameworks of Definition 23.5, a multigraph need not have a realization at the target rank with nondegenerate hinges: the two-vertex double edge has none (the paragraph before Theorem 23.38). The multigraph form, with hinges that need only lie in the panels, is Theorem 23.66 in Section 23.5. We state the theorem first with nonparallel panels and nondegenerate hinges, as Katoh and Tanigawa's are, for graphs with at least two vertices. We then derive the form with nonparallel panels for every simple graph with at least one vertex (Theorem 23.50), which the molecular conjecture (Theorem 23.56) uses.

Theorem 23.48 (KT Theorem 5.6 with nondegenerate hinges)**.** *Let $d \geq 2$, let K be an infinite field, and let $G = (V, E)$ be a simple graph with $|V| \geq 2$. Then G has a nonparallel full-rank realization (Definition 23.21), a panel-hinge realization at the target rank whose panels are nonparallel (Definition 23.10), all of whose hinges are nondegenerate.*

Proof. We follow Katoh and Tanigawa's proof (KT p. 670). Let $c = \mathrm{def}(\tilde{G})$.

The realization. By Lemma 20.34, G has a spanning subgraph G' that is a minimal c-dof-graph. It is simple, as a subgraph of G, and it has the $|V| \geq 2$ vertices of G. By KT Theorem 5.5 (Theorem 23.38), G' has a nonparallel full-rank realization (G', q) (Definition 23.21), with $\operatorname{rank} R(G', q) = D(|V| - 1) - c$. Let (G, p) be the panel-hinge framework on G with the same normals n_v. Its panels are those of (G', q), so they are nonparallel, and at each edge of G' it has the hinge of (G', q). Katoh and Tanigawa first apply a projective transformation so that the panels of any two bodies meet in a $(d-2)$-dimensional affine subspace. Infinitesimal rigidity is invariant under such a transformation. They then give each edge of G outside G' a hinge in the intersection of the panels of its ends. Here no transformation is needed, as the panels are nonparallel already. So the normals are in general position (Definition 23.10), and each edge of G has two distinct ends, as G is simple. So the hinge $C(n_u, n_v)$ of every edge uv of G is nonzero (Lemma 23.11).

The rank. The restriction of (G,p) to the edges of G' is (G',q), so $Z(G,p) \subseteq Z(G',q)$ (Lemma 23.18). Hence

$$\operatorname{rank} R(G,p) \;\geq\; \operatorname{rank} R(G',q) \;=\; D(|V|-1)-c.$$

The hinges of (G,p) are nondegenerate, so the upper bound on the rank of Lemma 20.15, which Proposition 23.19 uses, holds at (G,p), and $\operatorname{rank} R(G,p) \leq D(|V|-1)-c$. So (G,p) is at the target rank. □

Formalization note 23.49. Both pinned theorems assume that G spans the set W of bodies, and state the target rank as $\dim Z(G,p) = D + \operatorname{def}(\tilde{G})$ (Remark 23.23). The first assumes nothing of the edge labels: it relabels the edges, as the first pinned theorem of Theorem 23.38 does, and applies the second, which assumes the supply of fresh labels (Remark 23.40). The selector of the framework that they construct assigns each edge of G its two ends, as every panel-hinge framework's does (Formalization note 23.6). They ask for a nonzero hinge at every edge label, not only at the edges of G (Remark 23.58). A label outside $E(G)$ gets the hinge of two distinct bodies, which is why they assume $|V| \geq 2$.

Without the statement on the hinges, the theorem holds for every simple graph with at least one vertex.

Theorem 23.50 (Realization of a simple graph at the target rank; KT Theorem 5.6)**.** *Let $d \geq 2$, let K be an infinite field, and let $G = (V,E)$ be a simple graph with $V \neq \emptyset$. Then G has a nonparallel full-rank realization (G,p) (Definition 23.21): its panels are nonparallel (Definition 23.10), and*

$$\operatorname{rank} R(G,p) \;=\; D(|V|-1) - \operatorname{def}(\tilde{G}).$$

In particular, every hinge of (G,p) is nondegenerate (Lemma 23.11).

Proof. Let $c = \operatorname{def}(\tilde{G})$. If $|V| = 1$, then G has no edge, as it is simple, and $c = 0$. So every realization of G has rank $0 = D(|V|-1) - c$, and its panels are nonparallel, as it has no two distinct bodies. If $|V| \geq 2$, then the realization of Theorem 23.48 is a nonparallel full-rank realization. In both cases the normals are in general position (Definition 23.10), and each edge of G has two distinct ends, as G is simple, so every hinge is nondegenerate (Lemma 23.11). □

Formalization note 23.51. Both pinned theorems assume that G spans the set W of bodies, and they state the target rank as $\dim Z(G,p) = D + \operatorname{def}(\tilde{G})$ (Remark 23.23). Their selector assigns each edge of G its two ends, as every panel-hinge framework's does (Formalization note 23.6). For $|V| \geq 2$, the framework of the second is that of the second pinned theorem of Theorem 23.48. Neither states that the hinges are nondegenerate. At the edges of G this follows from the nonparallel panels and the selector, as in the proof, and neither states anything of the hinge at a label outside $E(G)$. The second pinned theorem assumes $D(|W|-1) < |L|$ (Remark 23.40) and proves the theorem by the argument above. The first assumes nothing of the edge labels: it relabels the edges into a set of more than $D(|W|-1)$ labels, applies the second there, and reads the realization on the labels of G (Lemma 23.44).

We record the case $d = 3$.

Theorem 23.52 (KT Theorem 5.6 at $d = 3$)**.** *Let $d = 3$, so that $D = 6$, let K be an infinite field, and let $G = (V, E)$ be a simple graph with $V \neq \emptyset$. Then G has a panel-hinge realization at the target rank $6(|V| - 1) - \operatorname{def}(\tilde{G})$ (Definition 23.21) whose panels are nonparallel (Definition 23.10).*

Proof. This is Theorem 23.50 at $d = 3$. □

Formalization note 23.53. As for Theorem 23.50, the pinned theorem assumes that G spans the set of bodies, and its selector assigns each edge of G its two ends. Its panels are nonparallel, so every hinge at an edge of G is nondegenerate, as for Theorem 23.50. It assumes nothing of the edge labels.

The realization in the proof of Theorem 23.48 is a generic full-rank realization (Definition 23.21). So the generic conclusion of KT Theorem 5.5 holds for every simple graph with at least two vertices, minimal or not. A later chapter uses its case $d = 2$ for a graph whose bodies are part of a larger set (Section 30.3). There the target rank is no longer equivalent to $\dim Z(G, p) = D + \operatorname{def}(\tilde{G})$, since each body outside the graph adds D dimensions of motions (Remark 23.23). So the proof compares ranks rather than motions.

Theorem 23.54 (Generic full-rank realizations of simple graphs; KT Theorem 5.6)**.** *Let $d \geq 2$, let K be an infinite field, and let $G = (V, E)$ be a simple graph with $|V| \geq 2$. Then G has a generic full-rank realization (G, p) (Definition 23.21): its normals are in general position (Definition 23.9), and*

$$\operatorname{rank} R(G, p) \;=\; D(|V| - 1) - \operatorname{def}(\tilde{G}).$$

In particular, every hinge of (G, p) is nondegenerate (Lemma 23.11).

Proof. The realization is the one built in the proof of Theorem 23.48. Let $c = \operatorname{def}(\tilde{G})$, and let G' be a spanning subgraph of G that is a minimal c-dof-graph (Lemma 20.34). Let (G', q) be a generic full-rank realization of G' (Theorem 23.38), and let (G, p) be the panel-hinge framework on G with the normals of (G', q). These normals are in general position, so every hinge of (G, p) is nondegenerate (Lemma 23.11).

The restriction of (G, p) to the edges of G' has the hinges of (G', q). So it has the same motions as (G', q), and hence the same row space (Lemma 19.25). Its rows are among those of $R(G, p)$, so $\operatorname{rank} R(G, p) \geq \operatorname{rank} R(G', q) = D(|V| - 1) - c$. Conversely, the hinges of (G, p) are nondegenerate, so $\operatorname{rank} R(G, p) \leq D(|V| - 1) - c$, also when the screw assignments are defined on a larger set of bodies (Lemma 20.16). □

Formalization note 23.55. The pinned theorems do not assume that G has every body as a vertex. The first two state their conclusion as a panel-hinge framework on G with normals in general position, a nonzero hinge at every edge label (Remark 23.58), and the target rank. Their selector assigns each edge of G its two ends, as every panel-hinge framework's does (Formalization note 23.6). The third restates the first as a generic full-rank realization (Formalization note 23.22). The second assumes the supply of fresh labels (Remark 23.40). The first and third assume nothing of the edge labels: the first relabels the edges, as the first pinned theorem of Theorem 23.38 does, and applies the second.

We now prove the molecular conjecture of Tay and Whiteley [TW84], recalled in Section 17.2, for simple graphs. Katoh and Tanigawa state and prove it for multigraphs (KT Conjecture 1.2, p. 648), and Formalization note 23.57 explains the restriction. On both sides the hinges are nondegenerate, as theirs are. A degenerate hinge welds the two bodies of its edge together. So without this condition the body-hinge side would hold exactly when G is connected, and the theorem would fail at the single edge, whose deficiency is 1. On the panel side the condition follows from the nonparallel panels, as G is simple (Lemma 23.11).

Theorem 23.56 (The molecular conjecture for simple graphs; KT Conjecture 1.2)**.** *Let $d \geq 2$, let K be an infinite field, and let $G = (V, E)$ be a simple graph with $V \neq \emptyset$. Then G has an infinitesimally rigid body-hinge realization (Definition 19.15) with nondegenerate hinges if and only if G has an infinitesimally rigid panel-hinge realization (Definition 23.5) with nondegenerate hinges and nonparallel panels (Definition 23.10).*

Proof. We follow Katoh and Tanigawa (KT p. 670). Suppose first that G has an infinitesimally rigid panel-hinge realization with nondegenerate hinges (and nonparallel panels, which this direction does not use). Its body-hinge interpretation (Definition 23.5) is an infinitesimally rigid body-hinge realization of G with the same hinges.

Conversely, let (G, p) be an infinitesimally rigid body-hinge realization of G with nondegenerate hinges. As it is infinitesimally rigid, $\dim Z(G, p) = D$ (Lemma 19.18). As its hinges are nondegenerate, $\dim Z(G, p) \geq D + \operatorname{def}(\tilde{G})$ (Lemma 20.15). Hence $\operatorname{def}(\tilde{G}) \leq 0$, and so $\operatorname{def}(\tilde{G}) = 0$ (Definition 20.3). By KT Theorem 5.6 (Theorem 23.50), G has a panel-hinge realization with nonparallel panels at the target rank $D(|V|-1)$ (Definition 23.21). It is infinitesimally rigid (Lemma 19.18). Its normals are in general position (Definition 23.10), and each edge of G has two distinct ends, as G is simple, so its hinges are nondegenerate (Lemma 23.11). □

Formalization note 23.57. Katoh and Tanigawa state Conjecture 1.2 for multigraphs (KT p. 648), and their panel-hinge frameworks ask only that the hinges at each body lie in a common hyperplane. The panel-hinge frameworks of Definition 23.5 take the hinge of each edge to be the meet of the panels of its ends, and for them the multigraph form of the theorem is false. The two-vertex double edge has an infinitesimally rigid body-hinge realization with nondegenerate hinges (Lemma 23.30). In a panel-hinge realization of it with nondegenerate hinges, both hinges are $\pm C(n_u, n_v)$, and the rank is $D - 1$ (the paragraph before Theorem 23.38). Katoh and Tanigawa realize it with two coincident panels and two distinct hinges in them (KT Lemma 5.3, p. 669). The multigraph form holds for hinges that need only lie in the panels (Theorem 23.70).

The pinned theorem assumes that G spans the set W of bodies, and nothing of the edge labels (Remark 23.40). On its panel side the endpoint selector assigns each edge of G its two ends, as every panel-hinge framework's does (Formalization note 23.6). Both of its sides ask for a nonzero hinge at each edge of G and at no other label (Remark 23.58).

By the proof, the two sides of Theorem 23.56 hold if and only if $\mathrm{def}(\tilde{G}) = 0$, that is, if and only if $\tilde{G}$ contains D edge-disjoint spanning trees (Lemma 20.14). Katoh and Tanigawa state their result in this form [KT11, p. 647].

Remark 23.58 (Nondegenerate hinges at every edge label)**.** Let W and L be the sets of bodies and edge labels of Remark 23.40. The pinned theorems of Theorem 23.48 and Theorem 23.54 ask for a nonzero hinge $C(p(e))$ at every label $e \in L$, not only at the edges of G. So does the predicate of Definition 23.63. The pinned theorem of Theorem 23.56 asks for one only at the edges of G, as does the body-hinge side of Theorem 23.70; its panel side asks for one at every label, as the predicate does. A label outside $E(G)$ imposes no constraint. A body-hinge framework can give it any nonzero extensor. A panel-hinge framework gives it the meet of the panels of two bodies (Formalization note 23.6), which is nonzero only when their normals are linearly independent.

On one body every hinge of a panel-hinge framework is $C(n_v, n_v) = 0$, so such a framework has a nonzero hinge at every label only when L is empty. A simple graph on one vertex has no edge, so both sides of the pinned theorem of Theorem 23.56 hold there, as a single body is infinitesimally rigid whatever its hinges.

Remark 23.59 (An instance of the hypotheses)**.** The hypotheses of the pinned theorem of Theorem 23.56 can all be met. Take $K = \mathbb{R}$, $d = 2$, one body and no edge label, and let G have the one body as its vertex and no edge. Then G is simple and spans the bodies. As the set of labels is empty, the instance also shows that the theorem assumes nothing of the labels (Remark 23.40). The formalization applies the theorem to this instance (`molecular_conjecture_witness`): a single body is infinitesimally rigid, so G has an infinitesimally rigid body-hinge realization, and by the theorem an infinitesimally rigid panel-hinge realization, whose panels are nonparallel, as there is one body.

Case III uses KT Lemma 5.4: a cycle on m bodies, for $3 \le m \le D$, has an infinitesimally rigid nonparallel panel-hinge realization (KT p. 670; Lemma 23.80 for $m \le d + 1$, in Section 23.6). Katoh and Tanigawa refer to Crapo and Whiteley [CW82] and to Whiteley [Whi99] for the computation in dimension 3, and note that it works in every dimension. We first treat the triangle, $m = 3$. The three-vertex realization of Case III (Lemma 23.198) uses the next two lemmas.

Lemma 23.60 (A triangle with independent hinges is rigid)**.** *Let (G, p) be a body-hinge framework, let u, v and w be bodies, and let e_1, e_2 and e_3 be edges of G that join u to v, v to w, and w to u. Suppose that $C(p(e_1))$, $C(p(e_2))$ and $C(p(e_3))$ are linearly independent. Then every infinitesimal motion S of (G, p) has $S(u) = S(v) = S(w)$.*

Proof. We argue as in the proof of KT Lemma 5.3 (Lemma 19.21). By the hinge constraints (Definition 19.3), there are scalars λ_1, λ_2 and λ_3 with $S(u) - S(v) = \lambda_1 C(p(e_1))$, $S(v) - S(w) = \lambda_2 C(p(e_2))$ and $S(w) - S(u) = \lambda_3 C(p(e_3))$. The three differences sum to zero, so $\lambda_1 C(p(e_1)) + \lambda_2 C(p(e_2)) + \lambda_3 C(p(e_3)) = 0$. By the linear independence, $\lambda_1 = \lambda_2 = \lambda_3 = 0$, so $S(u) = S(v) = S(w)$. □

Lemma 23.61 (Normals for a triangle with independent hinges)**.** *Let $d \geq 2$. Then there are $n_0, n_1, n_2 \in K^{d+1}$ such that the extensors $C(n_0, n_1)$, $C(n_1, n_2)$ and $C(n_2, n_0)$ are linearly independent. In particular, $n_0 \vee n_1$, $n_1 \vee n_2$ and $n_2 \vee n_0$ are nonzero.*

Proof. Let $e_0, \dots, e_d$ be the standard basis of K^{d+1}, and let $n_i = e_i$ for $i = 0, 1, 2$, which is possible as $d \geq 2$. The joins $e_a \vee e_b$ with $a < b$ are linearly independent (the remark after Definition 18.3). Up to sign, $e_0 \vee e_1$, $e_1 \vee e_2$ and $e_2 \vee e_0 = -e_0 \vee e_2$ are three distinct ones among them, so they are linearly independent, and in particular nonzero. The extensors $C(n_i, n_{i+1})$ are their images under the complement isomorphism $*$ (Definition 23.1). Since $*$ is an isomorphism, they are linearly independent too. Lemma 23.4 records this. □

For every m, a cycle on m bodies whose hinges have linearly independent extensors is infinitesimally rigid, by the argument of Lemma 23.60 (Lemma 23.77). The next lemma gives normals with such hinges for $m \leq d + 1$.

Lemma 23.62 (Normals for a cycle with independent hinges)**.** *Let $3 \leq m \leq d + 1$. Then there are $n_0, \dots, n_{m-1} \in K^{d+1}$ such that the m extensors $C(n_i, n_{i+1})$, with indices mod m, are linearly independent. In particular, each $n_i \vee n_{i+1}$ is nonzero.*

Proof. We argue as for Lemma 23.61, with $n_i = e_i$ for $0 \leq i \leq m - 1$; this uses $m \leq d + 1$. Up to sign, the joins $n_i \vee n_{i+1}$ are the $e_a \vee e_b$ with $a < b$ for the pairs $\{i, i + 1\}$ with $0 \leq i \leq m - 2$ and the pair $\{0, m - 1\}$, the edges of a cycle on $\{0, \dots, m - 1\}$. These m pairs are distinct, as $m \geq 3$. So the joins are linearly independent, and so are the extensors $C(n_i, n_{i+1}) = *(n_i \vee n_{i+1})$ (Lemma 23.4). □

23.5. **KT Theorem 5.6 and the molecular conjecture for multigraphs.** Katoh and Tanigawa state Theorem 5.6 and the molecular conjecture for multigraphs (KT p. 648, p. 670). Their panel-hinge frameworks ask only that the hinges at each body lie in a common hyperplane, the panel of the body (KT §5.1, p. 668). Two bodies may then share a panel, as in their realization of the double edge on two vertices (KT Lemma 5.3, p. 669). In the panel-hinge frameworks of Definition 23.5, a hinge is the meet of the two panels, and the double edge has no realization at its target rank with nondegenerate hinges (the paragraph before Theorem 23.38). So the multigraph form of the molecular conjecture fails for them (Formalization note 23.57). This subsection asks instead that each hinge lie in the panels of the ends of its edge (Definition 23.63). For such realizations both results hold for every multigraph (Theorem 23.66, Theorem 23.70). The proof is Katoh and Tanigawa's. It needs no simplicity, since KT Theorem 5.5 gives every minimal k-dof-graph a nondegenerate-hinge panel realization (Theorem 23.38).

The definition is that of a nondegenerate-hinge panel realization (Definition 23.24) without the condition on the rank.

Definition 23.63 (Hinge-coplanar panel realization; KT §5.1)**.** Let $G = (V, E)$ be a multigraph. A *hinge-coplanar panel realization* of G is a body-hinge framework (G, p) (Definition 19.3) together with a nonzero normal $n_v \in K^{d+1}$ for each body v, such that the hinge $C(p(e))$ at each edge e with ends u and v is nonzero and lies in the panels with normals n_u and n_v (Definition 23.24).

Formalization note 23.64. The pinned predicate asks for a nonzero hinge at every edge label, not only at the edges of G (Remark 23.58).

The hinges at a body v lie in the hyperplane $n_v^\perp$, the panel of v. So a hinge-coplanar panel realization is a panel-hinge framework in Katoh and Tanigawa's sense (KT p. 648), in homogeneous coordinates, with a panel chosen for every body. Let e be an edge with ends u and v. If n_u and n_v are linearly independent, then $n_u^\perp \cap n_v^\perp$ has dimension $d-1$. A nonzero extensor that lies in both panels is then a nonzero multiple of $C(n_u, n_v)$ (the remark after Definition 23.1). So the hinge at e imposes the same constraint as in the panel-hinge framework with normals n. If n_u and n_v are proportional, the two panels coincide, and their meet is zero. The hinge at e can then be any nonzero extensor in their common panel, as in KT Lemma 5.3.

Remark 23.65 (Loops and panels)**.** Katoh and Tanigawa's multigraphs may have parallel edges but no loops (KT §2.5, p. 654). The statements below admit loops. A loop e at a body v imposes the constraint $S(v) - S(v) \in \operatorname{span} C(p(e))$, which every screw assignment meets, so its rows of the rigidity matrix vanish. A hinge-coplanar panel realization gives it a nonzero extensor in the panel of v. Katoh and Tanigawa choose the panel $\Pi_{G,p}(v)$ of a body arbitrarily when it is not unique (KT §5.1, p. 668). A hinge-coplanar panel realization records a nonzero normal for every body, and such a normal can always be chosen. In homogeneous coordinates a panel or a hinge may lie in the hyperplane at infinity. Katoh and Tanigawa's panels and hinges are affine. Infinitesimal rigidity is invariant under projective transformations (KT p. 670).

We first prove KT Theorem 5.6.

Theorem 23.66 (Realization of a multigraph at the target rank; KT Theorem 5.6)**.** *Let $d \geq 2$, let K be an infinite field, and let $G = (V, E)$ be a multigraph, possibly with loops, with $V \neq \emptyset$. Then G has a hinge-coplanar panel realization (G, p) (Definition 23.63) at the target rank, that is, with*

$$\operatorname{rank} R(G,p) \;=\; D(|V|-1) - \operatorname{def}(\tilde{G}).$$

Proof. We follow Katoh and Tanigawa's proof (KT p. 670), as for Theorem 23.48. Let $c = \operatorname{def}(\tilde{G})$.

One body. Let $|V| = 1$. Every edge of G is a loop, and the only partition of V has one part, so $c = 0$. The rows of a loop vanish, so every realization of G has rank $0 = D(|V|-1) - c$. Give the body a nonzero normal n. The subspace $n^\perp$ contains $d-1$ linearly independent vectors $x_1, \dots, x_{d-1}$. Their join $x_1 \vee \cdots \vee x_{d-1}$ is nonzero (the remark after Definition 18.3), and it lies in the panel with normal n. Give every edge this hinge.

The realization. Let $|V| \geq 2$. By Lemma 20.34, G has a spanning subgraph G' that is a minimal c-dof-graph. By KT Theorem 5.5 (Theorem 23.38), G' has a nondegenerate-hinge panel realization (G', q), with normals n_v, whether or not G' is simple. Its rank is $D(|V|-1) - c$. We extend q to a realization (G, p) of G. At each edge of G' we keep the hinge of q. We give each other edge of G, with ends u and v, possibly equal, a nonzero extensor that lies in the panels with normals n_u and n_v. Such an extensor exists. The subspace $n_u^\perp \cap n_v^\perp$ of K^{d+1} is cut out by two

linear equations, so it contains $d-1$ linearly independent vectors, and their join is such an extensor (`exists_extensor_in_two_panels_grade`). With the normals n_v, (G,p) is a hinge-coplanar panel realization of G.

Katoh and Tanigawa first apply a projective transformation so that any two panels meet in a $(d-2)$-dimensional affine subspace. Infinitesimal rigidity is invariant under such a transformation. They then choose each new hinge in that intersection. In homogeneous coordinates the transformation is not needed, since any two panels contain a common nonzero extensor.

The rank. The restriction of (G,p) to G' (Definition 23.13) is (G',q). So $Z(G,p) \subseteq Z(G',q)$ (Lemma 23.18), and

$$\operatorname{rank} R(G,p) \;\geq\; \operatorname{rank} R(G',q) \;=\; D(|V|-1)-c.$$

The hinges of (G,p) are nondegenerate, so the upper bound on the rank of Lemma 20.15, which Proposition 23.19 uses, holds at (G,p), and $\operatorname{rank} R(G,p) \leq D(|V|-1)-c$. So (G,p) is at the target rank. □

Formalization note 23.67. The pinned theorems assume that G spans the set W of bodies, and they state the target rank as $\dim Z(G,p) = D + \operatorname{def}(\tilde{G})$ (Remark 23.23). The first and the third, at grade k and at dimension d, assume nothing of the edge labels: each relabels the edges, as the first pinned theorem of Theorem 23.38 does, and applies the theorem pinned after it. The second assumes the supply of fresh labels, and the fourth $D(|W|-1) < |L|$ (Remark 23.40).

We record the case $d = 3$.

Corollary 23.68 (KT Theorem 5.6 for multigraphs at $d = 3$)**.** *Let $d = 3$, so that $D = 6$, let K be an infinite field, and let $G = (V,E)$ be a multigraph, possibly with loops, with $V \neq \emptyset$. Then G has a hinge-coplanar panel realization at the target rank $6(|V|-1) - \operatorname{def}(\tilde{G})$ (Definition 23.21).*

Proof. This is Theorem 23.66 at $d = 3$. □

Formalization note 23.69. The pinned theorem assumes that G spans the set W of bodies, and nothing of the edge labels.

Katoh and Tanigawa prove the molecular conjecture of Tay and Whiteley [TW84] for multigraphs (KT Conjecture 1.2, p. 648). In the present form it follows from Theorem 23.66, as the simple case follows from Theorem 23.50.

Theorem 23.70 (The molecular conjecture for multigraphs; KT Conjecture 1.2)**.** *Let $d \geq 2$, let K be an infinite field, and let $G = (V,E)$ be a multigraph, possibly with loops, with $V \neq \emptyset$. Then G has an infinitesimally rigid body-hinge realization (Definition 19.15) with nondegenerate hinges at the edges of G if and only if G has an infinitesimally rigid hinge-coplanar panel realization (Definition 23.63).*

Proof. We argue as for Theorem 23.56. A hinge-coplanar panel realization of G is a body-hinge realization of G whose hinges are nondegenerate, at every label and so at the edges of G. So if G has an infinitesimally rigid one, then G has an infinitesimally rigid body-hinge realization with nondegenerate hinges at the edges.

Conversely, let (G, p) be an infinitesimally rigid body-hinge realization of G with nondegenerate hinges at the edges. As it is infinitesimally rigid, $\dim Z(G, p) = D$ (Lemma 19.18). As its hinges at the edges are nondegenerate, $\dim Z(G, p) \geq D + \operatorname{def}(\tilde{G})$ (Lemma 20.15). Hence $\operatorname{def}(\tilde{G}) \leq 0$, and so $\operatorname{def}(\tilde{G}) = 0$ (Definition 20.3). By KT Theorem 5.6 (Theorem 23.66), G has a hinge-coplanar panel realization at the target rank $D(|V| - 1)$. It is infinitesimally rigid (Lemma 19.18). □

Formalization note 23.71. The pinned theorem assumes that G spans the set W of bodies, and nothing of the edge labels (Remark 23.40). Its body-hinge side asks for a nonzero hinge only at the edges of G, as Theorem 23.56 does; its panel side asks for one at every edge label, as Definition 23.63 does (Remark 23.58). A hinge-coplanar panel realization is a body-hinge realization with nondegenerate hinges at the edges of G, and no more is needed for the lower bound.

By the proof above, both sides hold if and only if $\operatorname{def}(\tilde{G}) = 0$, that is, if and only if $\tilde{G}$ contains D edge-disjoint spanning trees (Lemma 20.14). This is the form of Katoh and Tanigawa's abstract [KT11, p. 647].

Remark 23.72 (One body)**.** The pinned theorems of Theorem 23.66 and Theorem 23.70 hold for one body, as their statements do. On one body every hinge of a panel-hinge framework is $C(n_v, n_v) = 0$ (Remark 23.58), but a hinge-coplanar panel realization on one body may give every label a nonzero extensor in the panel of the body, as in the proof of Theorem 23.66. The pinned theorem of Theorem 23.56 holds for one body too, as it asks for a nonzero hinge only at the edges of G.

23.6. **Case I: a proper rigid subgraph.** This subsection proves Case I of KT Theorem 5.5, in which the minimal k-dof-graph G has at least three vertices and a proper rigid subgraph H (KT §6.2). Katoh and Tanigawa contract H to a single vertex, realize H and the contraction $G/E(H)$ by induction, and combine the two realizations into one of G whose rank is the sum of their ranks (KT eq. (6.3)). To obtain a nonparallel realization when G is simple, they distinguish three cases: G is not simple (KT Lemma 6.2); G is simple and some proper rigid subgraph has a simple contraction (KT Lemma 6.3); and G is simple and no proper rigid subgraph has one (KT Lemma 6.5).

We first prove KT Lemma 5.4 on cycles, which Case III uses, and define the block pin, which pins all the bodies of a set at once. We then follow KT §6.2, which applies KT Lemma 3.5 to the contraction (Lemma 21.9): the rank count of KT eq. (6.3), with the block-triangular bound behind it, and its form for infinitesimal motions; KT Claim 6.4, with the rank lemmas it needs; and finally KT Lemmas 6.3, 6.2 and 6.5, with KT Claim 6.6 and the case split that combines them (Lemma 23.109).

Remark 23.73 (Katoh and Tanigawa's steps and the formal proof)**.** Several of the steps just listed follow Katoh and Tanigawa's argument, or restate it for infinitesimal motions, but the formal proof of KT Theorem 5.5 (Theorem 23.38) does not take them. We keep them beside the argument they belong to, and say here what the formal proof does instead.

Cycles. KT Lemma 5.4 realizes a cycle on m vertices for every $3 \leq m \leq D$ (KT p. 670). At most D hinges can have linearly independent extensors (Lemma 23.75),

and some D panel support extensors are linearly independent (Lemma 23.76). Katoh and Tanigawa apply KT Lemma 5.4 only to cycles on at most $d + 1$ vertices (the note after Lemma 23.80). The formal proof realizes these cycles with the normals of Lemma 23.62, and it uses neither of the two lemmas. For two bodies joined by two edges, Lemma 23.74 restates Lemma 23.30 for panel-hinge frameworks. The formal proof applies the body-hinge form, for instance in KT Lemma 6.2 (Lemma 23.102).

The rank count. Katoh and Tanigawa argue with the ranks of rigidity matrices throughout KT §6.2. The first diagonal block of KT eq. (6.3) consists of the rows of the edges of H. The second consists of the rows of the other edges, with the columns of $V(H)$ deleted. Katoh and Tanigawa read off that rank $R(G, p)$ is at least the sum of the ranks of the two blocks (KT pp. 673–675). Lemma 23.87 states the count as an identity. Its proof obtains the reverse inequality from the block pin (Lemma 23.84). For $k = 0$, the count has a form for infinitesimal motions (Lemma 23.89). Two more lemmas give properties of the block pin: deleting edges enlarges it (Lemma 23.83), and it detects rigidity (Lemma 23.85). The formal proof uses neither the identity nor these lemmas. Nor does it use the forms of the identity with the block pin (the note after Lemma 23.87).

Like Katoh and Tanigawa, the formal proof needs only the lower bound. No realization with nondegenerate hinges has a rank larger than the target rank $D(|V(G)|-1)-k$ (Lemma 20.16). The formal proof obtains the lower bound by counting rows at one choice of normals. Some $D(|V(H)|-1)$ rows of the edges of H are linearly independent, and so are some $D(|V(G)|-|V(H)|)-k$ rows of the other edges after the columns of $V(H)$ are deleted. By the block form, all these rows together are linearly independent (the proof of Lemma 23.100). This count gives the rank at every k. The form for infinitesimal motions gives only rigidity, the case $k = 0$.

KT Claim 6.4. In KT Lemma 6.3 the hinges of the edges between $V(H)$ and the other bodies change (KT eq. (6.6)). Katoh and Tanigawa compare the second block with its value when every body of H is given the panel of the vertex to which H is contracted. There it has the rank of the contraction. As the coefficients of their panels are algebraically independent, the second block has at least this rank (KT Claim 6.4 and eq. (6.9), p. 675). For $k = 0$ and a simple contraction, Lemma 23.98 gives this rank at some normals, and its proof takes these. At these normals the bodies of H share one normal. So the hinges of the edges of H are zero, and the normals are not in general position. The formal proof needs instead a nonzero polynomial in the normals, off whose zeros the second block has the rank of the contraction. Over an infinite field it has a common non-root with the polynomials for the rows of H and for general position (the proof of Lemma 23.100). The polynomial is that of Lemma 23.96, at every k, and its proof takes the steps of Lemma 23.98 at the same normals. It takes the place of Katoh and Tanigawa's algebraic independence (the passage after Lemma 23.100).

KT Lemma 5.4 states that a cycle on m vertices, for $3 \le m \le D$, has an infinitesimally rigid nonparallel panel-hinge realization [KT11, p. 670]. Katoh and Tanigawa note that a cycle can be analyzed directly from the definition of infinitesimal motions. For the computation in dimension 3 they refer to Crapo and Whiteley [CW82, Proposition 3.4] and to Whiteley [Whi99, Proposition 3], and they note that it works

in every dimension. Crapo and Whiteley show that a cycle of m panels and m hinges in dimension 3 is infinitesimally rigid if and only if the extensors of its hinges are linearly independent. We need one direction: in every dimension, a cycle whose hinges have linearly independent extensors is infinitesimally rigid (Lemma 23.77). Since $\bigwedge^{d-1} K^{d+1}$ has dimension D, the hinges of a cycle on m bodies can have linearly independent extensors only if $m \leq D$ (Lemma 23.75), the bound of KT Lemma 5.4. For $m \leq d+1$, the normals of Lemma 23.62 give a generic full-rank realization of the cycle (Lemma 23.80).

The case of two bodies joined by two edges is the double edge of KT Lemma 5.3, which Lemma 23.30 treats for body-hinge frameworks. The next lemma restates that lemma for panel-hinge frameworks.

Lemma 23.74 (Two bodies joined by two independent hinges)**.** *Let (G, p) be a panel-hinge framework, and let e_1 and e_2 be edges of G that both join the bodies u and v, with $C(p(e_1))$ and $C(p(e_2))$ linearly independent. Then every infinitesimal motion S of (G, p) has $S(u) = S(v)$. In particular, if u and v are distinct and are the only bodies, then (G, p) is infinitesimally rigid, of rank D.*

Proof. This is Lemma 23.30 for the body-hinge interpretation of (G, p) (Definition 23.5). An infinitesimally rigid framework on two bodies has rank $D(2-1) = D$ (Lemma 19.18). □

Lemma 23.75 (At most D independent hinges)**.** *Let (G, p) be a panel-hinge framework, and let $e_1, \dots, e_m$ be edges of G with $C(p(e_1)), \dots, C(p(e_m))$ linearly independent. Then $m \leq D$.*

Proof. The extensors lie in $\bigwedge^{d-1} K^{d+1}$, which has dimension $\binom{d+1}{d-1} = D$ (the remark after Lemma 18.9). □

The bound of Lemma 23.75 is attained by panel support extensors (Lemma 23.76).

Lemma 23.76 (Up to D independent panel support extensors)**.** *Let $m \leq D$. Then there are $a_i, b_i \in K^{d+1}$ for $1 \leq i \leq m$ such that the extensors $C(a_i, b_i)$ are linearly independent, and so are the joins $a_i \vee b_i$.*

Proof. Let $e_0, \dots, e_d$ be the standard basis of K^{d+1}. The $\binom{d+1}{2} = D$ joins $e_s \vee e_t$ with $s < t$ are linearly independent (the remark after Definition 18.3). Choose m distinct pairs $s_i < t_i$, and let $a_i = e_{s_i}$ and $b_i = e_{t_i}$. Then the joins $a_i \vee b_i$ are linearly independent, and so are the extensors $C(a_i, b_i)$ (Lemma 23.4). □

The pairs (a_i, b_i) of Lemma 23.76 need not be the normals at the ends of the edges of a cycle, whose consecutive hinges share a normal. For a cycle, the normals of Lemma 23.62 serve instead (Lemma 23.80).

Lemma 23.77 (A cycle with independent hinges is rigid)**.** *Let (G, p) be a body-hinge framework, let $m \geq 1$, and let $v_0, \dots, v_{m-1}$ be bodies, not necessarily distinct. Let $f_0, \dots, f_{m-1}$ be edges of G such that f_i joins v_i and v_{i+1}, with indices modulo m, and suppose that $C(p(f_0)), \dots, C(p(f_{m-1}))$ are linearly independent. Then every infinitesimal motion S of (G, p) has $S(v_0) = S(v_1) = \cdots = S(v_{m-1})$. In particular, if $v_0, \dots, v_{m-1}$ are all the bodies, then (G, p) is infinitesimally rigid.*

Proof. We argue as for KT Lemma 5.3 (Lemma 19.21) and Lemma 23.60, which are the cases $m = 2$ and $m = 3$ with distinct bodies. Let S be an infinitesimal motion. By the hinge constraints (Definition 19.3), there are scalars $\lambda_0, \dots, \lambda_{m-1}$ with $S(v_i) - S(v_{i+1}) = \lambda_i C(p(f_i))$ for each i.

Since the indices are taken modulo m, the m differences sum to zero. Hence $\sum_i \lambda_i C(p(f_i)) = 0$, and by the linear independence every λ_i is zero. So $S(v_i) = S(v_{i+1})$ for every i. If $v_0, \dots, v_{m-1}$ are all the bodies, then S is trivial (Definition 19.15). □

The formal realization of a cycle ends by choosing its normals once more, in general position, without decreasing the rank (the note after Lemma 23.80). Katoh and Tanigawa take steps of this kind by KT Lemma 5.2 [KT11, p. 669], a special case, as they note, of a lemma of Jackson and Jordán [JJ10, Lemma 7.1]. It says that if a panel-hinge realization of a simple graph has one pair of equal panels, and every other pair of panels is nonparallel, then the graph has a nonparallel realization whose rank is not smaller. They rotate one panel of that pair slightly, and argue by continuity over $\mathbb{R}$. In the proof of KT Lemma 6.10 they use it to turn realizations of full rank whose panels are not nonparallel into nonparallel ones (KT p. 691). The formalization takes such steps with polynomials in the normals in place of the rotation: for an infinitesimally rigid realization by the next lemma, and in general by the polynomial of Lemma 23.120, as in the proof of Lemma 23.135.

Lemma 23.78 (A rigid realization gives a generic one)**.** *Let K be an infinite field, and let G be a multigraph with $V(G) \neq \emptyset$ and $\operatorname{def}(\tilde{G}) = 0$. Suppose that some panel-hinge realization (G, q_0) of G has a nondegenerate hinge at every edge and is infinitesimally rigid on $V(G)$. Then G has a generic full-rank realization.*

Proof. The normals q_0 need not be in general position. As (G, q_0) is infinitesimally rigid on $V(G)$ and its hinges are nondegenerate, some $D(|V(G)|-1)$ rows of $R(G, q_0)$ are linearly independent (Lemma 23.124). As in the proof of Lemma 23.120, there is a polynomial Q in the normals with $Q(q_0) \neq 0$ such that the same rows are linearly independent at every q with $Q(q) \neq 0$. Off the zero set of another nonzero polynomial, the normals are in general position (Definition 23.9). As K is infinite, some normals q are a non-root of both.

Then $R(G, q)$ has $D(|V(G)| - 1)$ linearly independent rows. Let W be the set of bodies. The screw assignments form a space of dimension $D|W|$, so $\dim Z(G, q) \leq D|W| - D(|V(G)|-1) = D(|W \setminus V(G)|+1)$, and (G, q) is infinitesimally rigid on $V(G)$ (Lemma 23.117). Its rank is then $D(|V(G)| - 1)$, as in the proof of Lemma 23.124. As $\operatorname{def}(\tilde{G}) = 0$, it is at the target rank (Remark 23.23), and its normals are in general position. So it is a generic full-rank realization of G. □

Formalization note 23.79. The pinned theorem takes the framework (G, q_0) with an endpoint selector that assigns each edge of G its two ends (Formalization note 23.6), and it gives the generic full-rank realization the same selector.

Lemma 23.80 (Realizing a cycle; KT Lemma 5.4)**.** *Let K be an infinite field, let $3 \leq m \leq d + 1$, and let G be a cycle on m vertices (Definition 21.64) that is a*

minimal 0*-dof-graph. Then* G *has a generic full-rank realization (Definition 23.21), of rank* $D(m-1)$.

Proof. Let $v_0, \dots, v_{m-1}$ and $f_0, \dots, f_{m-1}$ be the vertices and the edges of G, where f_i joins v_i and v_{i+1}, with indices modulo m (Definition 21.64). Let $e_0, \dots, e_d$ be the standard basis of K^{d+1}, and give the body v_i the normal e_i, which is possible as $m \leq d+1$. Up to sign, the hinge at f_i is $C(e_i, e_{i+1})$, again with indices modulo m (Definition 23.5). These m extensors are linearly independent, by Lemma 23.62 and its proof, and so are the hinges.

So every infinitesimal motion is constant on $\{v_0, \dots, v_{m-1}\} = V(G)$ (Lemma 23.77). As G is a 0-dof-graph, $\operatorname{def}(\tilde{G}) = 0$, and the realization is at the target rank $D(m-1)$ (Remark 23.23). Any two distinct standard basis vectors are linearly independent, so the normals are in general position (Definition 23.9). Hence the realization is a generic full-rank realization of G. □

Formalization note 23.81. Katoh and Tanigawa state the lemma for $3 \leq m \leq D$ (KT p. 670). The statement has the range of the pinned theorem, $m \leq d+1$, which is theirs when $d = 2$, where $D = 3$. For $d \geq 3$ we have $d+1 < D$, and the cycles with $d+1 < m \leq D$ are not covered. No use needs them. Katoh and Tanigawa apply the lemma to the triangle in their proof of Lemma 6.7(i) (KT p. 677), and to cycles on at most d vertices in their proof of Lemma 6.13 (KT p. 692; Lemma 21.66). Case III applies it to the same cycles (Lemma 23.208).

The formal framework also has bodies outside $V(G)$ (Formalization note 23.22), and general position concerns their normals too. The pinned theorem gives those bodies the zero normal. It then chooses new normals for all the bodies, in general position, at which the rank of $R(G,p)$ is still $D(m-1)$. Such normals exist because K is infinite: each of the two conditions holds off the zero set of a nonzero polynomial in the normals, for the rank a minor of the rows that does not vanish at the given normals. This is the argument of Lemma 23.120, later in this chapter (Section 23.8), for one realization, and the step is Lemma 23.78.

Katoh and Tanigawa's Case I contracts a proper rigid subgraph H of G to one vertex (KT §6.2, p. 673). In terms of infinitesimal motions, the corresponding operation pins all the bodies of H at once, as pinned(v) does for one body in KT Lemma 5.1 (Lemma 19.20).

Definition 23.82 (The block pin of a set of bodies)**.** Let (G,p) be a body-hinge framework, and let s be a set of bodies. The *block pin* of s is the subspace

$$Z_s(G,p) \;=\; \{S \in Z(G,p) : S(v) = 0 \text{ for every } v \in s\}$$

of the infinitesimal motions that vanish on s.

For a single body, $Z_{\{v\}}(G,p)$ is the subspace pinned(v) of Lemma 19.20. For a nonempty set s, $Z_s(G,p)$ is the intersection of the subspaces pinned(v) with $v \in s$.

Lemma 23.83 (Deleting edges enlarges the block pin)**.** *Let* (G,p) *be a body-hinge framework, let* s *be a set of bodies, and let* (G',p) *be the restriction of* (G,p) *to a subgraph* G' *of* G *(Definition 23.13). Then* $Z_s(G,p) \subseteq Z_s(G',p)$*. In particular,* $\dim Z_s(G,p) \leq \dim Z_s(G',p)$.

Proof. The conditions $S(v) = 0$ for $v \in s$ do not involve the graph, and $Z(G,p) \subseteq Z(G',p)$ (Lemma 23.18). □

For one body, $\dim \text{pinned}(v) + D = \dim Z(G,p)$ (Lemma 19.20). For a larger set of bodies, only an inequality holds.

Lemma 23.84 (The block pin and the trivial motions)**.** *Let (G,p) be a body-hinge framework, and let s be a nonempty set of bodies. Then*

$$D + \dim Z_s(G,p) \;\leq\; \dim Z(G,p).$$

Proof. Let $v \in s$. Then $Z_s(G,p) \subseteq \text{pinned}(v)$, and $\text{pinned}(v)$ meets the space of trivial motions only in 0 (Lemma 19.20). The trivial motions form a D-dimensional subspace of $Z(G,p)$ (Lemma 19.18). So $Z(G,p)$ contains the direct sum of this subspace and $Z_s(G,p)$. □

A block pin also detects infinitesimal rigidity.

Lemma 23.85 (Rigidity and the block pin)**.** *Let (G,p) be a body-hinge framework, and let s be a nonempty set of bodies. If (G,p) is infinitesimally rigid, then $Z_s(G,p) = 0$. Conversely, if every infinitesimal motion of (G,p) is constant on s and $Z_s(G,p) = 0$, then (G,p) is infinitesimally rigid.*

Proof. Let (G,p) be infinitesimally rigid, and let $S \in Z_s(G,p)$. Then S is trivial, that is, constant (Definition 19.15), and it vanishes at a body of s. So $S = 0$.

Conversely, let S be an infinitesimal motion, let $v \in s$, and let T be the trivial motion with the value $S(v)$ at every body. Then $S - T$ is an infinitesimal motion, and it vanishes on s because S is constant on s. So $S - T \in Z_s(G,p) = 0$, and $S = T$ is trivial. □

We now follow Katoh and Tanigawa's sketch of Case I at the start of KT §6.2 (p. 673). Let G be a minimal k-dof-graph, let H be a proper rigid subgraph of G, and let $G/E(H)$ be its contraction to a vertex r of H (Definition 21.3). Then H is a minimal 0-dof-graph (KT Lemma 3.3, Lemma 20.25), and $G/E(H)$ is a minimal k-dof-graph with fewer vertices than G (KT Lemma 3.5, Lemma 21.9). So the induction hypothesis, KT eq. (6.1), gives a realization (H,p_1) of rank $D(|V(H)|-1)$ and a realization $(G/E(H),p_2)$ of rank $D(|V(G)|-|V(H)|)-k$. Katoh and Tanigawa replace the body r of $(G/E(H),p_2)$ by the rigid framework (H,p_1). They show that the rank of the resulting realization of G is the sum of the two ranks, which is $D(|V(G)|-1)-k$, the target rank of G. We prove this rank count, KT eq. (6.3), and then KT Claim 6.4, which Katoh and Tanigawa use when G and $G/E(H)$ are simple.

Katoh and Tanigawa read the rank count off the shape (6.3) of $R(G,p)$: with the columns of $V(H)$ first, the rows of H have zeros in all the other columns [KT11, pp. 673–674]. The next lemma is the bound that this shape gives, for any set W of bodies, where π_W deletes the columns of W (Section 19.3).

Lemma 23.86 (The block-triangular rank bound; KT eq. (6.3))**.** *Let (G,p) be a panel-hinge framework, let W be a set of bodies, and let $H = G[W]$. Let $R_W(G \setminus$*

$E(H), p)$ be the matrix of the rows of the edges of G not in H, with the columns of W deleted. Then

$$\operatorname{rank} R(H,p) + \operatorname{rank} R_W(G \setminus E(H), p) \;\leq\; \operatorname{rank} R(G,p).$$

Proof. Let $\mathcal{R}$, $\mathcal{R}_H$ and $\mathcal{R}'$ be the row spaces of $R(G,p)$, $R(H,p)$ and $R(G \setminus E(H), p)$. The rows of H and of $G \setminus E(H)$ are rows of G, so $\mathcal{R}_H$ and $\mathcal{R}'$ lie in $\mathcal{R}$. A row of an edge of H depends only on the screws at the bodies of W, so π_W maps it to 0. Hence $\mathcal{R}_H \subseteq \mathcal{R} \cap \ker \pi_W$. The rank of $R_W(G \setminus E(H), p)$ is $\dim \pi_W(\mathcal{R}')$, which is at most $\dim \pi_W(\mathcal{R})$. The bound follows, since $\dim(\mathcal{R} \cap \ker \pi_W) + \dim \pi_W(\mathcal{R}) = \dim \mathcal{R}$ by rank-nullity for π_W on $\mathcal{R}$. □

Lemma 23.87 (Case I: the rank adds over the contraction; KT eq. (6.3))**.** *Let G be a minimal k-dof-graph, let H be a proper rigid subgraph of G, and let $G/E(H)$ be its contraction to a vertex r of H. Let (G,p) be a body-hinge framework, and let $(G/E(H), p_2)$ be a realization of $G/E(H)$ at the target rank (Definition 23.21). Then (G,p) is at the target rank if and only if*

$$\operatorname{rank} R(G,p) \;=\; D(|V(H)| - 1) + \operatorname{rank} R(G/E(H), p_2).$$

Proof. The contraction keeps the deficiency, $\operatorname{def}(\widetilde{G/E(H)}) = \operatorname{def}(\tilde{G}) = k$ (Lemma 29.22; by KT Lemma 3.5, Lemma 21.9, $G/E(H)$ is even a minimal k-dof-graph). It has $|V(G)| - |V(H)| + 1$ vertices (Lemma 21.51), so $\operatorname{rank} R(G/E(H), p_2) = D(|V(G)| - |V(H)|) - k$. The right side of the identity is therefore $D(|V(G)| - 1) - k$, the target rank of G. This proves the lemma.

Katoh and Tanigawa attain the right side in their Lemmas 6.2 and 6.3 (pp. 673–675). There (H, p_1) is a realization of H of rank $D(|V(H)| - 1)$, and p has the hinges of p_1 at the edges of H (KT eqs. (6.2) and (6.6)). Put the columns of the bodies of H first. The hinge constraint at an edge of H involves only the screws at bodies of H, so $R(G,p)$ has the block form of KT eq. (6.3):

$$R(G,p) \;=\; \begin{pmatrix} R(H,p_1) & 0 \\ * & R_{V(H)}(G \setminus E(H), p) \end{pmatrix}.$$

The first block row consists of the rows of the edges of H, and the second of the rows of the other edges. Its right block $R_{V(H)}(G \setminus E(H), p)$ consists of these rows with the columns of $V(H)$ deleted. Suppose that some rows of the first block row are linearly independent, and that some rows of the second are linearly independent after the columns of $V(H)$ are deleted. Then all of them together are linearly independent. Indeed, in a vanishing linear combination only the rows of the second block row have nonzero entries in the columns of $V(G) \setminus V(H)$, so their coefficients vanish, and then so do the others. Hence

$$\operatorname{rank} R(G,p) \;\geq\; D(|V(H)| - 1) + \operatorname{rank} R_{V(H)}(G \setminus E(H), p)$$

(Lemma 23.86).

When the rows of H have rank $D(|V(H)| - 1)$, as here, this bound is an equality. A screw assignment that vanishes on $V(H)$ satisfies the hinge constraints at the edges of H. It satisfies those at the other edges exactly when its values at the other bodies lie in the null space of $R_{V(H)}(G \setminus E(H), p)$. So the block pin $Z_{V(H)}(G,p)$

(Definition 23.82) has dimension $D(|V(G)| - |V(H)|) - \operatorname{rank} R_{V(H)}(G \setminus E(H), p)$. Since $\operatorname{rank} R(G,p) = D|V(G)| - \dim Z(G,p)$, the bound $D + \dim Z_{V(H)}(G,p) \leq \dim Z(G,p)$ (Lemma 23.84) is the reverse inequality.

It remains to compare the second block with $R(G/E(H), p_2)$. The bodies of $G/E(H)$ meet $V(H)$ only in r, so deleting the columns of $V(H)$ from $R(G/E(H), p_2)$ deletes those of r. By KT Lemma 5.1 (Lemma 19.20), this does not change the rank. Katoh and Tanigawa note this at eq. (6.5), and Lemma 19.35 states it in this form. In KT Lemma 6.2, p has the hinges of p_2 at the edges not in H. The row of such an edge in $R(G,p)$ then differs from its row in $R(G/E(H), p_2)$ only in the columns of $V(H)$ (KT eq. (6.4)). So the second block has the rank of $R(G/E(H), p_2)$, and the identity holds.

In KT Lemma 6.3, G and $G/E(H)$ are simple, and p changes the hinges at the edges between $V(H)$ and the other bodies (KT eq. (6.6)). With KT Lemma 5.1, KT Claim 6.4 shows that the second block still has at least the rank of $R(G/E(H), p_2)$ (KT eq. (6.9); Lemma 23.96, and Lemma 23.98 at one point, below). The proof of the claim uses that p_1 and p_2 are generic, so that the rank is the largest over all choices of the panels. The general form of this argument is the assertion of Lemma 23.112 on the largest rank, in Section 23.7: where every hinge is nondegenerate, the largest rank is attained on a dense Zariski-open set of normals. Hence the rank of $R(G,p)$ is at least the target rank. It is also at most the target rank, as the hinges are nondegenerate (Lemma 20.16). □

Formalization note 23.88. The first pinned theorem states the identity with $\operatorname{def}(\tilde{G})$ in place of k, and does not assume G minimal: the first paragraph of the proof uses only the deficiency and the number of vertices of $G/E(H)$. Of the two frameworks it uses only their ranks, so their graphs need not be G and $G/E(H)$. The third and fourth state the lemma with the block pin in place of $R(G/E(H), p_2)$, for panel-hinge and for body-hinge frameworks. Let s be a nonempty set of bodies, and suppose that $\dim Z(G,p) \leq D + \dim Z_s(G,p)$. Then $\dim Z(G,p) = D + k'$, the form of the target rank in Remark 23.23, if and only if $\dim Z_s(G,p) = k'$. For $s = V(H)$, the hypothesis holds when the rows of H have rank $D(|V(H)| - 1)$, by the proof above, and $\dim Z_s(G,p) = k$ says that the second block has the rank of $R(G/E(H), p_2)$. The second pinned theorem assumes that every infinitesimal motion of (G,p) is constant on s, and lets $s \subseteq t$. Then every infinitesimal motion is constant on t if and only if $Z_s(G,p) \subseteq Z_t(G,p)$. It is the one used elsewhere, with s a single body, in Lemma 23.117, later in this chapter.

For $k = 0$ the rank count has a form in terms of infinitesimal motions. In it the contraction has the hinges of p: it is the framework $(G/E(H), p)$ of Definition 23.13. Katoh and Tanigawa's framework of eq. (6.7) is of this kind.

Lemma 23.89 (Rigidity from H and the contraction; KT eq. (6.3))**.** *Let (G,p) be a body-hinge framework, let H be a proper rigid subgraph of G, and let $G/E(H)$ be its contraction to a vertex r of H, with vertex set $s_c = (V(G) \setminus V(H)) \cup \{r\}$. Suppose that every infinitesimal motion of the restriction (H,p) is constant on $V(H)$, and that every infinitesimal motion of $(G/E(H), p)$ is constant on s_c. Then every infinitesimal motion of (G,p) is constant on $V(G)$.*

Proof. Let S be an infinitesimal motion of (G, p). As H is a subgraph of G, S is an infinitesimal motion of (H, p) (Lemma 23.18), so S is constant on $V(H)$. Let f map every vertex of H to r and fix the other bodies. Then $S(f(w)) = S(w)$ for every body w, as $r \in V(H)$. An edge e of $G/E(H)$ is an edge of G not in H. If its ends in G are u and v, then its ends in $G/E(H)$ are $f(u)$ and $f(v)$ (Definition 21.3), and $S(f(u)) - S(f(v)) = S(u) - S(v)$. So S satisfies the hinge constraint of $(G/E(H), p)$ at e, because it satisfies that of (G, p) (Definition 19.3). Hence S is an infinitesimal motion of $(G/E(H), p)$, and it is constant on s_c. The sets $V(H)$ and s_c share the body r and cover $V(G)$, so S is constant on $V(G)$. □

Formalization note 23.90. The pinned theorem joins the restrictions of (G, p) to two subgraphs G_1 and G_2 of G. Let s_1 and s_2 be sets of bodies with a common body, and suppose that every infinitesimal motion of (G_i, p) is constant on s_i for $i = 1, 2$. Then every infinitesimal motion of (G, p) is constant on every set $t \subseteq s_1 \cup s_2$. Its proof is the first and the last step of the proof above. The contraction $G/E(H)$ is not a subgraph of G when an edge not in H has an end in $V(H) \setminus \{r\}$. So the lemma is not an instance of the pinned theorem, and no pinned theorem gives the step from (G, p) to $(G/E(H), p)$. No main result uses the pinned theorem.

The lemma takes a different set of bodies for each of its two frameworks, and the contraction needs this. Contracting H identifies all of $V(H)$ with r, so the bodies of $V(H) \setminus \{r\}$ are not vertices of $G/E(H)$. Each edge of $G/E(H)$ is an edge of G not in H, and each of its ends in $V(H)$ is replaced by r, so both of its ends lie in s_c. Hence no hinge constraint of $(G/E(H), p)$ involves a body of $V(H) \setminus \{r\}$, and an infinitesimal motion of $(G/E(H), p)$ may take any values there. So the lemma can ask that the infinitesimal motions of $(G/E(H), p)$ be constant on s_c, but on no larger set, and in particular not on $V(G)$. The bodies of $V(H) \setminus \{r\}$ are held by the edges of H, which are the edges of the restriction (H, p).

The same two sets of bodies appear in the rank count of Lemma 23.87. The first block of KT eq. (6.3) has the columns of $V(H)$, and its rank is $D(|V(H)| - 1)$. The second has the columns of $V(G) \setminus V(H) = s_c \setminus \{r\}$. In KT Lemma 6.2 its rank is $\operatorname{rank} R(G/E(H), p_2) = D(|s_c| - 1) - k$, since deleting the columns of the one body r does not change the rank, and in KT Lemma 6.3 it is at least this. The two sets share exactly the body r, so $|V(H)| + |s_c| = |V(G)| + 1$ and

$$D(|V(H)| - 1) + D(|s_c| - 1) - k \;=\; D(|V(G)| - 1) - k,$$

the target rank of G (KT p. 675).

In KT Lemma 6.3 the graphs G and $G/E(H)$ are simple, and the realizations (H, p_1) and $(G/E(H), p_2)$ are nonparallel. Katoh and Tanigawa give an edge of G from a body $u \notin V(H)$ to a body $v \in V(H)$ the hinge in which the panel of u in p_2 meets the panel of v in p_1 (KT eq. (6.6)). On the edges not in H, these hinges form a body-hinge framework $(G/E(H), p')$ on the contraction, in which every body other than r is a panel and r need not be one (KT eq. (6.7), where p' is written $p_{E \setminus E'}$). KT Claim 6.4 states that $\operatorname{rank} R(G/E(H), p') \geq \operatorname{rank} R(G/E(H), p_2)$ (p. 675). Its proof has two steps. First, as Katoh and Tanigawa note after Jackson and Jordán [JJ10], the entries of $R(G/E(H), p')$ can be written in terms of the coefficients of the equations of the panels of p_1 and p_2. Second, if every body of H is given the panel

of r in p_2, then p' becomes p_2. As the coefficients of p_1 and p_2 are algebraically independent, $R(G/E(H), p')$ has the largest rank over all these choices, which is at least $\operatorname{rank} R(G/E(H), p_2)$. With KT Lemma 5.1 the claim gives KT eq. (6.9), the bound on the second block in the proof of Lemma 23.87. For panel normals, the first step is the next lemma. The lemmas after it give the rows outside H at every deficiency (Lemma 23.96), and Lemma 23.98 gives the rank at the placement of the second step when $G/E(H)$ is simple and rigid.

Lemma 23.91 (Rows polynomial in the normals; KT Claim 6.4)**.** *Let G be a multigraph, and let e be an edge of G. There is a finite family of linear forms on screw assignments, depending on the normals $n = (n_v)$, whose coordinates in the standard basis are polynomials of degree two in the coordinates of the normals. For every panel-hinge framework (G, n) whose hinge at e is nondegenerate, the family at n spans the same space as the rows of $R(G, n)$ at e (Definition 19.5).*

Proof. Let u and v be the ends of e, and let $C = C(n_u, n_v)$ (Definition 23.1). For $x \in \bigwedge^{d-1} K^{d+1}$, write x_t for its coordinate at the t-th standard basis vector. The family consists of the forms

$$S \;\mapsto\; C_t\, x_{t'} - C_{t'}\, x_t, \qquad \text{where } x = S(u) - S(v),$$

for all pairs of indices t and t'. As a form in x, each of them vanishes at $x = C$. If $C \neq 0$, say $C_{t_0} \neq 0$, then the forms $x \mapsto C_{t_0} x_{t'} - C_{t'} x_{t_0}$ with $t' \neq t_0$ are $D - 1$ linearly independent forms that vanish at C. So they span the space of forms that vanish at C, which has dimension $D - 1$. Hence the family spans the same space as the rows of $R(G, n)$ at e (Definition 19.5).

The coordinates of these forms are linear in those of C. The coordinates of the join $n_u \vee n_v$ (Definition 18.4) are the 2×2 minors $n_u(i)\, n_v(j) - n_u(j)\, n_v(i)$ with $i < j$, which are polynomials of degree two in the normals. The complement isomorphism $*$ maps each standard basis vector of $\bigwedge^2 K^{d+1}$ to a nonzero multiple of one of $\bigwedge^{d-1} K^{d+1}$ (the proof of Definition 22.5). So the coordinates of $C = *(n_u \vee n_v)$ are polynomials of degree two in the normals, and so are those of the forms. □

Formalization note 23.92. The pinned theorems state the lemma for the forms in x. The first gives the coordinate of the form $x \mapsto C_t x_{t'} - C_{t'} x_t$ at each standard basis vector as the value at the normals of a polynomial in the normals of u and v, and the second bounds the degree of that polynomial by two. The coordinates of the form in S are these, up to sign, at the columns of u and v. The third is the spanning: for $C \neq 0$, the forms in x span the forms that vanish at C. The pinned theorem of Lemma 23.115 takes a minor of these forms, as in the proof of Lemma 23.120.

The algebraic induction chooses its realizations off the zeros of polynomials in the panel normals. The rows of $R(G, q)$ can be chosen with entries that are polynomials in the normals q (Lemma 23.91). So if some N rows of $R(G, q_0)$ are linearly independent, one of their $N \times N$ minors is a polynomial Q in the normals with $Q(q_0) \neq 0$, and the rank of $R(G, q)$ is at least N wherever $Q(q) \neq 0$. This is, in polynomial form, the lower semicontinuity of the rank by which Katoh and Tanigawa prove KT Lemma 5.2 over $\mathbb{R}$ (KT p. 669; the passage before Lemma 23.78), and Lemma 23.120

states it in Section 23.8. The next lemma is its form for rows with the columns of a set of bodies deleted.

Lemma 23.93 (Independent rows with columns deleted, off the zeros of a polynomial)**.** *Let G be a graph, let t be a set of bodies, and let q_0 be panel normals at which some m rows of $R(G, q_0)$ are linearly independent after the columns of t are deleted. Then there is a polynomial Q in the panel normals with $Q(q_0) \neq 0$ such that at every q with $Q(q) \neq 0$, some m rows of $R(G, q)$ are linearly independent after the columns of t are deleted.*

Proof. Deleting the columns of t replaces some entries of a row by zeros, independently of the normals, so the rows of $R(G, q)$ with these columns deleted again have entries that are polynomials in q (Lemma 23.91). Take m rows that are independent after the deletion at q_0. Some $m \times m$ minor of these m rows, with the columns of t deleted, is nonzero at q_0. Let Q be this minor, as a polynomial in the normals. Wherever $Q(q) \neq 0$, the same m rows are independent after the deletion. This is the argument of Lemma 23.120. □

The induction hypothesis, KT eq. (6.1), supplies a realization of $G/E(H)$ whose rank is $D(|V|-1)-\text{def}$ (Definition 23.21) and whose normals are in general position: any two of them are linearly independent (Definition 23.9). Such a realization also gives one with nondegenerate hinges, the form of the hypothesis in Definition 23.24 (Lemma 23.25). Case I compares the rows of the realization of $G/E(H)$ with those of $G \setminus E(H)$ edge by edge, and the next lemma first orients the edges of $G/E(H)$ as those of $G \setminus E(H)$.

Lemma 23.94 (Reorienting a realization of the contraction)**.** *Let H be a subgraph of G, let $r \in V(H)$, let f map every body of $V(H)$ to r and fix every other body, and let $G_1 = G/E(H)$. Orient the edges of $G \setminus E(H)$, and orient an edge of G_1 from $f(u)$ to $f(v)$ if it is oriented from u to v in $G \setminus E(H)$. Suppose that G_1 has a panel-hinge realization whose normals are in general position and whose rank is $D(|V(G_1)|-1)-\text{def}(\tilde{G}_1)$. Then there are panel normals in general position at which the panel-hinge framework on G_1, with this orientation, has the same rank.*

Proof. Take the normals of the given realization; general position is a condition on the normals alone. The given realization orients each edge of G_1 one way or the other. Reversing an edge changes the sign of the extensor of its hinge, which is built from the normals of its ends in order (Definition 23.5), and so does not change its hinge constraint. The two orientations therefore give the same infinitesimal motions, and so the same row space and rank (Lemma 19.25). □

Formalization note 23.95. The formalization records an orientation of each edge with a panel-hinge framework (Definition 23.5), and compares the rows of two frameworks edge by edge only for equal orientations. The realization that the induction hypothesis supplies orients $G/E(H)$ in its own way; the lemma gives it the orientation of $G \setminus E(H)$.

The next lemma combines the last two with Lemma 19.35, as KT eq. (6.9) combines KT Lemma 5.1 and Claim 6.4.

Lemma 23.96 (The rows outside a contracted subgraph; KT Claim 6.4)**.** *Let H be a proper rigid subgraph of G, let $r \in V(H)$, and let $G_1 = G/E(H)$. Suppose that G_1 is a minimal k-dof-graph (Definition 20.8) with a panel-hinge realization whose normals are in general position and whose rank is $D(|V(G_1)|-1)-k$. Then there is a nonzero polynomial Q in the panel normals such that at every q with $Q(q) \neq 0$, some $D(|V(G_1)|-1)-k$ rows of $R(G \setminus E(H), q)$ are linearly independent after the columns of $V(H)$ are deleted.*

Proof. Let $N = D(|V(G_1)|-1)-k$, and let f map the bodies of $V(H)$ to r and fix the others. Orient $G \setminus E(H)$, and G_1 as in Lemma 23.94. By that lemma there are normals n in general position at which $R(G_1, n)$ has rank N. The bodies of G_1 meet $V(H)$ only in r, so deleting the columns of $V(H)$ keeps the rank N (Lemma 19.35). A minimal k-dof-graph has no loops: every base of $M(\tilde{G}_1)$ meets the copies of every edge (Definition 20.8), and the copies of a loop lie in no base (`Graph.loopless_of_isMinimalKDof`). So the hinges of G_1 at n are nonzero (Lemma 23.11), and some N rows of $R(G_1, n)$ are independent after the deletion.

Let q_0 give each body v the normal $n_{f(v)}$: the bodies of $V(H)$ get the normal of r, and the other bodies keep theirs. At q_0 an edge of $G \setminus E(H)$ from u to v has the hinge that the same edge of G_1, from $f(u)$ to $f(v)$, has at n. Its rows in the two matrices differ only in the columns of the bodies of $V(H)$. So after the columns of $V(H)$ are deleted, the rows of $R(G \setminus E(H), q_0)$ are those of $R(G_1, n)$, and N of them are independent. By Lemma 23.93 there is a polynomial Q with $Q(q_0) \neq 0$, so that $Q \neq 0$, such that at every q with $Q(q) \neq 0$ some N rows of $R(G \setminus E(H), q)$ are independent after the deletion. □

Formalization note 23.97. The rows in this lemma and in Lemma 23.93 are those of Lemma 23.91: for each edge, a fixed family of linear forms that vanish on the extensor of its hinge, with entries polynomial in the normals. They span the row space of the edge when its hinge is nonzero, which is why the proof needs $G/E(H)$ to have no loops: a loop has a zero hinge.

Katoh and Tanigawa reach the placement q_0 by giving every body of $V(H)$ the panel of the contracted body, and conclude from the algebraic independence of the coefficients of their panels that the rows of $G \setminus E(H)$ attain the rank of $G/E(H)$ [KT11, Claim 6.4, p. 675]. The polynomial Q takes the place of that argument, and Case I uses the lemma in Lemma 23.100. The next lemma gives the rank at one placement, q_0 in its proof, when $G/E(H)$ is simple and has deficiency zero. Its conclusion is that of Lemma 23.96 for $k = 0$ at one point, and it assumes $G/E(H)$ simple where that lemma assumes it minimal.

Lemma 23.98 (The rows outside H at one point; KT Claim 6.4 and eq. (6.9))**.** *Let G be a multigraph, let H be a proper rigid subgraph of G, and let $G/E(H)$ be its contraction to a vertex r of H, with vertex set $s_c = (V(G) \setminus V(H)) \cup \{r\}$. Suppose that $G/E(H)$ is a simple 0-dof-graph (Definition 20.8) with a generic full-rank realization (Definition 23.21). Then there are normals q_0 at which some $D(|s_c|-1)$ rows of $R(G \setminus E(H), q_0)$ are linearly independent after the columns of $V(H)$ are deleted.*

Proof. Let $N = D(|s_c|-1)$, and let f map every body of $V(H)$ to r and fix the other bodies. As $G/E(H)$ is a 0-dof-graph, its target rank is N. Its generic full-rank

realization has normals n in general position at which $R(G/E(H), n)$ has rank N, so it is infinitesimally rigid on s_c (Lemma 23.117). The bodies of $G/E(H)$ meet $V(H)$ only in r. So deleting the columns of $V(H)$ from $R(G/E(H), n)$ deletes those of the one body r, and by KT Lemma 5.1 (Lemma 19.20) the rank stays N (Lemma 19.35).

Now give each body v the normal $q_0(v) = n_{f(v)}$, so that the bodies of $V(H)$ get the normal of r. This is the choice in Katoh and Tanigawa's proof of Claim 6.4 that turns p' into p_2. Let e be an edge of $G \setminus E(H)$ with ends u and v. At q_0 its hinge is $C(n_{f(u)}, n_{f(v)})$, the hinge of e in $G/E(H)$ at n, as the ends of e in $G/E(H)$ are $f(u)$ and $f(v)$. Its two rows differ only in the columns of the bodies of $V(H)$. So after the columns of $V(H)$ are deleted, the rows of $R(G \setminus E(H), q_0)$ are those of $R(G/E(H), n)$, and they have rank N.

Finally, $G/E(H)$ is simple, so it has no loops. As n is in general position, the hinges of $G/E(H)$ at n, and so those of $G \setminus E(H)$ at q_0, are nondegenerate (Lemma 23.11). So the forms of Lemma 23.91 at q_0 span the rows of every edge of $G \setminus E(H)$, and some N of them are linearly independent after the deletion, as in Lemma 23.110. □

Formalization note 23.99. The rows of the statement are the forms of Lemma 23.91, as in Lemma 23.96. Of H the pinned theorem assumes only that $r \in V(H) \subseteq V(G)$, and of G nothing. It assumes that the endpoint selector gives each edge of $G \setminus E(H)$ its two ends (Formalization note 23.6). Its statement gives some normals q_0; its proof takes them of the form above, giving every body of $V(H)$ the normal of r. Its proof first orients each edge of $G/E(H)$ as the same edge of $G \setminus E(H)$, as in Lemma 23.94, so that the rows of the two graphs can be compared edge by edge. The polynomial Q, for every deficiency, is Lemma 23.96, whose proof takes the same steps and then those of Lemma 23.93. The formal proof of Case I uses that lemma, and not this one.

Katoh and Tanigawa need three cases to obtain a nonparallel realization when G is simple [KT11, pp. 673–676]. The reason is that the contraction $G/E(H)$ need not be simple when G is. The other graphs that Case I builds from G, such as $G \setminus E(H)$, H, and the graph G_v obtained by removing a vertex v (Definition 21.10), are subgraphs of G, and a subgraph of a simple graph is simple. The contraction instead identifies every vertex of H with r (Definition 21.3), and this can create a loop or two parallel edges. An edge of G that is not in H but has both ends in $V(H)$ becomes a loop at r. Two edges not in H become parallel when their ends are identified in pairs. For instance, two edges that join a vertex $w \notin V(H)$ to two vertices of H both join w and r in $G/E(H)$. The simplicity of G excludes neither configuration. So the simplicity of $G/E(H)$ is a hypothesis of KT Lemma 6.3, and it does not follow from that of G.

This gives the three cases. If G is not simple, then two parallel edges of G form a proper rigid subgraph, and KT Lemma 6.2 contracts it (Lemma 23.102). The induction asks for a nonparallel realization only of a simple graph, so this case needs no general position. If G is simple and some proper rigid subgraph H has a simple contraction, then the induction hypothesis gives generic full-rank realizations of H and $G/E(H)$, and KT Lemma 6.3 combines them (Lemma 23.100). If G is simple and no proper rigid subgraph has a simple contraction, then KT Lemma 6.5 removes

a vertex v of degree two instead (Lemma 23.106). The graph G_v is a subgraph of G, so it is simple, and the induction hypothesis gives a generic full-rank realization of it. By KT Claim 6.6 (Lemma 23.104), this last case occurs only for $k = 0$. We prove KT Lemma 6.3 first, then KT Lemma 6.2, KT Claim 6.6 and KT Lemma 6.5, and finally the case split.

Lemma 23.100 (Case I with a simple contraction; KT Lemma 6.3)**.** *Let $d \geq 2$, let K be an infinite field, and let G be a simple minimal k-dof-graph with $|V(G)| \geq 3$. Let H be a proper rigid subgraph of G whose contraction $G/E(H)$ to a vertex r of H is simple, and suppose that H and $G/E(H)$ have generic full-rank realizations (Definition 23.21). Then G has a generic full-rank realization, of rank $D(|V(G)| - 1) - k$.*

Proof. By KT Lemma 3.3 (Lemma 20.25), H is a minimal 0-dof-graph. By KT Lemma 3.5 (Lemma 21.9), $G/E(H)$ is a minimal k-dof-graph, with the vertex set $s_c = (V(G) \setminus V(H)) \cup \{r\}$. So the given realizations (H, p_1) and $(G/E(H), p_2)$ have the ranks $D(|V(H)| - 1)$ and $D(|s_c| - 1) - k$. In Katoh and Tanigawa's proof the induction hypothesis gives them, with algebraically independent coefficients, and KT eq. (6.6) combines them into a realization of G. Its rank is the target rank by KT eq. (6.3) and Claim 6.4, as in the proof of Lemma 23.87.

We choose instead one set of normals q_0 for all the bodies, off the zero sets of three nonzero polynomials in the normals. First, some $D(|V(H)| - 1)$ rows of $R(H, p_1)$ are linearly independent. The rows can be chosen with entries that are polynomials in the normals (Lemma 23.91). So one of the maximal minors of these rows is a polynomial Q_1 that does not vanish at the normals of p_1, and wherever $Q_1(q) \neq 0$, the same rows of $R(H, q)$ are linearly independent. This is the argument of Lemma 23.120, later in this chapter (Section 23.8).

Second, there is a nonzero polynomial Q_2 such that wherever $Q_2(q) \neq 0$, some $D(|s_c| - 1) - k$ rows of $R(G \setminus E(H), q)$ are linearly independent after the columns of $V(H)$ are deleted (Lemma 23.96, applied to $(G/E(H), p_2)$). This is KT Claim 6.4 with KT eq. (6.9), in polynomial form.

Third, write $q(a)_i$ for the i-th coordinate of the normal $q(a)$, from $i = 0$. Let P be the product, over the ordered pairs of distinct bodies a and b, of the minors $q(a)_0\, q(b)_1 - q(a)_1\, q(b)_0$. Wherever $P(q) \neq 0$, any two normals are linearly independent, so the normals are in general position (Definition 23.9). The polynomial P is not zero: at normals on the moment curve (Lemma 23.12), each factor is a difference $t_b - t_a \neq 0$. As K is infinite, the nonzero polynomial $Q_1 Q_2 P$ has a non-root q_0.

Let (G, q_0) be the panel-hinge framework on G with the normals q_0, in which every hinge is the meet of the panels of the ends of its edge, as in KT eq. (6.6). As $Q_1(q_0) \neq 0$, some $D(|V(H)| - 1)$ rows of the edges of H are linearly independent. As $Q_2(q_0) \neq 0$, some $D(|s_c| - 1) - k$ rows of the other edges are linearly independent after the columns of $V(H)$ are deleted. By the block form of KT eq. (6.3), all these rows together are linearly independent (the proof of Lemma 23.87). The sets $V(H)$ and s_c share only the body r and cover $V(G)$, so there are

$$D(|V(H)| - 1) + D(|s_c| - 1) - k \;=\; D(|V(G)| - 1) - k$$

of them. As $P(q_0) \neq 0$, the normals q_0 are in general position. The graph G has no loops, so every hinge of (G, q_0) is nondegenerate (Lemma 23.11). Hence the rank of $R(G, q_0)$ is at most $D(|V(G)| - 1) - k$ (Lemma 20.16), and (G, q_0) is a generic full-rank realization of G. □

Formalization note 23.101. The pinned theorem assumes the induction hypothesis KT eq. (6.1) for every graph with fewer vertices than G (Theorem 23.38), and uses it only for H and $G/E(H)$. The realization of H that the induction hypothesis gives orients the edges of H in its own way. The pinned theorem first gives that realization the orientation of G, which changes no infinitesimal motion, as for $G/E(H)$ in Lemma 23.94. It obtains the infinitesimal rigidity of that realization on $V(H)$ from its rank by the relative count of Lemma 23.117, later in this chapter, as the formal framework has bodies outside $V(H)$. Given Q_2, the choice of q_0 and the count of the rows are one step of its proof (`hasGenericFullRankRealization_of_couple_blockTriangular_ofNormals_set_kdof`).

Rank and general position. The normals q_0 satisfy conditions of two kinds: the rank conditions $Q_1(q_0) \neq 0$ and $Q_2(q_0) \neq 0$, and general position. Katoh and Tanigawa obtain both from one hypothesis [KT11, §5.1, p. 668]. They call two panels nonparallel if they meet in a $(d-2)$-dimensional affine subspace. The panel of a body v is defined by a vector $c(v) \in \mathbb{R}^d$, and two panels are nonparallel when their vectors are linearly independent. A nonparallel realization is generic if the coordinates of all the vectors $c(v)$ are algebraically independent over $\mathbb{Q}$. In the proof of KT Lemma 6.3 (pp. 674–675), p_1 and p_2 are taken so that the coefficients of the panels of p_1 at the bodies of H and of p_2 at the other bodies are algebraically independent over $\mathbb{Q}$. This choice makes any two of these panels nonparallel, so the realization of KT eq. (6.6) is nonparallel. In Claim 6.4 the same choice makes the rank of $R(G/E(H), p')$ the largest over all the realizations of the form of KT eq. (6.7). Katoh and Tanigawa note that their definition of a generic realization gives a smaller class of frameworks than the usual one, by the largest rank of the rigidity matrix, and that they use it to make their proof simpler (KT footnote 4, p. 668).

Over an infinite field that is algebraic over its prime field, such as the algebraic closure of $\mathbb{Q}$ or of a finite field, no coordinate is transcendental over the prime field, so no realization is generic in this sense. We therefore state the two kinds of condition separately, each as the non-vanishing of a polynomial in the normals, and use only that a nonzero polynomial over an infinite field has a non-root. A rank condition is the non-vanishing of a minor of the rows that does not vanish at a given realization (Lemma 23.120). Stated for open sets, this is the lower semicontinuity of the rank where every hinge is nondegenerate (Lemma 23.112, in Section 23.7). General position is a condition on pairs of normals that Katoh and Tanigawa's nonparallel panels satisfy (the remark after Definition 23.9), and it holds wherever P does not vanish. A product of nonzero polynomials is again a nonzero polynomial. At a non-root of the product all the conditions hold at once, and this takes the place of Katoh and Tanigawa's one hypothesis. The proof of KT Lemma 6.5 below uses the same product with a factor of a third kind.

Lemma 23.102 (Case I for a graph that is not simple; KT Lemma 6.2)**.** *Let $d \geq 2$, and let G be a minimal k-dof-graph with $|V(G)| \geq 3$ that is not simple. Suppose that for every k', every minimal k'-dof-graph with at least one and fewer than $|V(G)|$ vertices has a nondegenerate-hinge panel realization, and a generic full-rank realization if it is simple. Then G has a nondegenerate-hinge panel realization (Definition 23.24), of rank $D(|V(G)| - 1) - k$.*

Proof. A minimal k-dof-graph has no loops, as in the proof of Lemma 23.96. As G is not simple, it has two edges $e \neq f$ that join the same vertices $a \neq b$. Let H' be the subgraph with the vertices a and b and the edges e and f. It is a 0-dof-graph (Lemma 20.36; KT p. 673), and a proper rigid subgraph of G as $|V(G)| \geq 3$. Let $G' = G/E(H')$ be its contraction to a. By KT Lemma 3.5 (Lemma 21.9), G' is a minimal k-dof-graph with $|V(G)| - 1$ vertices. So, by the hypothesis, G' has a nondegenerate-hinge panel realization (G', p_2), of rank $D(|V(G)| - 2) - k$.

As Katoh and Tanigawa do, we give a and b the panel of the body a in (G', p_2), and every other body its panel in (G', p_2). We choose two linearly independent extensors C_e and C_f in the panel of a (Lemma 23.31, in the form for K^{d+1} given after it). Let p give e and f the hinges C_e and C_f, and every other edge its hinge in (G', p_2) (KT eq. (6.2)). Every hinge of p is nonzero. It lies in the panels of the ends of its edge: the ends of an edge in G' are its ends in G with b replaced by a (Definition 21.3), and a and b have the same panel.

We compute the rank as in the proof of Lemma 23.87 for KT Lemma 6.2. By KT Lemma 5.3 (Lemma 23.30), the restriction of (G, p) to H' is infinitesimally rigid, so the rows of e and f have rank D. The other edges have their hinges in (G', p_2). After the columns of a and b are deleted, their rows are those of $R(G', p_2)$ with the columns of a deleted (KT eq. (6.4)). By KT Lemma 5.1, these have the rank of $R(G', p_2)$ (KT eq. (6.5); Lemma 19.35). By the block form of KT eq. (6.3),

$$\operatorname{rank} R(G, p) \;\geq\; D + D(|V(G)| - 2) - k \;=\; D(|V(G)| - 1) - k$$

(Lemma 19.33). As every hinge is nonzero, the rank is at most this (Lemma 20.16). □

Formalization note 23.103. The pinned theorem proves more than the statement, which keeps the induction hypothesis in the form that Case I has. It assumes of the graphs with fewer vertices only that they have nondegenerate-hinge panel realizations, as the proof above uses no other. It is the form that the formal proof of KT Theorem 5.5 uses (Lemma 23.107, Lemma 23.109).

The third case rests on KT Claim 6.6 (p. 676), which Katoh and Tanigawa state under the hypotheses of KT Lemma 6.5, for every k.

Lemma 23.104 (A vertex of degree two; KT Claim 6.6)**.** *Let $D \geq 2$, and let G be a simple minimal k-dof-graph that has a proper rigid subgraph, and in which no proper rigid subgraph H has a simple contraction $G/E(H)$ (Definition 21.3). Then $k = 0$, and G has a vertex v of degree two, joined to two distinct vertices, such that G_v (Definition 21.10) is a simple minimal 0-dof-graph.*

Proof. Let G' be a proper rigid subgraph of G whose vertex set is maximal under inclusion. The subgraph of G induced on $V(G')$ contains G' and has the same

vertices, so its deficiency is at most that of G', which is zero (Definition 20.3). So we may take G' induced. Then an edge of G not in G' has at most one end in $V(G')$, and $G/E(G')$ has no loop. By the hypothesis, $G/E(G')$ is not simple, so it has two edges $e \neq f$ with the same ends. As G is simple, e and f are not parallel in G, so one of these ends is the vertex r to which G' is contracted. So each of e and f has exactly one end in $V(G')$, and they join a common vertex $v \notin V(G')$ to two vertices $a \neq b$ of G'. Katoh and Tanigawa read off v, e and f from the non-simplicity of $G/E(G')$ directly (KT p. 676). This needs G' induced: otherwise an edge of G with both ends in $V(G')$ could give a loop of $G/E(G')$ instead.

Let G'' be obtained from G' by adding v and the edges e and f. Removing v from G'' gives G', so $\operatorname{def}(\tilde{G''}) \leq \operatorname{def}(\tilde{G'}) = 0$, by KT Lemma 4.4 (Lemma 21.38). So G'' is a rigid subgraph of G with more vertices than G'. By the choice of G' it is not proper, so $V(G'') = V(G)$. As G contains G'' and has the same vertices, $k = \operatorname{def}(\tilde{G}) \leq \operatorname{def}(\tilde{G''}) = 0$ (Definition 20.3). As G has vertices, the partition $\{V(G)\}$ has value 0, so $\operatorname{def}(\tilde{G}) \geq 0$ and $k = 0$. Let e' be an edge of G not in G''. Then $G - e'$ contains G'' and has the same vertices, so its deficiency is zero, as is that of G. This contradicts the minimality of G (the remark after Definition 20.8), so $G = G''$. Hence v has degree two in G, with the neighbours $a \neq b$, and $G_v = G'$ is a minimal 0-dof-graph (KT Lemma 3.3, Lemma 20.25). It is simple, as a subgraph of G. □

Formalization note 23.105. The first pinned theorem gives $k = 0$, and the second, for $k = 0$, the vertex v.

Lemma 23.106 (Case I with no simple contraction; KT Lemma 6.5)**.** *Let $d \geq 2$, let K be an infinite field, and let G be a simple minimal 0-dof-graph that has a proper rigid subgraph, and in which no proper rigid subgraph H has a simple contraction $G/E(H)$. Suppose that every minimal 0-dof-graph with at least one and fewer than $|V(G)|$ vertices has a nondegenerate-hinge panel realization, and a generic full-rank realization if it is simple. Then G has a generic full-rank realization (Definition 23.21).*

Proof. By KT Claim 6.6 (Lemma 23.104), G has a vertex v of degree two, joined to two distinct vertices a and b, such that G_v is a simple minimal 0-dof-graph. A proper rigid subgraph of G has at least two vertices and not all of them, so $|V(G)| \geq 3$, and G_v has $|V(G)| - 1 \geq 2$ vertices. So it has a generic full-rank realization, of rank $D(|V(G)| - 2)$. Katoh and Tanigawa keep this realization, and give v a panel Π° that is nonparallel to the other panels and does not contain $\Pi(a) \cap \Pi(b)$. Then the hinges $\Pi^\circ \cap \Pi(a)$ and $\Pi^\circ \cap \Pi(b)$ of va and vb are distinct (KT p. 676).

We choose all the normals at once, as in the proof of Lemma 23.100, at a non-root q of the product of three nonzero polynomials. The first is a polynomial Q that does not vanish at the normals of this realization, such that $R(G_v, q)$ has rank at least $D(|V(G)| - 2)$ wherever $Q(q) \neq 0$ (Lemma 23.120). The polynomial Q is the minor by which the proof of Lemma 23.112 gives the lower semicontinuity of the rank. The second is the polynomial P of the proof of Lemma 23.100, so that the normals q are in general position.

The third is a 3×3 minor Δ of the matrix with the rows $q(v)$, $q(a)$ and $q(b)$, which is not the zero polynomial as $d + 1 \geq 3$. As $\Delta(q) \neq 0$, the normals $q(v)$, $q(a)$

and $q(b)$ are linearly independent. Since $q(a)$ and $q(b)$ are linearly independent, this says that the panel of v does not contain the intersection of the panels of a and b, as Katoh and Tanigawa ask of Π°.

Let (G, q) be the panel-hinge framework on G with the normals q. The edges of G_v have the same hinges in (G, q) and in (G_v, q), so their rows are those of $R(G_v, q)$, and these vanish in the columns of v. The hinges of va and vb are $C(q(v), q(a))$ and $C(q(v), q(b))$. They are linearly independent, because the joins $q(v) \vee q(a)$ and $q(v) \vee q(b)$ are (Lemma 23.4). Indeed, if $q(v)$, $q(a)$ and $q(b)$ are extended to a basis of K^{d+1}, the two joins belong to the corresponding basis of $\bigwedge^2 K^{d+1}$ (the remark after Definition 18.3). So, as in the proof of KT Lemma 5.3 (Lemma 19.21), the rows of va and vb have rank D in the columns of v. This is the block form of KT eq. (6.10).

Take D of these rows that are linearly independent in the columns of v, and $D(|V(G)| - 2)$ linearly independent rows of G_v. In a vanishing linear combination of them, only the first have nonzero entries in the columns of v, so their coefficients vanish, and then so do the others. Hence

$$\operatorname{rank} R(G, q) \;\geq\; D + D(|V(G)| - 2) \;=\; D(|V(G)| - 1),$$

and the rank is at most this (Lemma 19.18). As the normals q are in general position, (G, q) is a generic full-rank realization of G. □

Lemma 23.107 (The three cases of Case I; KT Lemmas 6.2, 6.3 and 6.5)**.** *Let $d \geq 2$, let K be an infinite field, and let G be a minimal* 0*-dof-graph that has a proper rigid subgraph. Suppose that for every k', every minimal k'-dof-graph with at least one and fewer than $|V(G)|$ vertices has a nondegenerate-hinge panel realization, and a generic full-rank realization if it is simple. Then G has a nondegenerate-hinge panel realization (Definition 23.24), and if G is simple, a generic full-rank realization (Definition 23.21).*

Proof. The hypothesis is the induction hypothesis KT eq. (6.1) for the graphs with fewer vertices than G. A proper rigid subgraph of G has at least two vertices and not all of them, so $|V(G)| \geq 3$. We follow Katoh and Tanigawa's three cases (KT p. 673). If G is not simple, then KT Lemma 6.2 gives a nondegenerate-hinge panel realization (Lemma 23.102). Let G be simple. Suppose that some proper rigid subgraph H has a simple contraction $G/E(H)$. Then H and $G/E(H)$ are simple minimal 0-dof-graphs with fewer vertices than G (KT Lemmas 3.3 and 3.5). The induction hypothesis gives generic full-rank realizations of both, and KT Lemma 6.3 gives one of G (Lemma 23.100). Otherwise, KT Lemma 6.5 gives one (Lemma 23.106). In both cases, G also has a nondegenerate-hinge panel realization (Lemma 23.25). □

Formalization note 23.108. The pinned theorem is Case I for $k = 0$; the next lemma is Case I for every k.

Case I for every k makes the same case split, and for $k > 0$ its third case does not occur, by the first assertion of KT Claim 6.6.

Lemma 23.109 (Case I; KT §6.2)**.** *Let $d \geq 2$, let K be an infinite field, and let G be a minimal k-dof-graph that has a proper rigid subgraph. Suppose that for every*

k', every minimal k'-dof-graph with at least one and fewer than $|V(G)|$ vertices has a nondegenerate-hinge panel realization, and a generic full-rank realization if it is simple. Then G has a nondegenerate-hinge panel realization (Definition 23.24), and if G is simple, a generic full-rank realization (Definition 23.21).

Proof. For $k = 0$ this is Lemma 23.107. Let $k > 0$. As there, $|V(G)| \geq 3$, since a proper rigid subgraph of G has at least two vertices and not all of them. If G is not simple, then KT Lemma 6.2 gives a nondegenerate-hinge panel realization (Lemma 23.102). Let G be simple. Suppose that some proper rigid subgraph H has a simple contraction $G/E(H)$. Then H is a simple minimal 0-dof-graph and $G/E(H)$ a simple minimal k-dof-graph, both with fewer vertices than G (KT Lemmas 3.3 and 3.5). The hypothesis gives generic full-rank realizations of both, and KT Lemma 6.3 gives one of G (Lemma 23.100), which is also a nondegenerate-hinge panel realization (Lemma 23.25). Otherwise no proper rigid subgraph of G has a simple contraction, and KT Claim 6.6 gives $k = 0$ (Lemma 23.104), a contradiction. □

23.7. **The rank at generic normals.** Katoh and Tanigawa call a nonparallel panel-hinge realization generic if the coordinates of the normals of its panels are algebraically independent over $\mathbb{Q}$. The entries of the rigidity matrix are polynomials in these coordinates, so a generic realization has the largest rank of all nonparallel realizations of its graph [KT11, §5.1, p. 668]. Jackson and Jordán made this observation for panel-and-hinge frameworks [JJ10, §7], and Katoh and Tanigawa refer to them for it. Their proof uses it in KT Claim 6.4 (p. 675) and for the realizations that the induction hypothesis gives (KT footnote 6, p. 684). We replace algebraic independence by the non-vanishing of polynomials in the normals, as explained after Lemma 23.100. Where every hinge is nondegenerate, a bound on the rank gives as many linearly independent rows of the form of Lemma 23.91 (Lemma 23.110), and so a minor of these rows, a polynomial in the normals, that does not vanish. Lemma 23.112 states the resulting lower semicontinuity of the rank on the Zariski-open set of such normals. The proofs of KT Lemmas 6.3 and 6.8 (Lemma 23.100, Lemma 23.135) keep the minor instead, in the form of Lemma 23.120, in the next subsection. Given normals at which the rank is at least N, that lemma gives a polynomial in the normals that does not vanish there, off whose zeros the rank is at least N. These proofs choose their normals off the zeros of a product of such polynomials, and bound the rank from above by Lemma 20.16.

Lemma 23.110 (Independent rows from a bound on the rank)**.** *Let (G, p) be a body-hinge framework whose hinges are nondegenerate. At each edge e of G, with ends u and w, take the linear forms $S \mapsto C_t\, x_{t'} - C_{t'}\, x_t$, where $C = C(p(e))$ and $x = S(u) - S(w)$, for all pairs of indices t and t', as in Lemma 23.91. If $N \leq \operatorname{rank} R(G, p)$, then some N of these forms are linearly independent.*

Proof. As the hinges are nondegenerate, the forms at each edge span the same space as the rows of $R(G, p)$ at that edge (the proof of Lemma 23.91). So all the forms span the row space of $R(G, p)$, whose dimension is $\operatorname{rank} R(G, p) \geq N$, and some N of them are linearly independent. □

Formalization note 23.111. The pinned theorem reads the ends of each edge from the endpoint selector (Formalization note 23.6), and indexes the forms by the edge and the pair of indices. The last step of the proof of Lemma 23.124, in the next subsection, is the case of a framework rigid on $V(G)$, with $N = D(|V(G)| - 1)$.

Lemma 23.112 (Lower semicontinuity of the rank at nondegenerate hinges; KT §5.1)**.** *Let K be an infinite field and let G be a multigraph. Regard the normals q of the panel-hinge frameworks (G, q) on a finite set W of bodies as the points of K^N, where $N = (d + 1)|W|$, and let U be the set of the q at which every hinge of (G, q) is nondegenerate (Definition 23.5). Then U is Zariski-open, and for each r, the set of the $q \in U$ with* $\operatorname{rank} R(G, q) \geq r$ *is Zariski-open; that is, the rank is lower semicontinuous on U. The rank is at least its value at a given $q_0 \in U$ on a dense Zariski-open set contained in U, and if U is not empty, the largest rank on U is attained on such a set. Equivalently,* $\dim Z(G, q) = D|W| - \operatorname{rank} R(G, q)$ *is upper semicontinuous on U, and its least value on U is attained on a dense Zariski-open set contained in U.*

Proof. The set U contains the normals of Katoh and Tanigawa's nonparallel realizations of a simple graph, as these normals are in general position (the remark after Definition 23.10 and Lemma 23.11). The remark after the proof shows that the lemma fails outside U. The set U is Zariski-open, as the coordinates of the hinge $C(q(u), q(w))$ of an edge with ends u and w are polynomials in q (Definition 23.1). Choose the rows of $R(G, q)$ at each edge as in Lemma 23.91. Their coordinates are polynomials of degree two in q, and at every $q \in U$ they span the row space of $R(G, q)$.

So for $q \in U$, the rank of $R(G, q)$ is at least r if and only if some r of these rows are linearly independent at q (Lemma 23.110), that is, if and only if some $r \times r$ minor of the matrix of these rows does not vanish at q. Each minor Δ is a polynomial in q. Hence the set of the $q \in U$ with $\operatorname{rank} R(G, q) \geq r$ is the intersection of U with the union of the finitely many sets $\{\Delta \neq 0\}$, and it is Zariski-open. If $\operatorname{rank} R(G, q_0) \geq r$ at some $q_0 \in U$, then some minor Δ has $\Delta(q_0) \neq 0$. The rank is then at least r on the open set $U \cap \{\Delta \neq 0\}$, which contains q_0. As K is infinite, a nonzero polynomial has a non-root, and so has a product of nonzero polynomials. Hence any two nonempty Zariski-open subsets of K^N meet, and each of them is dense. Taking for q_0 normals in U of the largest rank gives the assertion on the largest rank. The assertions on $Z(G, q)$ are the same, as $\dim Z(G, q) = D|W| - \operatorname{rank} R(G, q)$ (Definition 19.7). □

The restriction to U is needed. If the normals at the ends u and w of an edge are linearly dependent, its hinge is degenerate, the hinge constraint welds u and w (Definition 23.5), and the rows of the edge have rank D rather than $D - 1$. So for the graph with one edge uw, $\operatorname{rank} R(G, q)$ is D where $q(u)$ and $q(w)$ are linearly dependent and $D - 1$ elsewhere. The set where the rank is D is Zariski-closed and not open, and the largest rank is attained only there. The rows of Lemma 23.91 vanish at a degenerate hinge, and the rank of the matrix of these rows, unlike that of $R(G, q)$, is lower semicontinuous at every q.

Formalization note 23.113. The pinned theorem is that of Lemma 23.120, which this lemma states in the terms of KT §5.1. At normals $q_0 \in U$ with $\operatorname{rank} R(G, q_0) \geq$

r, it gives a polynomial Q with $Q(q_0) \neq 0$ such that $\operatorname{rank} R(G,q) \geq r$ wherever $Q(q) \neq 0$. So the set of the $q \in U$ with $\operatorname{rank} R(G,q) \geq r$ contains, with each of its points q_0, the points of U off the zeros of Q: it is open in U, which is the lower semicontinuity on U. The pinned theorem holds over every field. The other clauses are the proof's: U is Zariski-open, as the coordinates of the hinges are polynomials in q; the open sets are dense, as K is infinite; the largest rank is the case of normals $q_0 \in U$ of the largest rank; and the assertions on $Z(G,q)$ follow by the remark after Definition 19.7. The pinned theorem reads the ends of each edge from the endpoint selector (Formalization note 23.6).

23.8. **Counting rows and motions.** As in Remark 23.23, let the frameworks on a multigraph G have a finite set $W \supseteq V(G)$ of bodies, those outside $V(G)$ on no edge. Each body of $W \setminus V(G)$ adds D dimensions to $Z(G,q)$, and every infinitesimal motion is constant on $V(G)$ exactly when $\dim Z(G,q) = D(|W \setminus V(G)| + 1)$. The first four lemmas count in this form. They compute the motions that vanish on $V(G)$ (Lemma 23.114), and pass from independent rows, or from a bound on $\dim Z(G,q)$, to motions constant on $V(G)$. Then comes the form of Lemma 23.112 that the realization lemmas use, which keeps the minor of its proof as a polynomial (Lemma 23.120), with a superseded application of it to the contraction of Case I (Lemma 23.121). The subsection ends with the count of KT eq. (6.12) when the graph obtained by splitting off a vertex of degree two is rigid (Lemma 23.128), and the four lemmas on rows before it. The remark at the start of Case II (Remark 23.130) says which of these lemmas the formal proof of KT Theorem 5.5 does not use, and what it uses instead.

Lemma 23.114 (The block pin of $V(G)$)**.** *Let (G,p) be a body-hinge framework whose screw assignments are defined on a finite set W of bodies with $V(G) \subseteq W$. Then*

$$\dim Z_{V(G)}(G,p) \;=\; D\,|W \setminus V(G)|.$$

Proof. A screw assignment that vanishes on $V(G)$ satisfies every hinge constraint, as both ends of every edge lie in $V(G)$. So the block pin $Z_{V(G)}(G,p)$ (Definition 23.82) consists of all the screw assignments that vanish on $V(G)$. Such an assignment takes an arbitrary value in K^D at each body of $W \setminus V(G)$, so these assignments form a space of dimension $D|W \setminus V(G)|$. □

Lemma 23.115 (Independent rows bound the motions)**.** *Let K be an infinite field, and let G be a multigraph with $V(G) \neq \emptyset$, whose panel-hinge frameworks have a finite set $W \supseteq V(G)$ of bodies. Suppose that at some normals q_0, some $m \geq D(|V(G)|-1)$ rows of $R(G,q_0)$, chosen as in Lemma 23.91, are linearly independent. Then there are normals q with*

$$\dim Z(G,q) \;\leq\; D\,(|W \setminus V(G)| + 1).$$

Proof. The normals q_0 will do. The m rows lie in the row space of $R(G,q_0)$, so

$$\dim Z(G,q_0) \;=\; D|W| - \operatorname{rank} R(G,q_0) \;\leq\; D|W| - D(|V(G)|-1),$$

and $|W| = |V(G)| + |W \setminus V(G)|$. □

Formalization note 23.116. The pinned theorem takes an $m \times m$ minor of the m rows that does not vanish at q_0, as in the proof of Lemma 23.120, and normals q at which it does not vanish (`exists_good_realization_ofParam`). At q the m rows are again linearly independent, and they lie in the row space of $R(G, q)$ (Lemma 23.91), so $m + \dim Z(G, q) \leq D|W|$. It then substitutes $|W| = |V(G)| + |W \setminus V(G)|$, as above. It assumes that every edge label is an edge of G, whose ends the endpoint selector gives (Formalization note 23.6). No main result uses it.

Lemma 23.117 (Rigidity on $V(G)$ from a bound on the motions)**.** *Let (G, p) be a body-hinge framework whose screw assignments are defined on a finite set W of bodies with $\emptyset \neq V(G) \subseteq W$. If*

$$\dim Z(G, p) \;\leq\; D\,(|W \setminus V(G)| + 1),$$

then every infinitesimal motion of (G, p) is constant on $V(G)$.

Proof. Fix a body $v_0 \in V(G)$. A motion that vanishes on $V(G)$ vanishes at v_0, so $Z_{V(G)}(G, p) \subseteq \text{pinned}(v_0)$. By KT Lemma 5.1 (Lemma 19.20), the hypothesis and Lemma 23.114,

$$\dim \text{pinned}(v_0) \;=\; \dim Z(G, p) - D \;\leq\; D\,|W \setminus V(G)| \;=\; \dim Z_{V(G)}(G, p).$$

So the two subspaces are equal: a motion that vanishes at v_0 vanishes on $V(G)$. Every motion S is the sum of the trivial motion with value $S(v_0)$ and a motion that vanishes at v_0 (Lemma 19.20), so S is constant on $V(G)$. This is the criterion in the note after Lemma 23.87, with $s = \{v_0\}$ and $t = V(G)$. □

Lemma 23.118 (A rigid realization from independent rows)**.** *Let K be an infinite field, and let G be a multigraph with $V(G) \neq \emptyset$. Suppose that at some normals q_0, some $m \geq D(|V(G)| - 1)$ rows of $R(G, q_0)$, chosen as in Lemma 23.91, are linearly independent. Then G has a panel-hinge realization of rank $D(|V(G)| - 1)$, that is, one whose infinitesimal motions are all constant on $V(G)$ (Remark 23.23).*

Proof. Let W be the set of bodies. By Lemma 23.115 and its proof, the normals $q = q_0$ satisfy $\dim Z(G, q) \leq D(|W \setminus V(G)| + 1)$. By Lemma 23.117, every infinitesimal motion of (G, q) is then constant on $V(G)$. □

Formalization note 23.119. The pinned theorem takes q from the pinned theorem of Lemma 23.115, under the same assumption on the endpoint selector. Its conclusion is the first predicate of Formalization note 23.22. No main result uses it.

Lemma 23.120 (A rank bound off the zeros of a polynomial)**.** *Let G be a multigraph, and let q_0 be normals at which every hinge of the panel-hinge framework (G, q_0) is nondegenerate. If $\operatorname{rank} R(G, q_0) \geq N$, then there is a polynomial Q in the normals with $Q(q_0) \neq 0$ such that $\operatorname{rank} R(G, q) \geq N$ at every q with $Q(q) \neq 0$.*

Proof. Choose the rows of $R(G, q)$ at each edge as in Lemma 23.91. As the hinges of (G, q_0) are nondegenerate, these rows span the row space of $R(G, q_0)$, so some N of them are linearly independent at q_0. The matrix of their coordinates then has an $N \times N$ minor that does not vanish at q_0. Its entries are polynomials in the normals, and we let Q be this minor, as a polynomial in the normals.

Wherever $Q(q) \neq 0$, the same N rows are linearly independent at q. Each of them is a form $S \mapsto \varphi(S(u) - S(w))$ at an edge with ends u and w, where φ vanishes on the hinge of the edge at q (Lemma 23.91). So it lies in the row space of $R(G, q)$ (Definition 19.5), and $\operatorname{rank} R(G, q) \geq N$. This is the argument of Lemma 23.112, with the minor kept. □

The next lemma is superseded and kept for the record; no other node uses it. It gives the rank of the reoriented realization of $G/E(H)$ itself (Lemma 23.94) off the zeros of a polynomial, for Lemma 19.33 with $\pi = \pi_{V(H)}$. At normals in general position for G, however, an edge of $G \setminus E(H)$ at a body of $V(H)$ has different hinges in G and in $G/E(H)$, so the second hypothesis of that lemma need not hold. Case I uses Lemma 23.96 instead.

Lemma 23.121 (The rank of the reoriented contraction off the zeros of a polynomial (superseded))**.** *Let H, r, f and $G_1 = G/E(H)$, with their orientations, be as in Lemma 23.94. Suppose that G_1 has no loops and has a panel-hinge realization whose normals are in general position and whose rank is $N = D(|V(G_1)| - 1) - \operatorname{def}(\tilde{G}_1)$. Then there is a nonzero polynomial Q in the panel normals such that at every q with $Q(q) \neq 0$, the panel-hinge framework on G_1 with normals q and this orientation has rank at least N.*

Proof. By Lemma 23.94 there are normals n in general position at which this framework has rank N. As G_1 has no loops, its hinges at n are nonzero (Lemma 23.11). So Lemma 23.120, at n with the bound N, gives a polynomial Q with $Q(n) \neq 0$, so that $Q \neq 0$, such that the rank is at least N wherever Q does not vanish. □

We turn to KT eq. (6.12) when $k = 0$. Let G be a multigraph, let v be a vertex of G of degree two whose edges join it to vertices a and b, and let G_v and G_v^{ab} be obtained from G by removing v and by splitting off v (Definition 21.10). In Case II, Katoh and Tanigawa put v back into a realization of G_v^{ab}, with the hinge of vb equal to that of ab, and the rank rises by $D-1$ (Lemma 23.135). In Case III, where $k = 0$, the graph G_v^{ab} is a minimal 0-dof-graph (KT Lemma 4.8(i), Lemma 21.52), and the same count gives one row less than the target rank $D(|V(G)| - 1)$ (KT p. 680). Case III supplies the missing row (Section 23.10). Lemma 23.128 states this count, and three of the next four lemmas give its parts. The third, on rows shared by two frameworks, states the consequence of KT eq. (6.13), which the proof replaces by a row operation (Remark 23.130). In all four, as in Lemma 23.120, the rows of $R(G, q)$ at an edge are the linear forms of Lemma 23.91, indexed by the edge and a pair of coordinates. At a nondegenerate hinge they span the rows of the edge. The second and the fourth are stated in the generality in which other chapters use them: for every framework whose motions are constant on its vertex set, and for any two families of linear forms.

Lemma 23.122 (The $D-1$ rows of a nondegenerate hinge)**.** *Let (G, q) be a panel-hinge framework, and let e be an edge of G with distinct ends whose hinge is nondegenerate. Then some $D-1$ of the rows of $R(G, q)$ at e are linearly independent.*

Proof. Let u and w be the ends of e, and let $C \neq 0$ be its hinge. The linear forms on K^D that vanish on C form a space of dimension $D-1$ (Definition 19.5). The map

$S \mapsto S(u) - S(w)$ takes the screw assignments onto K^D, as $u \neq w$. So composing with it gives a space of linear forms on the screw assignments of dimension $D-1$ (Lemma 19.9). The rows of $R(G,q)$ at e span this space (Lemma 23.91), and $D-1$ of them are linearly independent. □

Formalization note 23.123. The first pinned theorem gives $D-1$ linearly independent forms in the span of the rows at e, and the second gives $D-1$ of the rows themselves. Both hold for every body-hinge framework, and read the ends of e from the endpoint selector (Formalization note 23.6).

Lemma 23.124 (The motions and independent rows of a framework rigid on $V(G)$)**.** *Let (G,p) be a body-hinge framework whose screw assignments are defined on a finite set W of bodies with $\emptyset \neq V(G) \subseteq W$, and suppose that every infinitesimal motion of (G,p) is constant on $V(G)$. Then*

$$\dim Z(G,p) \;=\; D\,(|W \setminus V(G)| + 1).$$

If moreover $C(p(e)) \neq 0$ at every edge e of G, that is, every hinge is nondegenerate, then some $D(|V(G)|-1)$ rows of $R(G,p)$ are linearly independent.

Proof. Fix a body $w \in V(G)$. A motion that vanishes at w vanishes on $V(G)$, as it is constant there. So $\text{pinned}(w) = Z_{V(G)}(G,p)$, and by KT Lemma 5.1 (Lemma 19.20) and Lemma 23.114,

$$\dim Z(G,p) \;=\; D + D\,|W \setminus V(G)|.$$

Hence $\operatorname{rank} R(G,p) \;=\; D|W| - \dim Z(G,p) \;=\; D(|V(G)|-1)$. If the hinges are nondegenerate, the rows of $R(G,p)$ span its row space (the proof of Lemma 23.91), and $D(|V(G)|-1)$ of them are linearly independent. □

Formalization note 23.125. The second pinned theorem is the dimension of $Z(G,p)$, and the first the independent rows. The first reads the ends of each edge from the endpoint selector (Formalization note 23.6), and it asks both the selector's ends and a nondegenerate hinge at every edge label, so that every label is an edge. No main result uses it.

Lemma 23.126 (Rows shared by two frameworks)**.** *Let (G_1,q_1) and (G_2,q_2) be panel-hinge frameworks. Index the rows of each rigidity matrix by an edge and a pair of coordinates, as in Lemma 23.91. Let s_1 be a set of indices at which the rows of $R(G_1,q_1)$ are linearly independent, and let $f\colon s_2 \to s_1$ be injective, such that the row of $R(G_2,q_2)$ at each $i \in s_2$ is the row of $R(G_1,q_1)$ at $f(i)$. Then the rows of $R(G_2,q_2)$ at the indices in s_2 are linearly independent.*

Proof. These rows are the rows of $R(G_1,q_1)$ at the indices $f(i)$, which are distinct as f is injective. A subfamily of a linearly independent family is linearly independent.

The row at an index depends only on the ends of its edge and their normals (Lemma 23.91). So the hypothesis holds, with f the identity, for indices at edges that G_1 and G_2 share with the same ends, when q_1 and q_2 agree at these ends. For instance, G and G_v^{ab} share the edges of G_v (Lemma 21.12). □

Lemma 23.127 (Forms independent at v and forms that vanish at v)**.** *Let v be a body, and let $\rho_1, \dots, \rho_m$ and $\sigma_1, \dots, \sigma_l$ be linear forms on the screw assignments,*

the maps from the bodies to the screw space (Definition 19.1). Suppose that the restrictions of $\rho_1, \dots, \rho_m$ to the screw assignments that vanish off v are linearly independent, and that $\sigma_1, \dots, \sigma_l$ are linearly independent and vanish on these screw assignments. Then $\rho_1, \dots, \rho_m, \sigma_1, \dots, \sigma_l$ are linearly independent.

Proof. Suppose that $\sum_i a_i\rho_i + \sum_j b_j\sigma_j = 0$, and let S be a screw assignment that vanishes off v. Then $\sigma_j(S) = 0$ for every j, and hence $\sum_i a_i\rho_i(S) = 0$. By the hypothesis on the ρ_i, every a_i is 0. Then $\sum_j b_j\sigma_j = 0$, and every b_j is 0. □

For instance, a row of a rigidity matrix at an edge whose ends differ from v vanishes on the screw assignments that vanish off v. The proof of Lemma 23.128 applies the lemma to rows of two frameworks, and Lemma 19.34 to forms in the row space of a rigidity matrix.

Lemma 23.128 (KT eq. (6.12) when G_v^{ab} is rigid: one row short)**.** *Let G be a multigraph, let v be a vertex of G of degree two whose edges join it to vertices a and b, and let G_v^{ab} be obtained by splitting off v (Definition 21.10). Let (G_v^{ab}, q) be a panel-hinge framework with nondegenerate hinges, every infinitesimal motion of which is constant on $V(G_v^{ab})$. Let $t \neq 0$, and let q_0 be the normals with $q_0(v) = q(a) + t\,q(b)$ and $q_0(w) = q(w)$ for every body $w \neq v$. Then some $D(|V(G)| - 1) - 1$ rows of $R(G, q_0)$ are linearly independent.*

Proof. We first make the count from a realization (G_v^{ab}, q) with nondegenerate hinges whose rigidity matrix has rank at least N, for any N; the proof of Lemma 23.135, in the next subsection, takes it with $t = 1$. Let $n_a = q(a)$ and $n_b = q(b)$. The hinge $C = C(n_a, n_b)$ of the edge ab of (G_v^{ab}, q) is nonzero. As C is bilinear and $C(n, n) = 0$ (Definition 23.1),

$$C(n_a + t\,n_b,\, n_b) \;=\; C, \qquad C(n_a + t\,n_b,\, n_a) \;=\; -t\,C.$$

So, up to sign, the edges vb and va of (G, q_0) have the hinges C and $-tC$, both nondegenerate as $t \neq 0$. As v has degree two, every other edge of G is an edge of G_v, and so of G_v^{ab} (Lemma 21.12). Its ends differ from v, so it has the same hinge, up to sign, in (G, q_0) as in (G_v^{ab}, q), and this hinge is nondegenerate.

The normals q_0 and q differ only at v, which lies on no edge of G_v^{ab}. So (G_v^{ab}, q_0) has the hinges and the infinitesimal motions of (G_v^{ab}, q) (Lemma 23.16), and so its row space (Lemma 19.25). Some N rows $\sigma_1, \dots, \sigma_N$ of $R(G_v^{ab}, q_0)$ are linearly independent (Lemma 23.110). Each of them lies in the row space of $R(G, q_0)$. A row at an edge of G_v is, up to sign, a row of $R(G, q_0)$ at that edge, as the edge has the same hinge in both frameworks up to sign. A row at the edge ab is, up to sign, a form $S \mapsto \varphi(S(a) - S(b))$, where φ is a linear form on K^D that vanishes on C. The forms $S \mapsto \varphi(S(v) - S(b))$ and $S \mapsto \varphi(S(v) - S(a))$ lie in the row spaces of the edges vb and va of (G, q_0), as φ vanishes on their hinges, and

$$\varphi(S(a) - S(b)) \;=\; \varphi(S(v) - S(b)) - \varphi(S(v) - S(a))$$

(Lemma 23.161). This is the row form of Katoh and Tanigawa's column operation, which adds the columns of v to those of a (KT eqs. (6.14)–(6.16)).

By Lemma 23.122, some $D-1$ rows $\rho_1, \dots, \rho_{D-1}$ of $R(G, q_0)$ at vb are linearly independent. Up to sign, they are the forms $S \mapsto \varphi_i(S(v) - S(b))$ for linearly independent

forms φ_i on K^D. On the screw assignments that vanish off v, they are the forms $S \mapsto \varphi_i(S(v))$, which are again linearly independent. The σ_j vanish there, as no edge of G_v^{ab} has v as an end. So the ρ_i and the σ_j are linearly independent together (Lemma 23.127), and they lie in the row space of $R(G, q_0)$. Hence (Lemma 19.34)

$$\operatorname{rank} R(G, q_0) \;\geq\; N + D - 1.$$

Now let every infinitesimal motion of (G_v^{ab}, q) be constant on $V(G_v^{ab})$, whose vertices are those of G other than v (Definition 21.10). Then $R(G_v^{ab}, q)$ has rank $N = D(|V(G)| - 2)$, by the dimension of its motion space (Lemma 23.124), and

$$\operatorname{rank} R(G, q_0) \;\geq\; D(|V(G)| - 2) + (D - 1) \;=\; D(|V(G)| - 1) - 1.$$

As the hinges of (G, q_0) are nondegenerate, its rows span its row space. So some $D(|V(G)| - 1) - 1$ of them are linearly independent (Lemma 23.110). Katoh and Tanigawa note this count for $k = 0$ on p. 680 [KT11]. □

The rows at ab cannot be left out of this count when $|V(G)| \geq 3$ and G has no proper rigid subgraph (Definition 20.17), as in Case III. Suppose that some $D(|V(G)|-2)$ rows of $R(G_v^{ab}, q)$ at the edges of G_v were linearly independent. They are rows of $R(G_v, q)$, so the realization (G_v, q), whose hinges are nondegenerate, would have rank at least $D(|V(G_v)| - 1)$. Then $\operatorname{def}(\tilde{G}_v) = 0$ (Lemma 20.16), and G_v, with its $|V(G)| - 1 \geq 2$ vertices, would be a proper rigid subgraph of G. This is why v is put back with the hinge of vb equal to that of ab.

Formalization note 23.129. The first pinned theorem is the lemma. It reads the ends of the edges of G and of G_v^{ab} from two endpoint selectors (Formalization note 23.6), and asks that the label of the new edge ab not be an edge of G_v, so that the graph obtained is G_v^{ab} (Definition 21.10). The second pinned theorem is the count from a realization of G_v^{ab} with nondegenerate hinges whose rigidity matrix has rank at least N, for any N: the hinges of (G, q_0) are nondegenerate, and $\operatorname{rank} R(G, q_0) \geq N+D-1$. The proof of Lemma 23.135 applies it with $t = 1$, at the rank $D(|V(G)|-2)-(k-1)$ that the induction hypothesis gives. Both place the rows of G_v^{ab} in the row space of $R(G, q_0)$, as above, and do not use Lemma 23.126. No main result uses the first.

23.9. **Case II: no proper rigid subgraph and $k > 0$.** This subsection proves Case II of KT Theorem 5.5, in which the minimal k-dof-graph G is 2-edge-connected, has at least three vertices and no proper rigid subgraph, and $k > 0$ (KT §6.3). Katoh and Tanigawa split off a vertex v of degree two, whose edges join it to vertices a and b. The graph G_v^{ab} is a minimal $(k-1)$-dof-graph with fewer vertices, simple by KT Lemma 6.7 (Lemma 21.58), and the induction hypothesis realizes it. They then put v back, with the hinge of the edge vb equal to that of the new edge ab, and the rank rises by $D - 1$ (KT Lemma 6.8). Katoh and Tanigawa note that this is essentially the idea of Whiteley's proof that the 1-extension preserves the infinitesimal rigidity of bar-joint frameworks [Whi96, Theorem 2.2.2] (KT p. 650).

The first two lemmas concern putting back the body v. The first pins one body (Lemma 23.131). The second says that the infinitesimal motions of a framework on G are constant on $V(G)$ exactly when they are constant on the other vertices and

agree there with their value at v (Lemma 23.133). The proof of KT Lemma 6.8 (Lemma 23.135) counts rows instead, and does not use them.

Remark 23.130 (Katoh and Tanigawa's steps in Case II and the formal proof)**.** Two lemmas of this subsection and three of Section 23.8 belong to Katoh and Tanigawa's proof of KT Lemma 6.8, or state facts about it for infinitesimal motions. The formal proof of KT Theorem 5.5 (Theorem 23.38) does not take them. We keep them beside the argument they belong to, and say here what the formal proof does instead.

Putting back v. Let (G_v^{ab}, q) be the realization that the induction hypothesis gives, of rank $D(|V(G)| - 2) - (k - 1)$ (KT eq. (6.11), p. 678). Katoh and Tanigawa put v back with the hinge of vb equal to the hinge $q(ab)$ of the edge ab. This gives a realization (G, p_1) (KT eq. (6.12), p. 678). In the formal proof the panel of v has the normal $q(a) + q(b)$, and the realization is (G, q_0) (the proof of Lemma 23.135). The first two lemmas of this subsection state facts about putting back v for infinitesimal motions, which Katoh and Tanigawa do not use. Lemma 23.131 reads KT Lemma 5.1 at a given rank: $\dim Z(G, p) = D + k'$ exactly when the motions that vanish at v form a space of dimension k'. By its second assertion, replacing the normal of v in a panel-hinge framework in which v lies on no edge, as in (G_v^{ab}, q), does not change the dimension of this space. Every motion is constant on $V(G)$ exactly when it is constant on the other vertices and takes there its value at v (Lemma 23.133).

The induction hypothesis gives the rank of $R(G_v^{ab}, q)$. The last step of the formal proof starts from a lower bound on the rank of $R(G, q_0)$ (Lemma 23.120). So the formal proof counts rows throughout, as Katoh and Tanigawa do, and it does not pass to motions. Besides, Lemma 23.133 concerns the rank $D(|V(G)| - 1)$, at which every motion is constant on $V(G)$ (Remark 23.23). In Case II, where $k > 0$, the target rank is smaller.

The rows of G_v^{ab}. The edges of G_v have the same hinges in (G, p_1) as in (G_v^{ab}, q), so the two rigidity matrices have the same rows at these edges (KT eq. (6.13), p. 679). Lemma 23.126 states the consequence for any two frameworks: linearly independent rows of one, each of which is also a row of the other, are linearly independent rows of the other. The edge ab of G_v^{ab} is not an edge of G, however, and its rows are not rows of $R(G, p_1)$. Katoh and Tanigawa add the columns of v to those of a. The rows of vb then have, outside the columns of v, the entries of the rows of $R(G_v^{ab}, q)$ at ab (KT eqs. (6.15)–(6.17), p. 679). The formal proof uses a row operation instead. At q_0 the hinges of vb, va and ab agree up to sign, so each row at ab is the difference of a row at vb and a row at va. Hence every row of $R(G_v^{ab}, q)$ lies in the row space of $R(G, q_0)$ (the proof of Lemma 23.128).

The rank then adds, as in KT eq. (6.16). Some $D - 1$ rows at vb are linearly independent on the screw assignments that vanish off v, and the rows of $R(G_v^{ab}, q)$ vanish there (Lemma 19.34). The linearly independent rows of $R(G_v^{ab}, q)$ that this count takes may include rows at ab. So the formal proof does not use Lemma 23.126.

One row short. For $k = 0$, the graph G_v^{ab} is a minimal 0-dof-graph, and the same count gives one row less than the target rank (KT p. 680). Lemma 23.128 states this count from a realization of G_v^{ab} whose motions are all constant on its vertex set, and its proof makes it first from a realization of any rank. For $k > 0$, the proof of Lemma 23.135 takes the count in this form. In Case III, where $k = 0$, the formal

proof obtains the full rank of each of its frameworks from the count of KT eq. (6.29) (Lemma 23.193).

The rotation. In (G, p_1) the hinges at v lie in the panel of a (KT p. 678). Katoh and Tanigawa rotate the panel of v slightly, which does not lower the rank, to make the realization nonparallel (KT Lemma 5.2, KT p. 679). For the rank $D(|V(G)| - 1)$, two lemmas of Section 23.8 pass instead from linearly independent rows to a realization. Suppose that at some normals, at least $D(|V(G)| - 1)$ rows of the rigidity matrix are linearly independent. Then Lemma 23.115 gives normals at which $\dim Z(G, q)$ is at most its value at that rank. Lemma 23.118 gives a realization of G of that rank. Both hold at the given normals, by their proofs. Their formal counterparts pass to normals off the zeros of a minor of these rows (the notes after them).

Neither lemma serves the formal proof. In Case II the count gives only $D(|V(G)|-1) - k$ rows, with $k > 0$, and the lemmas say nothing of the position of the normals. The formal proof needs a generic full-rank realization: normals in general position, at the target rank (Definition 23.21). Instead of rotating a panel, it chooses all the normals at once. It keeps a minor of the rows as a polynomial in the normals, nonzero at q_0, off whose zeros the rank is at least $D(|V(G)|-1)-k$ (Lemma 23.120). At a common non-root of this polynomial and of the polynomial for general position, the normals are in general position and the rank is at least $D(|V(G)| - 1) - k$ (the proof of Lemma 23.135). As the hinges there are nondegenerate, the rank there is at most the target rank (Lemma 20.16).

By Remark 23.23, a realization of G whose bodies are the vertices of G is at the target rank (Definition 23.21) exactly when $\dim Z(G, p) = D + \operatorname{def}(\tilde{G})$. The first assertion of the next lemma restates KT Lemma 5.1 (Lemma 19.20) in this form.

Lemma 23.131 (Pinning one body)**.** *Let (G, p) be a body-hinge framework, let v be a body, and let k' be an integer. Then* $\dim Z(G, p) = D + k'$ *if and only if the space* $\operatorname{pinned}(v)$ *of the infinitesimal motions S with $S(v) = 0$ has dimension k' (Lemma 19.20). Let (G, n) be a panel-hinge framework in which v lies on no edge, and let (G, n') be obtained by replacing the normal of v (Definition 23.15). Then* $\dim Z(G, n') = D + k'$ *if and only if* $\operatorname{pinned}(v)$ *has dimension k' in (G, n).*

Proof. By Lemma 19.20, $\dim Z(G, p) = D + \dim \operatorname{pinned}(v)$, which gives the first assertion. If v lies on no edge of G, then (G, n) and (G, n') have the same subspace $\operatorname{pinned}(v)$ (Lemma 23.16). So the second assertion is the first, applied to (G, n'). □

Formalization note 23.132. The first pinned theorem is the assertion for body-hinge frameworks, and the second the one for panel-hinge frameworks. As in Lemma 23.16, the second assumes that the pair of bodies that the endpoint selector assigns to each edge of G does not contain v (Formalization note 23.6).

Lemma 23.133 (Putting back a body of degree two)**.** *Let G be a minimal k-dof-graph with $k > 0$, let v be a vertex of G of degree two, and let $t = V(G) \setminus \{v\}$. Then every infinitesimal motion of a body-hinge framework (G, p) is constant on $V(G)$ if and only if every infinitesimal motion S is constant on t and satisfies $S(v) = S(w)$ for every $w \in t$.*

Proof. As v is a vertex of G, we have $V(G) = t \cup \{v\}$. A screw assignment S is constant on $t \cup \{v\}$ if and only if it is constant on t and $S(v) = S(w)$ for every $w \in t$. □

Formalization note 23.134. The pinned theorem is the assertion for every body-hinge framework, every set t of bodies and every body v, with $t \cup \{v\}$ in place of $V(G)$; the statement is its case $t = V(G) \setminus \{v\}$.

Lemma 23.135 (Case II: splitting off a vertex of degree two; KT Lemma 6.8)**.** *Let $d \geq 2$, let K be an infinite field, and let G be a 2-edge-connected minimal k-dof-graph with $k > 0$, $|V(G)| \geq 3$ and no proper rigid subgraph. Suppose that for every k', every minimal k'-dof-graph with at least one and fewer than $|V(G)|$ vertices has a nondegenerate-hinge panel realization, and a generic full-rank realization if it is simple. Then G has a generic full-rank realization (Definition 23.21), of rank $D(|V(G)| - 1) - k$.*

Proof. We follow Katoh and Tanigawa's proof of KT Lemma 6.8 (pp. 677–679). The graph G is simple (Lemma 21.57). As $d \geq 2$, we have $D \geq 3$, so G has a vertex v of degree two (Lemma 21.50; KT Lemma 4.6). Its edges e_a and e_b join it to vertices a and b, and $a \neq b$ as G is simple.

Next, G_v^{ab} is simple, by KT Lemma 6.7 (Lemma 21.58). If $|V(G)| \geq 4$, this is its last assertion. Let $|V(G)| = 3$. If an edge of G joined a and b, then this edge, e_a and e_b would be all the edges of the simple graph G. So G would be a triangle, and a 0-dof-graph by the first assertion of KT Lemma 6.7, contradicting $k > 0$. So no edge of G joins a and b, and G_v^{ab} is simple, as in the proof of Lemma 21.58.

By KT Lemma 4.8(ii) (Lemma 21.53), G_v^{ab} is a minimal $(k-1)$-dof-graph. Its vertices are the $|V(G)| - 1 \geq 2$ vertices of G other than v (Lemma 21.51). So the induction hypothesis gives a generic full-rank realization (G_v^{ab}, q), with

$$\operatorname{rank} R(G_v^{ab}, q) \;=\; D(|V(G)| - 2) - (k-1),$$

which is KT eq. (6.11). Its normals are in general position, so the normals $n_a = q(a)$ and $n_b = q(b)$ are linearly independent.

Katoh and Tanigawa keep the hinges of (G_v^{ab}, q) at the edges of G_v. They give the edge vb the hinge $q(ab)$ of the edge ab, and the edge va a $(d-2)$-dimensional affine subspace L of the panel $\Pi(a)$ (KT eq. (6.12)). With the panel $\Pi(a)$ also at v, this is a panel-hinge realization (G, p_1) for every such L (KT Claim 6.9). They add the columns of v to those of a (KT eqs. (6.14) and (6.15)). As $p_1(vb) = q(ab)$, the rows of vb and of the edges of G_v then have, outside the columns of v, the entries of $R(G_v^{ab}, q)$ (KT eqs. (6.16) and (6.17)). The rows of va vanish outside the columns of v, and they have rank $D-1$ there. So

$$\operatorname{rank} R(G, p_1) \;\geq\; (D-1) + \operatorname{rank} R(G_v^{ab}, q) \;=\; D(|V(G)| - 1) - k.$$

Finally they rotate the panel of v slightly, which does not lower the rank, to make the realization nonparallel (KT Lemma 5.2, KT p. 679).

In a panel-hinge framework the hinge of an edge is the meet of the panels of its ends (Definition 23.5), while Katoh and Tanigawa ask only that it lie in both panels. For the hinge of vb to be $q(ab) = \Pi(a) \cap \Pi(b)$, the panel $\Pi(v)$ must contain

$\Pi(a) \cap \Pi(b)$. It must also differ from $\Pi(a)$, as otherwise the hinge of va would be degenerate. Then the hinge $\Pi(v) \cap \Pi(a)$ of va is again $\Pi(a) \cap \Pi(b)$, so that $L = q(ab)$.

The normal $n_a + n_b$ gives such a panel, as n_a and n_b are linearly independent. Let q_0 be the normals with $q_0(v) = n_a + n_b$ and $q_0(w) = q(w)$ for every body $w \neq v$, and let (G, q_0) be the panel-hinge framework on G with these normals. As C is bilinear and $C(n, n) = 0$ (Definition 23.1), the edges vb and va of (G, q_0) have, up to sign, the hinge $C(n_a, n_b)$ of the edge ab of (G_v^{ab}, q). Every other edge of G is an edge of G_v, and so of G_v^{ab} (Lemma 21.12). Its ends differ from v, so it has the same hinge in both frameworks, up to sign.

We now count rows as Katoh and Tanigawa do, with a row operation in place of their column operation. The hinges of (G_v^{ab}, q) are nondegenerate, as its normals are in general position (Lemma 23.11). So the count of Lemma 23.128, with $t = 1$ and $N = D(|V(G)| - 2) - (k - 1)$, shows that the hinges of (G, q_0) are nondegenerate and that

$$\operatorname{rank} R(G, q_0) \;\geq\; (D-1) + D(|V(G)| - 2) - (k-1) \;=\; D(|V(G)| - 1) - k.$$

Its proof places every row of $R(G_v^{ab}, q)$ in the row space of $R(G, q_0)$, a row at ab being the difference of a row at vb and a row at va, and adds $D-1$ rows at vb. These are linearly independent on the screw assignments that vanish off v, where the rows of $R(G_v^{ab}, q)$ vanish. For $k = 0$, the graph G_v^{ab} would be a minimal 0-dof-graph, and the same count would give one row less than the target rank (KT p. 680). Case III supplies the missing row (Section 23.10).

The normals q_0 need not be in general position. Instead of rotating a panel, we choose all the normals at once, as in the proof of Lemma 23.100 and the passage after it. Every hinge of (G, q_0) is nonzero, and $R(G, q_0)$ has the rank above. So there is a polynomial Q in the normals with $Q(q_0) \neq 0$ such that $\operatorname{rank} R(G, q) \geq D(|V(G)| - 1) - k$ wherever $Q(q) \neq 0$. This is Lemma 23.120, and Q is the minor by which the proof of Lemma 23.112 gives the lower semicontinuity of the rank. Let P be the polynomial of the proof of Lemma 23.100, so that the normals q are in general position wherever $P(q) \neq 0$. As K is infinite, the nonzero polynomial QP has a non-root q'.

The rank of $R(G, q')$ is then at least $D(|V(G)| - 1) - k$, and the normals q' are in general position. As G has no loops, every hinge of (G, q') is nondegenerate (Lemma 23.11). So the rank is at most $D(|V(G)| - 1) - k$ (Lemma 20.16), and (G, q') is a generic full-rank realization of G. □

Formalization note 23.136. The pinned theorem also assumes that some edge label lies outside $E(G)$, the label of the new edge ab of G_v^{ab} (Remark 23.40). It then lies outside $E(G_v)$, as the count of Lemma 23.128 asks (Lemma 21.12).

23.10. **Case III: no proper rigid subgraph and $k = 0$.** This subsection proves Case III of KT Theorem 5.5, in which the minimal 0-dof-graph G has at least three vertices and no proper rigid subgraph (KT §6.4). As in Case II, Katoh and Tanigawa split off a vertex v of degree two, whose edges join it to vertices a and b. The graph G_v^{ab} is a minimal 0-dof-graph with fewer vertices (KT Lemma 4.8), and the induction hypothesis gives a realization (G_v^{ab}, q) of it. Putting v back as in Case II gives a framework (G, p_1) of rank at least $D(|V(G)| - 1) - 1$, one less

than the target rank (KT eq. (6.12), p. 680; Lemma 23.128). Katoh and Tanigawa find the missing row in three steps. By KT Claim 6.11, one of the $D-1$ rows of $R(G_v^{ab}, q)$ at the new edge ab is redundant. Row and column operations then bring $R(G, p_1)$ to a block form whose rank is $D(|V(G)|-1)$ if a $D \times D$ block is nonsingular (KT eqs. (6.24)–(6.29)). This block depends on the hinge of va, and KT Claim 6.12 shows that one of several such frameworks has a nonsingular block.

The first two subsubsections prove KT Claim 6.11 and give the row and column operations. The third proves the case $d = 3$, KT Lemma 6.10, with its three frameworks and KT Claim 6.12. The fourth proves KT Lemma 6.13, for every d and including the graphs with three vertices. This is the form that the formal proof of KT Theorem 5.5 uses (Lemma 23.208).

Remark 23.137 (Katoh and Tanigawa's steps in Case III and the formal proof)**.** Several lemmas of this subsection state single steps of Katoh and Tanigawa's proof of KT Lemma 6.10. The formal proof of KT Theorem 5.5 (Theorem 23.38) takes these steps in another form, or not at all. We keep the lemmas beside the argument they belong to, and say here what the formal proof does instead.

The row operations. Katoh and Tanigawa bring $R(G, p_1)$ to the block form of KT eq. (6.29) by column and row operations (KT pp. 679 and 685–686). After the columns of v are added to those of a, the new row has entries only in the columns of v, where it is a linear form r on K^D (KT eq. (6.28)). The top-left $D \times D$ block of KT eq. (6.29) consists of r and of the entries of the rows of va in these columns. Four lemmas of the second subsubsection state steps of this computation. Lemma 23.150 compares the rows of an edge in two panel-hinge frameworks, as in KT eq. (6.13). Lemma 23.148 writes a form that vanishes where $S(v) = 0$ as a form in $S(v)$. Lemma 23.163 computes the new row of KT eq. (6.27). Lemma 23.159 gives the independence from which the rank of KT eq. (6.29) follows.

The formal proof makes one count, at (G, p_1) itself (Lemma 23.193). There the edges of $G_v = G - v$ keep their hinges by the definition of (G, p_1) (Definition 23.147). So the formal proof does not compare two panel-hinge frameworks. After the column operation, the rows of va and the new row vanish where $S(v) = 0$ (Lemma 23.149). The formal proof needs only this, and not that they are forms in $S(v)$. It obtains the new row as a row at vb minus a form in the row space of $R(G_v, q)$, by the identity of Lemma 23.161. Lemma 23.163 computes the new row instead at normals q_0 that agree with q off v and have $q_0(v) = q(a) + t\, q(b)$, as in the proof of Lemma 23.135 for $t = 1$. For $t \neq 0$, the hinge of va is then a multiple of the hinge $q(ab)$ of ab, on which r vanishes, and the block of KT eq. (6.29) is singular (the note after Lemma 23.167). The bottom rows of KT eq. (6.29) at vb have entries in the columns of v. So Lemma 23.159, which takes forms that vanish on the screw assignments that vanish off v, applies only after the proof of Lemma 23.167 has replaced them by rows of $R(G_v^{ab}, q)$. The formal count takes Katoh and Tanigawa's form instead, in which the bottom rows are linearly independent outside the columns of v. The note after Lemma 23.167 lists the theorems of that subsubsection that the formal proof uses.

The ranks of the frameworks. Katoh and Tanigawa read the rank of each of their three frameworks off its block form (KT eqs. (6.29), (6.30) and (6.41), pp. 686

and 689). Lemma 23.167 states this count for (G, p_1), and Lemma 23.181 and Lemma 23.189 state it for the other two. As the remark at the start of Case II says (Remark 23.130), the formal proof obtains the rank of each of its frameworks from one count. The three lemmas are this count at the three frameworks (Lemma 23.193 and the notes after them). In the general case the formal proof applies the count at the d frameworks of KT Lemma 6.13, along a chain of length d (Definition 21.62). It does not first count the rows of KT eq. (6.12) one short of the target rank (KT p. 680; the note after Lemma 23.206).

The third framework. In the case $d = 3$, the vertex a has degree two as well, with the neighbours v and c (KT p. 680). Katoh and Tanigawa build their third framework (G, p_3) on a realization (G_a^{vc}, q_ρ) of the graph G_a^{vc} obtained by splitting off at a along vc. This realization has the panels of (G_v^{ab}, q), through an isomorphism ρ from G_a^{vc} to G_v^{ab} (KT pp. 686–687 and eq. (6.31)). Lemma 23.183, Lemma 23.185 and Lemma 23.187 give the isomorphism, the realization and its rows. The formal proof takes the case $d = 3$ as a case of KT Lemma 6.13 (Lemma 23.208). There it renames the bodies by a permutation σ_i for each $2 \leq i \leq d-1$ (Lemma 23.202). For $d = 3$ and $i = 2$, σ_2 is the transposition of v and a, and the renamed realization is (G_a^{vc}, q_ρ) (the note after Lemma 23.206).

The top-left block M_3 of KT eq. (6.41), for (G, p_3), has the last row $-r$ (KT eq. (6.44), Lemma 23.157). Katoh and Tanigawa obtain this by restricting the relation of KT eq. (6.24) to the screw assignments that vanish off a, which gives KT eq. (6.43) (KT pp. 690–691). As a has degree two, only the rows at ab and ac remain. Lemma 23.156 states KT eq. (6.43), which holds as the relation is zero. The step to KT eq. (6.44) restricts each row to these screw assignments (the note after Lemma 23.157). The formal proof makes this step in the general case, at interior vertices of the chain, for the relation of KT eq. (6.52) grouped by edge (KT eq. (6.66), Lemma 23.204). It does not state KT eq. (6.43) as a step of its own.

KT Claim 6.12. Katoh and Tanigawa note that r is not zero, as it is a combination of linearly independent forms with a coefficient 1 (KT p. 690). Lemma 23.179 states this. The formal proof obtains $r \neq 0$ with the relation itself (Lemma 23.154). Katoh and Tanigawa then choose four affinely independent points with given incidences with the panels of a, b and c, which they can as (G_v^{ab}, q) is generic (KT p. 691). Three lemmas concern this choice. Two of them treat points that depend on some variables (Lemma 23.173, Lemma 23.174). Lemma 23.176 gives four points with these incidences for the coordinate planes. The formal proof chooses no points. The span of KT eq. (6.45) needs only linearly independent vectors in homogeneous coordinates, which exist for any linearly independent normals (Lemma 23.177 and the passage before it). The formal proof takes them at every d, in the general case (the note after that lemma).

23.10.1. *KT Claim 6.11: a redundant row at the new edge.* In Katoh and Tanigawa's proof, the induction hypothesis gives a generic realization (G_v^{ab}, q) of G_v^{ab} of rank $D(|V(G)| - 2)$ (KT eq. (6.18), p. 682). They call a row of $R(G_v^{ab}, q)$ redundant if removing it does not lower the rank. KT Claim 6.11 says that one of the $D - 1$ rows at the new edge ab is redundant [KT11, p. 684]. Its proof finds the redundancy first in the matroid $M(\widetilde{G_v^{ab}})$ and then in the rows, in three steps. First, $M(\widetilde{G_v^{ab}})$ has

a base that does not contain every copy of ab (KT Lemma 4.3(ii), Lemma 21.41). Second, it follows that the graph $G_v = G - v$, which is G_v^{ab} without the edge ab, is a minimal k'-dof-graph with $k' \leq D-2$ (Lemma 23.138). So the induction hypothesis gives $\operatorname{rank} R(G_v, q) \geq D(|V(G)| - 2) - (D-2)$ (KT eq. (6.22), Lemma 23.140). Third, the rows of $R(G_v^{ab}, q)$ are those of $R(G_v, q)$ and the rows at ab. The rows at ab therefore raise the rank by at most $D-2$, and one of them is redundant (KT eq. (6.23), Lemma 23.145).

The first step is the forward half of KT Lemma 4.3(ii) (Lemma 21.41). Let $\widetilde{ab}$ be the set of the $D-1$ copies in $\widetilde{G_v^{ab}}$ of the new edge ab. If G_v^{ab}, like G, is a k-dof-graph, then $M(\widetilde{G_v^{ab}})$ has a base B' with $|\widetilde{ab} \cap B'| < D-1$. Katoh and Tanigawa prove it by the surgery of KT Lemma 4.1 [KT11, p. 661], and the proof of Lemma 21.41 gives their argument in full. A base B of $M(\tilde{G})$ is the union of D pairwise disjoint forests, each with an edge at v (Lemma 21.21). Rerouting them across v (Lemma 21.24) gives an independent set of $M(\widetilde{G_v^{ab}})$ with $|B| - D$ elements and fewer than $D-1$ copies of ab. As both graphs are k-dof-graphs, KT eq. (2.4) (Theorem 20.12) shows that this set is a base. A copy of ab outside B' can be removed from $\widetilde{G_v^{ab}}$ without lowering the rank of its matroid. Katoh and Tanigawa call such a copy a redundant edge (p. 684).

In the second step, G is a minimal 0-dof-graph, as in Case III.

Lemma 23.138 (The graph $G-v$ is a minimal k'-dof-graph with $k' \leq D-2$; from the proof of KT Claim 6.11)**.** *Let $D \geq 2$, and let G be a minimal 0-dof-graph with a vertex v of degree two, whose edges e_a and e_b join v to distinct vertices $a, b \neq v$. Let $G_v = G - v$ and $k' = \operatorname{def}(\tilde{G}_v)$. Then G_v is a minimal k'-dof-graph, and*

$$0 \;\leq\; k' \;\leq\; D-2.$$

Proof. We follow Katoh and Tanigawa [KT11, p. 684]. The graph G_v is a subgraph of G, so it is a minimal k'-dof-graph by KT Lemma 3.3 (Lemma 20.25). It has the vertex a, so the partition of $V(G_v)$ into one part shows that $k' \geq 0$ (Definition 20.3). For the same reason $\operatorname{def}(\widetilde{G_v^{ab}}) \geq 0$. By KT Lemma 4.3(i) (Lemma 21.33), $\operatorname{def}(\widetilde{G_v^{ab}}) \leq \operatorname{def}(\tilde{G}) = 0$, so G_v^{ab} is a 0-dof-graph.

By the forward half of KT Lemma 4.3(ii) (Lemma 21.41), at $k = 0$, $M(\widetilde{G_v^{ab}})$ has a base B' with $h = |\widetilde{ab} \cap B'| < D-1$. As G_v^{ab} has the $|V(G)|-1$ vertices of G other than v, KT eq. (2.4) (Theorem 20.12) gives $|B'| = D(|V(G)| - 2)$. The graph G_v is G_v^{ab} without the edge ab (Lemma 21.12). So $M(\tilde{G}_v)$ is the restriction of $M(\widetilde{G_v^{ab}})$ to $E(\tilde{G}_v)$ (Lemma 20.24), in which $B' \setminus \widetilde{ab}$ is an independent set of $|B'| - h$ elements. As G_v and G_v^{ab} have the same vertices, KT eq. (2.4) gives

$$k' \;=\; D\,(|V(G)| - 2) - \operatorname{rank} M(\tilde{G}_v) \;\leq\; |B'| - (|B'| - h) \;=\; h \;\leq\; D-2.$$

□

Formalization note 23.139. The pinned theorem also assumes that some edge label lies outside $E(G)$, the label of the new edge ab of G_v^{ab}, which the proof uses (Remark 23.40).

Katoh and Tanigawa prove KT eq. (6.22) at the normals q of KT eq. (6.18), from the genericity of q (the remark below). As explained after Lemma 23.100, we replace genericity by the non-vanishing of polynomials in the normals. So we state KT eq. (6.22) as a bound off the zeros of a polynomial.

Lemma 23.140 (The rank of $G - v$ off the zeros of a polynomial; KT eq. (6.22))**.** *Let $d \geq 2$, let G be a simple minimal 0-dof-graph with a vertex v of degree two, whose edges join v to distinct vertices $a, b \neq v$, and let $G_v = G - v$. Then there is a nonzero polynomial P in the normals such that*

$$\operatorname{rank} R(G_v, q) \;\geq\; D\,(|V(G_v)| - 1) - (D - 2)$$

for all normals q with $P(q) \neq 0$.

Proof. We assume, as in Case III, that the induction hypothesis KT eq. (6.1) holds for the graphs with fewer vertices than G (the note after the proof). The graph G_v is a minimal k'-dof-graph with $k' = \operatorname{def}(\tilde{G}_v) \leq D - 2$ (Lemma 23.138). It is simple, as a subgraph of G, and it has at least one and fewer than $|V(G)|$ vertices. So the induction hypothesis gives a generic full-rank realization (G_v, q_0) (Definition 23.21), with

$$\operatorname{rank} R(G_v, q_0) \;=\; D\,(|V(G_v)| - 1) - k'.$$

Its normals are in general position and G_v has no loops, so its hinges are nondegenerate (Lemma 23.11). By Lemma 23.120, there is a polynomial P in the normals with $P(q_0) \neq 0$ such that $\operatorname{rank} R(G_v, q) \geq D(|V(G_v)| - 1) - k'$ wherever $P(q) \neq 0$. In particular $P \neq 0$. As $k' \leq D - 2$, this is the bound of the lemma. □

Formalization note 23.141. The pinned theorem also assumes the induction hypothesis for every k' and every minimal k'-dof-graph with at least one and fewer than $|V(G)|$ vertices, which the statement omits. Over an infinite field the statement holds without it: G_v is simple, and KT Theorem 5.5 (Theorem 23.38) gives it the generic full-rank realization that the proof takes from the induction hypothesis. The pinned theorem also assumes that some edge label lies outside $E(G)$ (Remark 23.40). It fixes an endpoint selector that assigns each edge of G_v^{ab} its ends (Formalization note 23.6) before it gives P.

The hypothesis that G is simple is needed. Let $d = 2$, so that $D = 3$, and let G have the vertices v, a, b, c, the edges va, vb and cb, and two edges ac. Comparing the partitions of $V(G)$ (Definition 20.3) shows that $\operatorname{def}(\tilde{G}) = 0$. Removing an edge ac gives the value 1 to the partition into singletons. Removing va or vb gives it to the partition $\{\{v\}, \{a, b, c\}\}$, and removing cb to $\{\{b\}, \{a, c, v\}\}$. So G is a minimal 0-dof-graph, and v has degree two. In (G_v, q) the two edges ac have the same hinge, and so the same rows. Let Q be the product of a 2×2 minor of the normals at a and c and one of the normals at c and b. Where $Q(q) \neq 0$, the hinges of (G_v, q) are nondegenerate, and

$$\operatorname{rank} R(G_v, q) \;\leq\; 2(D-1) \;=\; 4 \;<\; 5 \;=\; D\,(|V(G_v)| - 1) - (D - 2).$$

Over an infinite field, PQ has a non-root for every nonzero polynomial P, so no polynomial P as in the conclusion of the lemma exists. The two edges ac form a proper rigid subgraph of G, so this graph falls under Case I.

Remark 23.142 (KT eq. (6.22) and KT footnote 6)**.** We recall Katoh and Tanigawa's proof of KT eq. (6.22) [KT11, pp. 684–685]. The realization (G_v^{ab}, q) of KT eq. (6.18) is generic: the coordinates of the vectors $c(u)$ that define its panels are algebraically independent over $\mathbb{Q}$ (KT §5.1, p. 668; the passage after Lemma 23.100). By KT eq. (6.1), the simple minimal k'-dof-graph G_v has a nonparallel realization of rank $D(|V(G_v)|-1)-k'$. A generic nonparallel realization of G_v has the largest rank of all nonparallel realizations of G_v (KT §5.1). No realization with nondegenerate hinges has a rank larger than $D(|V(G_v)|-1)-k'$ (Lemma 20.16). So every generic nonparallel realization of G_v has rank $D(|V(G_v)|-1)-k'$ (KT footnote 6, p. 684). The realization (G_v, q) has the bodies and the panels of (G_v^{ab}, q), so it is a generic nonparallel realization of G_v. Hence $\operatorname{rank} R(G_v, q) = D(|V(G_v)|-1)-k'$.

In our proof the normals of KT eq. (6.18) are not given first. Case III chooses them off the zeros of a product of nonzero polynomials, fixed before the choice (the proof of Lemma 23.200, later in this subsection). One factor is the polynomial P of Lemma 23.140. Another is the polynomial that Lemma 23.120 gives for the realization of G_v^{ab} from the induction hypothesis, and a third puts the normals in general position. Off the zeros of the second, $\operatorname{rank} R(G_v^{ab}, q) \geq D(|V(G_v)|-1)$. Off those of the third, the hinges are nondegenerate, and as $\operatorname{def}(\widetilde{G_v^{ab}}) = 0$, the rank is at most $D(|V(G_v)|-1)$ (Lemma 20.16). So KT eqs. (6.18) and (6.22) hold at the same normals, as in Katoh and Tanigawa's proof. They hold by the choice of the normals, not by their genericity.

The third step is a count in linear algebra, which we state for one hinge of any body-hinge framework.

Lemma 23.143 (A redundant row at a hinge; from the proof of KT Claim 6.11)**.** *Let (G, p) be a body-hinge framework, let e be an edge of G with distinct ends whose hinge is nondegenerate, and let R_e be the span of the rows of $R(G, p)$ at e. Let W be a space of linear forms on the screw assignments such that*

$$\dim(W + R_e) \;<\; \dim W + (D-1).$$

Then R_e has a basis $r_1, \dots, r_{D-1}$ with $r_i \in W + \operatorname{span}\{r_j : j \neq i\}$ for some i.

Proof. By Lemma 23.122, R_e contains $D-1$ linearly independent forms $r_1, \dots, r_{D-1}$. The space R_e has dimension $D-1$, as in the proof of that lemma, so they form a basis of R_e. Suppose that no r_i lies in $W + \operatorname{span}\{r_j : j \neq i\}$. If $\sum_j c_j r_j \in W$ with $c_i \neq 0$, then dividing by c_i puts r_i in $W + \operatorname{span}\{r_j : j \neq i\}$. So $\sum_j c_j r_j \in W$ only if every c_j is 0, and the images of the r_j in the quotient by W are linearly independent. Then $\dim(W + R_e) = \dim W + (D-1)$, contrary to the hypothesis. □

Formalization note 23.144. As for Lemma 23.122, the pinned theorem holds for every body-hinge framework, and it reads the ends of e from the endpoint selector (Formalization note 23.6).

We apply this count at the edge ab of G_v^{ab}, where W is the row space of $R(G_v, q)$.

Lemma 23.145 (KT Claim 6.11: a redundant row at ab; KT eq. (6.23))**.** *Let $d \geq 2$, let G be a multigraph with a vertex v of degree two, whose edges join v to distinct*

vertices $a, b \neq v$, and let $G_v = G - v$. Let q be normals at which the hinge of the new edge ab of (G_v^{ab}, q) is nondegenerate. Suppose that, for some $k' \leq D-2$,

$$\operatorname{rank} R(G_v^{ab}, q) = D\,(|V(G_v)| - 1) \quad \textit{and} \quad \operatorname{rank} R(G_v, q) = D\,(|V(G_v)| - 1) - k',$$

as in KT eqs. (6.18) and (6.22). Then one of the $D-1$ rows of $R(G_v^{ab}, q)$ at ab lies in the span of the rows of $R(G_v, q)$ and the other rows at ab.

Proof. We follow Katoh and Tanigawa [KT11, p. 685]. The edges of G_v^{ab} are those of G_v, with the same ends, and the new edge ab (Lemma 21.12). The two frameworks have the same normals, so an edge of G_v has the same hinge, and the same rows, in both. Let W be the row space of $R(G_v, q)$, and let R_{ab} be the span of the rows of $R(G_v^{ab}, q)$ at ab. Then the row space of $R(G_v^{ab}, q)$ is $W + R_{ab}$, and the two rank equations give

$$\dim(W + R_{ab}) \;=\; D\,(|V(G_v)| - 1) \;=\; \dim W + k'.$$

As $k' \leq D-2 < D-1$, the rows at ab raise $\dim W$ by fewer than $D-1$.

The $D-1$ rows $r_1, \dots, r_{D-1}$ of $R(G_v^{ab}, q)$ at ab form a basis of R_{ab} (Definition 19.5). The proof of Lemma 23.143 applies to every basis of R_{ab}. So some r_i lies in $W + \operatorname{span}\{r_j : j \neq i\}$. The rows of $R(G_v, q)$ and the r_j with $j \neq i$ then span $W + R_{ab}$. So removing the row r_i does not lower the rank of $R(G_v^{ab}, q)$, which is KT eq. (6.23). □

Formalization note 23.146. The pinned theorem gives a basis of the span of the rows at ab, one member of which lies in the span of the rows of $R(G_v, q)$ and the other members. This is the lemma for one choice of the rows at ab (Definition 19.5). The pinned theorem holds for any two multigraphs whose edges, with their ends, agree except for one edge ab of the first, and with any number in place of $|V(G_v)|$. Their frameworks share one endpoint selector, which assigns ab its ends (Formalization note 23.6). The formal proof of Case III applies it with $k' = D(|V(G_v)| - 1) - \operatorname{rank} R(G_v, q)$, for which the second rank equation holds by definition, and Lemma 23.140 gives $k' \leq D-2$.

23.10.2. *The row operations of KT eqs. (6.24)–(6.29).* Let G, v, a, b and (G_v^{ab}, q) be as in KT Claim 6.11 (Lemma 23.145), and write $q(e)$ for the hinge of an edge e in (G_v^{ab}, q). Katoh and Tanigawa put v back as in Case II. Their framework (G, p_1) of KT eq. (6.12) keeps the hinge $q(e)$ of each edge e of G_v, gives vb the hinge $q(ab)$, and gives va a $(d-2)$-dimensional affine subspace L of the panel $\Pi(a)$. For every such L, the rank of $R(G, p_1)$ is at least $D(|V(G)| - 1) - 1$, one less than the target rank (KT p. 680; Lemma 23.128). Katoh and Tanigawa find the missing row by row and column operations [KT11, pp. 685–686]. They give the argument for $d = 3$, where $D = 6$, and repeat it for every d in KT eqs. (6.50)–(6.52) (p. 693). We follow the first, with D in place of 6.

The formal proof builds (G, p_1) as a member of a family of body-hinge frameworks, with the hinge of va given by a second normal.

Definition 23.147 (The frameworks F_t and (G, p_1); KT eq. (6.12))**.** Let G be a multigraph with normals q, let e_c and e_r be edge labels, and let $n_1, n', n_2 \in K^{d+1}$. For $t \in K$, let F_t be the body-hinge framework on G (Definition 19.3) whose supporting extensor is $C(n_1 + t\,n', n_2)$ at e_r, is $C(n_1, n')$ at e_c if $e_c \neq e_r$

(Definition 23.1), and at every other edge label is its extensor in the panel-hinge framework (G, q) (Definition 23.5). For a vertex v of G whose edges va and vb join it to vertices a and b, we write (G, p_1) for F_0 with $e_c = va$, $e_r = vb$, $n_1 = q(a)$ and $n_2 = q(b)$.

Let v, a, b and (G_v^{ab}, q) be as above. In (G, p_1) the hinge of vb has the extensor $C(q(a), q(b))$ of the hinge $q(ab)$ of ab in (G_v^{ab}, q), and an edge e of G_v has the same ends in G as in G_v^{ab}, so it keeps its hinge $q(e)$. A $(d-2)$-dimensional affine subspace L of the panel $\Pi(a)$ is the meet of $\Pi(a)$ and a hyperplane, with normal n' say. Then $q(a)$ and n' are linearly independent, and $C(q(a), n')$ is a nonzero multiple of the extensor of L (the remarks after Definition 23.1). So (G, p_1) is Katoh and Tanigawa's framework of KT eq. (6.12) for this L [KT11, p. 678]. For $t \neq 0$, the extensor $C(q(a), n')$ at va is $-t^{-1} C(q(a) + t\, n', q(a))$. So F_t has the rows of the panel-hinge framework in which v has the normal $q(a) + t\, n'$, a panel through L, and every other body keeps its normal. This is how the formal proof rotates the panel of v about L (the proof of Lemma 23.195), as Katoh and Tanigawa do to make (G, p_1) nonparallel (KT p. 678 and Lemma 5.2).

The redundant row of KT Claim 6.11 gives a linear relation among the rows of $R(G_v^{ab}, q)$ in which a row $(ab)i^*$ at ab has the coefficient 1 (KT eqs. (6.24) and (6.25)). The rows of $R(G, p_1)$ at vb correspond to those of $R(G_v^{ab}, q)$ at ab, and the edges of G_v have the same rows in both matrices. Katoh and Tanigawa add to the row $(vb)i^*$ of $R(G, p_1)$ the other terms of the relation, with the rows at vb in place of those at ab (KT eqs. (6.26) and (6.27)). After the columns of v are added to those of a (KT eqs. (6.14) and (6.15)), the new row has nonzero entries only in the columns of v (KT eq. (6.28)), as do the rows of va. The matrix then has the block form of KT eq. (6.29). Its rank is $D(|V(G)| - 1)$ if its top-left $D \times D$ block, formed by the new row and the rows of va, is nonsingular. Whether it is depends on L. Katoh and Tanigawa show that one of three such frameworks has a nonsingular block (KT Claim 6.12, Lemma 23.191, in the next subsubsection).

We write the rows of a rigidity matrix as linear forms on the screw assignments, as in the paragraphs after Definition 19.7. The entries of a form in the columns of a body u are its restriction to the screw assignments that vanish off u.

Lemma 23.148 (Forms that vanish where $S(v) = 0$; the form of KT eq. (6.28))**.** *Let v be a body, and let f be a linear form on the screw assignments such that $f(S) = 0$ for every screw assignment S with $S(v) = 0$. For a screw y, let S_y be the screw assignment with the value y at v and 0 elsewhere. Then $f(S) = f(S_{S(v)})$ for every screw assignment S.*

Proof. The screw assignment $S - S_{S(v)}$ has the value 0 at v, so $f(S - S_{S(v)}) = 0$. By linearity, $f(S) = f(S_{S(v)})$. □

In terms of forms, adding the columns of v to those of a is a change of variables.

Lemma 23.149 (The column operation of KT eqs. (6.14) and (6.15))**.** *Let $v \neq a$ be bodies, and let Φ be the map on screw assignments with $\Phi(S)(v) = S(v) + S(a)$ and $\Phi(S)(u) = S(u)$ for every body $u \neq v$. Then Φ is a linear automorphism. For*

every linear form x on K^D, the form $S \mapsto x(S(v) - S(a))$ composed with Φ is the form $S \mapsto x(S(v))$, which vanishes at every S with $S(v) = 0$.

Proof. Each value of $\Phi(S)$ is a linear function of S, so Φ is linear. Let Ψ be the map with $\Psi(S)(v) = S(v) - S(a)$ and $\Psi(S)(u) = S(u)$ for $u \neq v$. As $a \neq v$, both maps keep the value at a. Hence $\Psi(\Phi(S))(v) = (S(v) + S(a)) - S(a) = S(v)$ and $\Phi(\Psi(S))(v) = (S(v) - S(a)) + S(a) = S(v)$, and both composites keep the values at the other bodies. So Ψ is the inverse of Φ. As $\Phi(S)(a) = S(a)$, we have

$$x(\Phi(S)(v) - \Phi(S)(a)) \;=\; x(S(v) + S(a) - S(a)) \;=\; x(S(v)),$$

which is 0 when $S(v) = 0$. So the composite depends only on $S(v)$ (Lemma 23.148).

Let us check that composing with Φ is Katoh and Tanigawa's column operation. Let f be a linear form on the screw assignments. On the screw assignments that vanish off v, Φ is the identity, so $f \circ \Phi$ has the entries of f in the columns of v. If S vanishes off a, then $\Phi(S)$ has the value $S(a)$ at both v and a. So the entries of $f \circ \Phi$ in the columns of a are the sums of those of f in the columns of a and in the columns of v. The entries in the other columns do not change. Replacing each row f of $R(G, p_1)$ by $f \circ \Phi$ therefore adds the columns of v to those of a (KT eqs. (6.14) and (6.15)). As Φ is invertible, $f \mapsto f \circ \Phi$ is an injective linear map, so it does not change the rank. The rows of va are forms $S \mapsto x(S(v) - S(a))$, so they become forms $S \mapsto x(S(v))$, with zero entries outside the columns of v, as in KT eq. (6.15). □

The framework (G, p_1) keeps the hinges of the edges of G_v, so these edges have the same rows in $R(G, p_1)$ as in $R(G_v^{ab}, q)$ (KT eq. (6.13), p. 679). For two panel-hinge frameworks this reads as follows.

Lemma 23.150 (The rows at an edge whose ends keep their normals; KT eq. (6.13))**.** *Let (G, q_0) and (G', q) be panel-hinge frameworks, and let e be an edge of G and of G' with the same ends u and w in both. If $q_0(u) = q(u)$ and $q_0(w) = q(w)$, then e has the same rows in $R(G, q_0)$ as in $R(G', q)$, indexed as in Lemma 23.91.*

Proof. The hinge of e is $C(q_0(u), q_0(w))$ in (G, q_0) and $C(q(u), q(w))$ in (G', q) (Definition 23.5), and these are equal. The rows at e are the forms $S \mapsto x(S(u) - S(w))$ for the linear forms x on K^D that Lemma 23.91 assigns to the hinge. They depend only on the hinge and on u and w, not on the rest of the graph. □

Formalization note 23.151. The pinned theorem takes the two frameworks with one endpoint selector, which gives e the same ends in both (Formalization note 23.6), and it holds for every edge label e.

We now turn to the relation that KT Claim 6.11 gives. The rows of $R(G_v^{ab}, q)$ at ab depend on a choice of linear forms on K^D (Definition 19.5), so we state the relation for one such choice.

Lemma 23.152 (The linear relation of KT eqs. (6.24) and (6.25))**.** *Under the hypotheses of Lemma 23.145, let W be the row space of $R(G_v, q)$. Then there are linearly independent linear forms $r_1, \dots, r_{D-1}$ on K^D that vanish on $C(q(a), q(b))$, an index i^* and scalars $\lambda_1, \dots, \lambda_{D-1}$ with $\lambda_{i^*} = 1$, such that the form $S \mapsto \sum_j \lambda_j r_j(S(a)-$*

$S(b)$) lies in W. That is, writing $R(G_v^{ab}, q; (ab)j)$ for the row $S \mapsto r_j(S(a) - S(b))$ at ab and $R(G_v^{ab}, q; e_j)$ for the rows of an edge e of G_v, there are scalars λ_{ej} with

$$\sum_{1 \le j \le D-1} \lambda_j \, R(G_v^{ab}, q; (ab)j) \; + \sum_{e \in E(G_v)} \sum_j \lambda_{ej} \, R(G_v^{ab}, q; e_j) \; = \; 0.$$

Proof. We follow Katoh and Tanigawa [KT11, p. 685]. Let R_{ab} be the span of the rows at ab. As $a \neq b$, the map $x \mapsto (S \mapsto x(S(a) - S(b)))$ is injective. It takes the linear forms on K^D that vanish on $C(q(a), q(b))$ onto R_{ab} (Definition 19.5). So every basis of R_{ab} consists of the rows at ab for one choice of $r_1, \dots, r_{D-1}$. By Lemma 23.145, for one such choice a row $R(G_v^{ab}, q; (ab)i^*)$ lies in the span of W and the other rows at ab. Write it as $g - \sum_{j \neq i^*} \lambda_j R(G_v^{ab}, q; (ab)j)$ with $g \in W$, and let $\lambda_{i^*} = 1$. Then $\sum_j \lambda_j R(G_v^{ab}, q; (ab)j) = g$ lies in W.

The edges of G_v have the same rows in $R(G_v, q)$ as in $R(G_v^{ab}, q)$ (Lemma 21.12). Writing g as a combination $-\sum_{e,j} \lambda_{ej} R(G_v^{ab}, q; e_j)$ of these rows gives the relation, which is KT eq. (6.24) with $\lambda_{(ab)j} = \lambda_j$, and $\lambda_{i^*} = 1$ is KT eq. (6.25). □

Formalization note 23.153. The pinned theorem gives a basis of R_{ab}, and it states the relation as $R(G_v^{ab}, q; (ab)i^*) = g + h$, with $g \in W$ and h in the span of the other members of the basis. Then $g = \sum_j \lambda_j R(G_v^{ab}, q; (ab)j)$, where $\lambda_{i^*} = 1$ and the other λ_j are the coefficients of $-h$. As for Lemma 23.145, the pinned theorem holds for any two multigraphs whose edges, with their ends, agree except for one edge ab of the first.

In the formal proof of Case III, the normals q are chosen so that KT eqs. (6.18) and (6.22) hold (the remark on KT footnote 6): the motions of (G_v^{ab}, q) are constant on its vertex set, and the bound of Lemma 23.140 holds. The next lemma derives the relation of KT eqs. (6.24) and (6.25) at such normals.

Lemma 23.154 (The relation of KT eqs. (6.24) and (6.25) when (G_v^{ab}, q) is rigid)**.** *Let $d \ge 2$, let G be a multigraph with a vertex v of degree two, whose edges join v to distinct vertices $a, b \neq v$, and let $G_v = G - v$. Let q be normals at which the hinge of the new edge ab of (G_v^{ab}, q) is nondegenerate, and every infinitesimal motion of (G_v^{ab}, q) is constant on $V(G_v^{ab})$. Suppose that*

$$\operatorname{rank} R(G_v, q) \; \ge \; D\,(|V(G_v^{ab})| - 1) - (D - 2).$$

Then there are linearly independent forms $r_1, \dots, r_{D-1}$ that span the same space as the rows of $R(G_v^{ab}, q)$ at ab, an index i^* and scalars $\lambda_1, \dots, \lambda_{D-1}$ with $\lambda_{i^*} = 1$, such that the form $\sum_j \lambda_j r_j$ is not zero and lies in the row space of $R(G_v, q)$.

Proof. We follow Katoh and Tanigawa [KT11, pp. 682–685]. Let $m = |V(G_v^{ab})|$, which is at least 1, as a is a vertex of G_v^{ab}. As the motions of (G_v^{ab}, q) are constant on its vertex set, $\operatorname{rank} R(G_v^{ab}, q) = D(m-1)$, as in the proof of Lemma 23.124. This is KT eq. (6.18). The edges of G_v are edges of G_v^{ab} with the same ends (Lemma 21.12), and so with the same rows (Lemma 23.91). So the row space of $R(G_v, q)$ lies in that of $R(G_v^{ab}, q)$, and $\operatorname{rank} R(G_v, q) \le D(m-1)$. Let $k' = D(m-1) - \operatorname{rank} R(G_v, q)$. Then $\operatorname{rank} R(G_v, q) = D(m-1) - k'$, which is KT eq. (6.22), and the hypothesis gives

$k' \leq D-2$. As G_v and G_v^{ab} have the same vertices, the hypotheses of Lemma 23.145 hold.

So Lemma 23.152 gives the forms, the index i^* and the scalars of KT eqs. (6.24) and (6.25), where r_j is the row $S \mapsto r'_j(S(a) - S(b))$ for the linear forms r'_j on K^D of that lemma. The form $\sum_j \lambda_j r_j$ is not zero, as the r_j are linearly independent and $\lambda_{i^*} = 1$. □

Formalization note 23.155. As for Lemma 23.145, the pinned theorem holds for any two multigraphs whose edges, with their ends, agree except for one edge ab of the first, here with $|V(G_v^{ab})| \geq 1$, and their frameworks share one endpoint selector, which assigns ab its ends (Formalization note 23.6). It takes $D \geq 2$ in place of $d \geq 2$.

Katoh and Tanigawa use one more consequence of KT eq. (6.24), in the proof of Claim 6.12 for the third of their frameworks (p. 690; Lemma 23.191, in the next subsubsection).

Lemma 23.156 (The columns of a in KT eq. (6.24); KT eq. (6.43))**.** *Let the rows $R(G_v^{ab}, q; e_j)$ and the scalars λ_j and λ_{ej} be as in Lemma 23.152, and let $\lambda_{(ab)j} = \lambda_j$. For a body u, write $R(G_v^{ab}, q; e_j, u)$ for the restriction of the row e_j to the screw assignments that vanish off u. Then for every body u,*

$$\sum_{e \in E(G_v) \cup \{ab\}} \sum_j \lambda_{ej}\, R(G_v^{ab}, q; e_j, u) \;=\; 0.$$

For $u = a$ this is KT eq. (6.43).

Proof. By KT eq. (6.24) (Lemma 23.152), the combination is the zero form. So is its restriction to the screw assignments that vanish off u. □

In the case $d = 3$, the vertex a has degree two as well, and its edges are va and ac (KT p. 680). Then only the edges ab and ac of G_v^{ab} meet a, and KT eq. (6.43) has nonzero terms only at these two edges (KT p. 691). The top-left block M_3 of Katoh and Tanigawa's third framework has as its last row the combination of the rows at ac with the coefficients of KT eq. (6.24) (KT eq. (6.42), p. 690). By the next lemma, this row is $-r$, where r is the combination of the rows at ab with these coefficients.

Lemma 23.157 (The last row of M_3; KT eq. (6.44))**.** *Under the hypotheses of Lemma 23.145, suppose that the edges of G at a are va and ac, where v, a, b and c are distinct. Let the forms r_j and the scalars λ_j and λ_{ej} be as in Lemma 23.152, and let $r = \sum_j \lambda_j r_j$. Write the rows $R(G_v^{ab}, q; (ac)j)$ at ac as the forms $S \mapsto s_j(S(a) - S(c))$. Then*

$$r \;=\; -\sum_{1 \leq j \leq D-1} \lambda_{(ac)j}\, s_j.$$

Proof. We follow Katoh and Tanigawa [KT11, pp. 690–691]. By KT eq. (6.43) (Lemma 23.156), the combination of the rows of $R(G_v^{ab}, q)$ in KT eq. (6.24) vanishes on the screw assignments that vanish off a. Let S be such a screw assignment, with $S(a) = y$. A row $S \mapsto x(S(u) - S(w))$ at an edge with ends $u, w \neq a$ vanishes

at S. As the edges of G at a are va and ac, the edges of G_v^{ab} at a are ab and ac (Definition 21.10). Their rows take the values $r_j(y)$ and $s_j(y)$ at S. So

$$0 \;=\; \sum_j \lambda_j\, r_j(y) + \sum_j \lambda_{(ac)j}\, s_j(y) \;=\; r(y) + \sum_j \lambda_{(ac)j}\, s_j(y)$$

for every $y \in K^D$, which is KT eq. (6.44). □

Formalization note 23.158. The first pinned theorem is the computation of the proof, for any forms at ab and at ac and any further form that vanishes on the screw assignments that vanish off a. The second and third give the restriction of a form $S \mapsto x(S(u) - S(w))$ to these screw assignments: it is x if $u = a \neq w$, and 0 if $a \notin \{u, w\}$. The formal proof of the case $d = 3$ uses the lemma in a span form, `BodyHingeFramework.acolumn_mem_hingeRowBlock_of_span_rigidityRows`. Let ac be the only edge at a of a body-hinge framework, and let f be a form in the span of its rows. Then the restriction of f to the screw assignments that vanish off a, a linear form x on K^D, vanishes on the hinge of ac, and $S \mapsto x(S(a) - S(c))$ lies in the span of the rows (`BodyHingeFramework.hingeRow_acolumn_mem_span_rigidityRows`). For the framework (G_v, q) and the form $f\colon S \mapsto r(S(a) - S(b))$, the span form says that r vanishes on $C(q(a), q(c))$, as KT eq. (6.44) shows.

The rank in KT eq. (6.29) comes from the following independence. It extends Lemma 23.127 by one more form in the first family. A bound on the rank follows as in Lemma 19.34.

Lemma 23.159 (Two blocks of independent forms; KT eq. (6.29))**.** *Let v be a body, and let $\varphi_1, \dots, \varphi_{D-1}$, ω and $\psi_1, \dots, \psi_l$ be linear forms on the screw assignments. Suppose that the restrictions of $\varphi_1, \dots, \varphi_{D-1}$ and ω to the screw assignments that vanish off v are linearly independent, and that $\psi_1, \dots, \psi_l$ are linearly independent and vanish on these screw assignments. Then the forms $\varphi_1, \dots, \varphi_{D-1}, \omega, \psi_1, \dots, \psi_l$ are linearly independent.*

Proof. We argue as in the proof of Lemma 23.127. Suppose that $\sum_i c_i\varphi_i + c\,\omega + \sum_j c'_j\psi_j = 0$. Every ψ_j vanishes on the screw assignments that vanish off v, so $\sum_i c_i\varphi_i + c\,\omega$ vanishes on them. By the first hypothesis, c and every c_i are 0. Then $\sum_j c'_j\psi_j = 0$, and every c'_j is 0. □

Formalization note 23.160. The pinned theorem allows any finite number of forms in place of $\varphi_1, \dots, \varphi_{D-1}$.

The next two lemmas compute the new row of KT eq. (6.27) at the panel-hinge framework (G, q_0) whose normals q_0 agree with q off v and have $q_0(v) = q(a) + t\,q(b)$, as in the proof of Lemma 23.135, where $t = 1$. Only the hinge of vb enters, and it is $q(ab)$, as in (G, p_1).

Lemma 23.161 (The row operation at one row; KT eqs. (6.26) and (6.27))**.** *Let G be a multigraph with a vertex v of degree two, whose edges join v to distinct vertices $a, b \neq v$, and orient vb from v to b and ab from a to b. Let q be normals, $t \in K$, and q_0 normals with $q_0(v) = q(a) + t\,q(b)$ and $q_0(b) = q(b)$. Then each row of $R(G, q_0)$ at vb, minus the row of $R(G_v^{ab}, q)$ at ab with the same index (Lemma 23.91), is a*

form $S \mapsto x(S(v) - S(a))$ for a linear form x on K^D that vanishes on $C(q(a), q(b))$. Moreover, for every linear form x on K^D, the form $S \mapsto x(S(v) - S(a))$ composed with the map Φ of Lemma 23.149 depends only on $S(v)$.

Proof. As C is bilinear and $C(n, n) = 0$ (Definition 23.1),

$$C(q_0(v), q_0(b)) \;=\; C(q(a) + t\, q(b),\, q(b)) \;=\; C(q(a), q(b)).$$

So vb has in (G, q_0) the hinge that ab has in (G_v^{ab}, q). The two rows with the same index are therefore the forms $S \mapsto x(S(v) - S(b))$ and $S \mapsto x(S(a) - S(b))$ for one form x, which vanishes on $C(q(a), q(b))$ (Lemma 23.91). As $(S(v) - S(b)) - (S(a) - S(b)) = S(v) - S(a)$, their difference is $S \mapsto x(S(v) - S(a))$. By Lemma 23.149, the composite in the last claim is $S \mapsto x(S(v))$. □

Formalization note 23.162. The first pinned theorem is the first claim, for edge labels that the endpoint selector (Formalization note 23.6) orients so, in any two multigraphs. The second is the identity $x(S(u) - S(w)) - x(S(u') - S(w)) = x(S(u) - S(u'))$, and the third is the last claim.

The previous lemma gives KT eq. (6.27) term by term. As the rows at ab are the forms $S \mapsto r_j(S(a) - S(b))$, the whole relation of KT eq. (6.24) can be treated at once, with the single form $r = \sum_j \lambda_j r_j$.

Lemma 23.163 (The new row of KT eq. (6.27))**.** *Let $r = \sum_j \lambda_j r_j$, where the forms r_j and the scalars λ_j are those of Lemma 23.152, under the hypotheses of Lemma 23.145. Let $t \in K$, and let q_0 be normals with $q_0(v) = q(a) + t\, q(b)$ and $q_0(b) = q(b)$. Then the combination of the rows at ab with the coefficients λ_j is the form $S \mapsto r(S(a) - S(b))$, the form $S \mapsto r(S(v) - S(b))$ is a row of $R(G, q_0)$ at vb, and*

$$r(S(v) - S(b)) - r(S(a) - S(b)) \;=\; r(S(v) - S(a)).$$

Proof. The first claim holds as $R(G_v^{ab}, q; (ab)j)$ is the form $S \mapsto r_j(S(a) - S(b))$. Each r_j vanishes on $C(q(a), q(b))$, and so does r. Moreover $r \neq 0$, as the r_j are linearly independent and $\lambda_{i^*} = 1$. As in the proof of Lemma 23.161, the hinge of vb in (G, q_0) is $C(q(a), q(b))$. So $S \mapsto r(S(v) - S(b))$ is a row of $R(G, q_0)$ at vb (Definition 19.5). The identity holds as $(S(v) - S(b)) - (S(a) - S(b)) = S(v) - S(a)$.

This is Katoh and Tanigawa's row operation, applied to the whole relation at once [KT11, pp. 685–686]. In (G, p_1), as in (G, q_0), the hinge of vb is $q(ab)$. So the rows $R(G, p_1; (vb)j)$ of $R(G, p_1)$ at vb are the forms $S \mapsto r_j(S(v) - S(b))$, and they correspond to the rows $(ab)j$ of $R(G_v^{ab}, q)$ (KT eq. (6.16)). The edges of G_v have the same rows in both matrices (KT eq. (6.13)). Katoh and Tanigawa add to the row $(vb)i^*$ the combination of the rows $(vb)j$ with $j \neq i^*$ and of the rows of the edges of G_v, with the coefficients of KT eq. (6.24) (KT eq. (6.26)). As $\lambda_{i^*} = 1$, the result is the row of KT eq. (6.27):

$$\begin{aligned}\sum_j \lambda_j\, R(G, p_1; (vb)j) + \sum_{e \in E(G_v)} \sum_j \lambda_{ej}\, R(G, p_1; e_j) \\ = r(S(v) - S(b)) - r(S(a) - S(b)) = r(S(v) - S(a)),\end{aligned}$$

as forms in S, by KT eq. (6.24). We call this row w. It reads $S(a)$ as well as $S(v)$: on the screw assignments that vanish off a it is the form $S \mapsto -r(S(a))$, which is not zero. Its entries outside the columns of v vanish only after the column operation, which turns it into the form $S \mapsto r(S(v))$ (Lemma 23.149). This is the row $(r, 0)$ of KT eq. (6.28). □

Formalization note 23.164. The pinned theorem holds for every form g in the span of the rows at ab in place of their combination with the coefficients λ_j. It finds the form r with $g(S) = r(S(a) - S(b))$ as in the proof of Lemma 23.152. It holds for any two multigraphs whose edges the endpoint selector assigns their ends, with vb and ab oriented by it (Formalization note 23.6).

We now return to Katoh and Tanigawa's framework (G, p_1), with an arbitrary subspace L. After the column operation, the rows of va and the row w have entries only in the columns of v, where they form the top-left $D \times D$ block of KT eq. (6.29). Katoh and Tanigawa call it M_1, and they test its singularity by the extensor of L (KT eq. (6.42) and p. 690). In these columns the rows of va are $D-1$ linearly independent linear forms on K^D that vanish on $C(L)$ (Definition 19.5). A nondegenerate hinge of a panel-hinge framework gives such forms (Lemma 23.122).

Lemma 23.165 (When the blocks of KT eq. (6.42) are nonsingular)**.** *Under the hypotheses of Lemma 23.145, let r be the linear form of Lemma 23.163. Let L be a $(d-2)$-dimensional affine subspace of the panel of a vertex of (G_v^{ab}, q), and let $r'_1, \dots, r'_{D-1}$ be linearly independent linear forms on K^D that vanish on $C(L)$. Then $r'_1, \dots, r'_{D-1}, r$ are linearly independent if and only if $r(C(L)) \neq 0$.*

Proof. The extensor $C(L)$ is not zero (Definition 18.8). So the linear forms on K^D that vanish on $C(L)$ form a space of dimension $D - 1$ (Definition 19.5), and $r'_1, \dots, r'_{D-1}$ is a basis of it. A form vanishes on span $C(L)$ if and only if it vanishes at $C(L)$ (the remark after Definition 19.5). So $r(C(L)) = 0$ if and only if r is a linear combination of the r'_i, that is, if and only if the D forms are linearly dependent. □

Formalization note 23.166. The pinned theorem holds for every linear form r and every edge of a body-hinge framework, with any linearly independent forms that span the forms vanishing on its extensor.

The next lemma gives the rank of $R(G, p_1)$ when this block is nonsingular. It is the count of Lemma 23.193, in the next subsubsection, at (G, p_1).

Lemma 23.167 (The rank of KT eq. (6.29); from the proof of KT Lemma 6.10)**.** *Under the hypotheses of Lemma 23.145, let the forms r_j, the index i^* and the scalars λ_j be as in Lemma 23.152, and let $r = \sum_j \lambda_j r_j$. Let L be a $(d-2)$-dimensional affine subspace of the panel $\Pi(a)$ of (G_v^{ab}, q), and let (G, p_1) be the body-hinge framework with the hinge L at va, the hinge $q(ab)$ of ab at vb, and at each edge e of G_v the hinge $q(e)$ of e in (G_v^{ab}, q) (KT eq. (6.12)): the framework of Definition 23.147 for a normal n' of a hyperplane that meets $\Pi(a)$ in L. If r does not vanish on $C(L)$, that is, if the top-left $D \times D$ block of KT eq. (6.29) is nonsingular, then* $\operatorname{rank} R(G, p_1) \geq D(|V(G)| - 1)$.

Proof. We follow Katoh and Tanigawa [KT11, pp. 685–686]. Let W be the row space of $R(G_v, q)$. The edges of G_v have the same hinges in (G, p_1) as in (G_v^{ab}, q), so W lies in the row space of $R(G, p_1)$. The hinge of vb is $q(ab)$, so the form $S \mapsto r(S(v) - S(b))$ is a row at vb, as in Lemma 23.163. By KT eq. (6.24), the form $S \mapsto r(S(a) - S(b))$ lies in W. So their difference, the row w of KT eq. (6.27), the form $S \mapsto r(S(v) - S(a))$ (Lemma 23.161), lies in the row space of $R(G, p_1)$.

Let $r'_1, \dots, r'_{D-1}$ be linearly independent linear forms on K^D that vanish on $C(L)$, so that the rows of va are the forms $S \mapsto r'_i(S(v) - S(a))$ (Definition 19.5). After the column operation (Lemma 23.149), the rows of va and w are the forms $S \mapsto r'_i(S(v))$ and $S \mapsto r(S(v))$. They have entries only in the columns of v. There they form the top-left $D \times D$ block of KT eq. (6.29), with the rows $r'_1, \dots, r'_{D-1}$ and r. This block is nonsingular exactly when r is not a combination of the r'_i, that is, when r does not vanish on $C(L)$ (KT p. 690, Lemma 23.165). The other rows of KT eq. (6.29) are the rows $(vb)j$ with $j \neq i^*$ and the rows of the edges of G_v. Outside the columns of v they have the entries of $R(G_v^{ab} \setminus (ab)i^*, q)$, the matrix $R(G_v^{ab}, q)$ without the row $(ab)i^*$. By KT eq. (6.24) that row lies in the span of the others, so KT eq. (6.23) holds:

$$\operatorname{rank} R(G_v^{ab} \setminus (ab)i^*, q) \;=\; \operatorname{rank} R(G_v^{ab}, q) \;=\; D(|V(G)| - 2).$$

Now suppose that r does not vanish on $C(L)$. Then the r'_i and r span the linear forms on K^D. So for every linear form x on K^D, the form $S \mapsto x(S(v) - S(a))$ is a combination of the rows of va and of w, and it lies in the row space of $R(G, p_1)$. Hence so does each row $(ab)j$ of $R(G_v^{ab}, q)$:

$$r_j(S(a) - S(b)) \;=\; r_j(S(v) - S(b)) - r_j(S(v) - S(a)).$$

In KT eq. (6.29) this is the row operation, with the rows of the top-left block, that clears the entries below it. So every row of $R(G_v^{ab} \setminus (ab)i^*, q)$ lies in the row space of $R(G, p_1)$, and we choose $D(|V(G)| - 2)$ linearly independent ones. They vanish on the screw assignments that vanish off v, as no edge of G_v^{ab} has the end v (as in the proof of Lemma 23.157). On these screw assignments the rows of va and w are the forms $S \mapsto r'_i(S(v))$ and $S \mapsto r(S(v))$, which are linearly independent. By Lemma 23.159, the $(D - 1) + 1 + D(|V(G)| - 2) = D(|V(G)| - 1)$ forms are linearly independent. They lie in the row space of $R(G, p_1)$, so its rank is at least $D(|V(G)| - 1)$ (Lemma 19.34). □

Formalization note 23.168. The pinned theorem is that of Lemma 23.193, in the next subsubsection, and the lemma is that lemma at (G, p_1): the extensor $C(q(a), n')$ of its hinge at va is a nonzero multiple of $C(L)$ (the paragraph after Definition 23.147). Its hypotheses hold here. The form r vanishes on $C(q(a), q(b))$, and $S \mapsto r(S(a) - S(b))$ lies in the row space of $R(G_v, q)$ (Lemma 23.152). The rows of $R(G_v^{ab}, q)$ are those of $R(G_v, q)$ and the forms $S \mapsto r'(S(a) - S(b))$ at ab, for the linear forms r' that vanish on $C(q(a), q(b))$, and $D(|V(G_v)| - 1)$ of them are linearly independent, as $R(G_v^{ab}, q)$ has this rank. The theorem counts in Katoh and Tanigawa's form, with the bottom rows of KT eq. (6.29) linearly independent outside the columns of v, where the proof above first puts the rows of $R(G_v^{ab} \setminus (ab)i^*, q)$ in the row space of $R(G, p_1)$. At the normals q_0 of Lemma 23.163, if $q_0(a) = q(a)$ and $t \neq 0$, the hinge of va is $-t\, C(q(a), q(b))$. This is the case $L = q(ab)$, where r vanishes on $C(L)$

and the block is singular. Of the theorems pinned in this subsubsection, the proofs of the main results use the definition of (G, p_1) (Definition 23.147), the theorems of Lemma 23.149, Lemma 23.152, Lemma 23.154 and Lemma 23.165, the identity of Lemma 23.161, the restrictions of rows of Lemma 23.157, and the theorem pinned here.

23.10.3. *The case $d = 3$: three frameworks and KT Claim 6.12 (KT §6.4.1).* Katoh and Tanigawa prove Case III first for $d = 3$, where $D = 6$ (KT Lemma 6.10), and then for every d (KT Lemma 6.13) [KT11, §6.4]. The formal proof of KT Theorem 5.5 uses the general case at every d, including $d = 3$ (Lemma 23.208). We give the case $d = 3$ in full as well, as it has the same idea with fewer parts.

Let G be a 2-edge-connected minimal 0-dof-graph with no proper rigid subgraph, as in KT Lemma 6.10. If $|V(G)| = 3$, then G is a triangle, which KT Lemma 6.7(i) realizes by KT Lemma 5.4 (p. 677). So let $|V(G)| \geq 4$, as Katoh and Tanigawa do (p. 682). At $d = 3$, KT Lemma 4.6 then gives two adjacent vertices v and a of degree two, with $N_G(v) = \{a, b\}$ and $N_G(a) = \{v, c\}$ (KT p. 680). The vertices v, a, b and c are distinct, as otherwise v, a and b would span a triangle, a proper rigid subgraph, as in the proof of KT Lemma 6.7(ii). By KT Lemma 4.8, the graph G_v^{ab} and the graph G_a^{vc} obtained by splitting off at a along vc are both minimal 0-dof-graphs.

The argument then has fewer parts than for general d. The case of a short cycle in KT Lemma 4.6, which KT Lemma 6.13 treats with KT Lemma 5.4, does not arise. There are three frameworks, one for each of the panels of a, b and c, in place of d frameworks along a chain of length d. The third is built on G_a^{vc}, through one isomorphism $G_a^{vc} \cong G_v^{ab}$ that sends v to a (KT p. 686, Lemma 23.183). For general d there are $d-2$ such isomorphisms, and the rows are matched along the chain (KT eqs. (6.54) and (6.62)). The rank of each of the three frameworks is read off its block form (KT eqs. (6.29), (6.30) and (6.41)). The last step is the span of the six joins of four points in $\bigwedge^2 K^4$, in place of the D joins of $d-1$ of $d+1$ points in $\bigwedge^{d-1} K^{d+1}$ (KT eq. (6.67), p. 698).

Let (G_v^{ab}, q) and the linear form $r = \sum_j \lambda_j r_j$ on K^D be as in Lemma 23.163, and let $\Pi(u)$ be the panel of a vertex u in (G_v^{ab}, q). The first framework is (G, p_1) of KT eq. (6.12), with a line $L \subset \Pi(a)$ at va. By Lemma 23.167, $\operatorname{rank} R(G, p_1) \geq D(|V(G)| - 1)$ if the top-left block M_1 of KT eq. (6.29) is nonsingular. The second framework, (G, p_2), exchanges the roles of a and b, with a line $L' \subset \Pi(b)$ at vb (KT eqs. (6.19) and (6.30)). For $d = 2$, one of M_1 and M_2 is nonsingular for some choice of L and L', but for $d \geq 3$ this may fail (KT p. 686). The third framework, (G, p_3), has a line $L'' \subset \Pi(c)$ at ac (KT eqs. (6.31) to (6.41)). KT Claim 6.12 says that for some choice of L, L' and L'', at least one of the three blocks M_1, M_2, M_3 is nonsingular (KT p. 690).

Katoh and Tanigawa prove the claim by contradiction [KT11, pp. 690–691]. The form r is not zero (Lemma 23.179). The block M_1 is singular if and only if r vanishes on $C(L)$ (Lemma 23.165), and likewise M_2 for L'. The last row of M_3 is $-r$ (KT eq. (6.44), Lemma 23.157), so M_3 is singular if and only if r vanishes on $C(L'')$. If no choice makes a block nonsingular, then r vanishes on the extensors of all lines in $\Pi(a)$, $\Pi(b)$ and $\Pi(c)$, the set of KT eq. (6.45). Four points with suitable incidences

with the three panels show that these extensors span K^D, by KT Lemma 2.1. So $r = 0$, a contradiction. The rest of this subsubsection proves the span and chooses the points. It then gives the three frameworks and Claim 6.12 (Lemma 23.191). The case $d = 3$ of Case III follows (Lemma 23.195), and then the general case.

The proof of KT Claim 6.12 ends with KT Lemma 2.1 (Lemma 18.9), which holds for any linearly independent vectors (the remark after it).

Lemma 23.169 (The omit-two extensors of $d+1$ linearly independent vectors span $\bigwedge^{d-1} K^{d+1}$; from KT Lemma 2.1)**.** *Let $\bar{p}_1, \dots, \bar{p}_{d+1} \in K^{d+1}$ be linearly independent. For $1 \le i < j \le d+1$, let E_{ij} be the join of the $d-1$ vectors $\bar{p}_l$ with $l \ne i, j$, in increasing order of l. Then the D extensors E_{ij} span $\bigwedge^{d-1} K^{d+1}$. In particular, for $d = 3$ the six joins $\bar{p}_i \vee \bar{p}_j$ ($1 \le i < j \le 4$) span $\bigwedge^2 K^4$.*

Proof. The extensors E_{ij} are those of Lemma 18.9. By the remark after that lemma, they form a basis of $\bigwedge^{d-1} K^{d+1}$. At $d = 3$, $E_{ij} = \bar{p}_k \vee \bar{p}_l$, where $k < l$ and $\{k, l\} = \{1, 2, 3, 4\} \setminus \{i, j\}$. So the six joins are the six E_{ij}, and they span $\bigwedge^2 K^4$. □

Formalization note 23.170. The pinned theorem is the first assertion, for every d; the second is its case $d = 3$.

Lemma 23.171 (A linear form that vanishes on a spanning set is zero; from the proof of KT Claim 6.12)**.** *Let S be a subset of $\bigwedge^{d-1} K^{d+1}$ that spans it, and let r be a linear form on $\bigwedge^{d-1} K^{d+1}$ that vanishes on S. Then $r = 0$.*

Proof. The linear forms r and 0 agree on S, hence on its span, which is $\bigwedge^{d-1} K^{d+1}$. □

Katoh and Tanigawa take four points of K^3 with

$$p_1 \in \Pi(a) \cap \Pi(b) \cap \Pi(c), \qquad p_2 \in \Pi(a) \cap \Pi(b) \setminus \Pi(c),$$
$$p_3 \in \Pi(b) \cap \Pi(c) \setminus \Pi(a), \qquad p_4 \in \Pi(c) \cap \Pi(a) \setminus \Pi(b),$$

which they can choose affinely independent, as (G_v^{ab}, q) is generic and nonparallel (KT p. 691). The points p_i are not the frameworks (G, p_i); we keep Katoh and Tanigawa's notation. Any two of the points lie on a common panel. So the line through them lies in $\Pi(a) \cup \Pi(b) \cup \Pi(c)$, and its extensor $\bar{p}_i \vee \bar{p}_j$ (Definition 18.8) is in the set of KT eq. (6.45). By Lemma 23.169, the six joins span K^D.

For d generic panels in K^d, Katoh and Tanigawa use that any j of them meet in an affine subspace of dimension $d-j$ (KT p. 698). The point p_1 is the case $j = d = 3$. A point x lies on the panel with normal n if and only if $\bar{x} \cdot n = 0$ (the remarks after Definition 23.1). In homogeneous coordinates, fewer than $d+1$ panels always have a common nonzero vector.

Lemma 23.172 (A common nonzero vector of at most d panels; from the proof of KT Claim 6.12)**.** *Let $m < d+1$, and let $n_1, \dots, n_m \in K^{d+1}$. Then there is a nonzero $x \in K^{d+1}$ with $x \cdot n_i = 0$ for $1 \le i \le m$.*

Proof. The map $K^{d+1} \to K^m$ that sends x to $(x \cdot n_1, \dots, x \cdot n_m)$ is linear, and $m < d+1$. By rank–nullity its kernel has dimension at least $d+1-m > 0$, so it contains a nonzero vector. □

The next three lemmas concern the affine choice of the points. The formal proof does not use them: it chooses vectors of K^4 in place of the points (Lemma 23.177). If the determinant of four points that depend on some variables is a nonzero polynomial in them, the points are affinely independent for some values of the variables (Lemma 23.173). The determinant is a polynomial in the variables when the coordinates of the points are (Lemma 23.174). The coordinate planes of K^3 are three planes with four affinely independent points with the incidences above (Lemma 23.176).

Lemma 23.173 (Affine independence off the zeros of a polynomial; for the points of KT Claim 6.12)**.** *Let K be infinite, and let σ be a set of variables. For each $t \in K^\sigma$, let $p_1(t), \dots, p_4(t)$ be points of K^3. Suppose that a nonzero polynomial $P \in K[\sigma]$ satisfies $P(t) = \det(\bar{p}_1(t), \dots, \bar{p}_4(t))$, the determinant of the matrix with these rows, for every t. Then $p_1(t), \dots, p_4(t)$ are affinely independent for some t.*

Proof. As K is infinite and $P \neq 0$, there is a $t \in K^\sigma$ with $P(t) \neq 0$. At this t the vectors $\bar{p}_1(t), \dots, \bar{p}_4(t)$ have a nonzero determinant, so the points are affinely independent (Lemma 18.2). □

Lemma 23.174 (The determinant of points with polynomial coordinates; for the points of KT Claim 6.12)**.** *Let σ be a set of variables, and let $P_{ij} \in K[\sigma]$ for $1 \le i \le 4$ and $1 \le j \le 3$. For $t \in K^\sigma$, let $p_i(t) = (P_{i1}(t), P_{i2}(t), P_{i3}(t)) \in K^3$. Then some $P \in K[\sigma]$ satisfies $P(t) = \det(\bar{p}_1(t), \dots, \bar{p}_4(t))$ for every $t \in K^\sigma$.*

Proof. Let P be the determinant of the 4×4 matrix of polynomials with the rows $(P_{i1}, P_{i2}, P_{i3}, 1)$. Evaluation at t is a ring homomorphism $K[\sigma] \to K$, so it commutes with the determinant. It takes the row $(P_{i1}, P_{i2}, P_{i3}, 1)$ to $(p_i(t), 1) = \bar{p}_i(t)$ (Definition 18.1). So $P(t)$ is the determinant of $\bar{p}_1(t), \dots, \bar{p}_4(t)$. □

Formalization note 23.175. The pinned theorems of this lemma and of Lemma 23.173 hold for $d+1$ points of K^d, for every d.

Lemma 23.176 (Three planes with the points of KT Claim 6.12, at the coordinate planes)**.** *There are linearly independent $n_a, n_b, n_c \in K^4$ and affinely independent points $p_1, \dots, p_4 \in K^3$ such that*

$$p_1 \in \Pi(a) \cap \Pi(b) \cap \Pi(c), \qquad p_2 \in \Pi(a) \cap \Pi(b) \setminus \Pi(c),$$
$$p_3 \in \Pi(b) \cap \Pi(c) \setminus \Pi(a), \qquad p_4 \in \Pi(c) \cap \Pi(a) \setminus \Pi(b),$$

where $\Pi(u)$ is the plane $\{x \in K^3 : \bar{x} \cdot n_u = 0\}$.

Proof. Let n_a, n_b, n_c be the first three standard basis vectors of K^4, so that $\Pi(a)$, $\Pi(b)$ and $\Pi(c)$ are the coordinate planes $x_1 = 0$, $x_2 = 0$ and $x_3 = 0$ of K^3. Let $p_1 = 0$, $p_2 = e_3$, $p_3 = e_1$ and $p_4 = e_2$, where e_1, e_2, e_3 is the standard basis of K^3. The point p_1 lies on the three planes. The point e_3 lies on $\Pi(a)$ and $\Pi(b)$ but not on $\Pi(c)$, and e_1 and e_2 have the incidences required of p_3 and p_4 in the same way. Subtracting $\bar{p}_1 = (0, 0, 0, 1)$ from the other rows of the matrix with the rows

$\bar{p}_1, \dots, \bar{p}_4$ leaves a permutation matrix, so its determinant is ± 1. Hence the points are affinely independent (Lemma 18.2). □

The formal proof of the case $d = 3$ chooses the normals q of (G_v^{ab}, q) off the zeros of a product of polynomials, as in the passage after Lemma 23.100. One factor makes $q(a)$, $q(b)$ and $q(c)$ linearly independent (Lemma 23.195, below). The proof needs the four points for these normals. It takes them in homogeneous coordinates: vectors $\bar{p}_i \in K^4$, whose last coordinate need not be 1, with $\bar{p}_i \cdot q(u) = 0$ exactly for the panels $\Pi(u)$ that contain p_i. The span of Lemma 23.169 needs only that these vectors are linearly independent. A line in the panel $\Pi(u)$ is given in the formal proof as the intersection of $\Pi(u)$ with a second plane, and its extensor as their panel support extensor. If $\bar{p}_i$ and $\bar{p}_j$ are orthogonal to both normals, this extensor is a multiple of $\bar{p}_i \vee \bar{p}_j$ (Lemma 22.16). The vectors exist for any three linearly independent normals.

Lemma 23.177 (The points of KT Claim 6.12 in homogeneous coordinates, for given normals)**.** *Let $n_a, n_b, n_c \in K^4$ be linearly independent. Then there are linearly independent $\bar{p}_1, \dots, \bar{p}_4 \in K^4$ with $\bar{p}_1 \cdot n_a = \bar{p}_1 \cdot n_b = \bar{p}_1 \cdot n_c = 0$ and*

$$\bar{p}_2 \cdot n_a = \bar{p}_2 \cdot n_b = 0 \neq \bar{p}_2 \cdot n_c,$$
$$\bar{p}_3 \cdot n_b = \bar{p}_3 \cdot n_c = 0 \neq \bar{p}_3 \cdot n_a,$$
$$\bar{p}_4 \cdot n_c = \bar{p}_4 \cdot n_a = 0 \neq \bar{p}_4 \cdot n_b.$$

Proof. The linear map $K^4 \to K^3$ that sends x to $(x \cdot n_a, x \cdot n_b, x \cdot n_c)$ has the linearly independent rows n_a, n_b, n_c, so it has rank 3 and is onto. Let $\bar{p}_2$, $\bar{p}_3$ and $\bar{p}_4$ be vectors that it sends to $(0, 0, 1)$, $(1, 0, 0)$ and $(0, 1, 0)$. Let $\bar{p}_1$ be a nonzero vector orthogonal to n_a, n_b and n_c (Lemma 23.172, with $m = 3$). These vectors have the incidences of the claim.

Suppose that $\sum_i g_i \bar{p}_i = 0$. The dot product with n_a gives $g_3 = 0$, that with n_b gives $g_4 = 0$, and that with n_c gives $g_2 = 0$. Then $g_1 \bar{p}_1 = 0$, and $g_1 = 0$ as $\bar{p}_1 \neq 0$. So the four vectors are linearly independent. □

Formalization note 23.178. The pinned theorem holds for every d. For d linearly independent normals in K^{d+1}, it gives $d + 1$ linearly independent vectors: first one orthogonal to all the normals, and then, for each normal in turn, one orthogonal to all the others and not to it. The lemma is its case $d = 3$, with the normals n_c, n_a, n_b in this order. The formal proof of the general case uses the pinned theorem (Lemma 23.200), and that of the case $d = 3$ uses the lemma, derived from it.

Lemma 23.179 (The form r is not zero; from the proof of KT Claim 6.12)**.** *Under the hypotheses of Lemma 23.145, let the forms r_j and the scalars λ_j be as in Lemma 23.152. Then the linear form $r = \sum_j \lambda_j r_j$ on K^D is not zero.*

Proof. We follow Katoh and Tanigawa [KT11, p. 690]. The forms $r_1, \dots, r_{D-1}$ are linearly independent, and $\lambda_{i^*} = 1$ (KT eq. (6.25), Lemma 23.152). A linear combination of linearly independent vectors with a nonzero coefficient is not zero. □

Formalization note 23.180. The pinned theorem holds for any linearly independent family and any scalars with $\lambda_{i^*} = 1$.

The form r serves for the second framework as well, as it depends only on (G_v^{ab}, q) (KT p. 686). Like Lemma 23.167, the next lemma is a case of the count of Lemma 23.193.

Lemma 23.181 (The second framework; KT eqs. (6.19) and (6.30))**.** *Under the hypotheses of Lemma 23.145, let r be the linear form of Lemma 23.163. Let L' be a $(d-2)$-dimensional affine subspace of the panel $\Pi(b)$ of (G_v^{ab}, q), and let (G, p_2) be the body-hinge framework with the hinge $q(ab)$ at va, the hinge L' at vb, and the hinge $q(e)$ at each edge e of G_v (KT eq. (6.19)): the framework (G, p_1) of Definition 23.147 with the roles of a and b exchanged, for a normal of a hyperplane that meets $\Pi(b)$ in L'. If r does not vanish on $C(L')$, that is, if the top-left $D \times D$ block M_2 of KT eq. (6.30) is nonsingular, then* $\operatorname{rank} R(G, p_2) \geq D(|V(G)| - 1)$.

Proof. We argue as in the proof of Lemma 23.167, with the roles of a and b exchanged (KT pp. 682 and 686). Let W be the row space of $R(G_v, q)$, which lies in the row space of $R(G, p_2)$. The hinge of va is $q(ab)$, on which r vanishes, so the form $S \mapsto r(S(v) - S(a))$ is a row at va. By KT eq. (6.24), the form $S \mapsto r(S(a) - S(b))$ lies in W. So their sum, the form $S \mapsto r(S(v) - S(b))$, lies in the row space of $R(G, p_2)$.

Let $r'_1, \dots, r'_{D-1}$ be linearly independent forms that vanish on $C(L')$, so that the rows of vb are the forms $S \mapsto r'_i(S(v) - S(b))$. Katoh and Tanigawa add the columns of v to those of b (KT p. 682). By Lemma 23.149, with b in place of a, this turns the rows of vb and the form $S \mapsto r(S(v) - S(b))$ into the forms $S \mapsto r'_i(S(v))$ and $S \mapsto r(S(v))$. These make up the block M_2, with the rows $r'_1, \dots, r'_{D-1}$ and r, which is nonsingular exactly when $r(C(L')) \neq 0$ (Lemma 23.165). The count in the proof of Lemma 23.167, with b in place of a, then gives the bound. In it the rows of the edges of G_v^{ab} vanish on the screw assignments that vanish off v, as no edge of G_v^{ab} has the end v (as in the proof of Lemma 23.157). □

Formalization note 23.182. The pinned theorem is that of Lemma 23.193, and the lemma is that lemma at (G, p_2), with a and b exchanged and the form $-r$. Its hypotheses hold as in the note after Lemma 23.167: $-r$ vanishes on $C(q(b), q(a)) = -C(q(a), q(b))$, the form $S \mapsto (-r)(S(b) - S(a))$ is $S \mapsto r(S(a) - S(b))$, and the rows at ab are the forms $S \mapsto (-r')(S(b) - S(a))$. The formal proofs of Lemma 23.195 and, for $i = 1$, of Lemma 23.206 apply it so.

The next three lemmas give Katoh and Tanigawa's realization of G_a^{vc} with the panels of (G_v^{ab}, q), in which the panel of a becomes that of v (KT pp. 686–687). The third framework is built on it. The formal proof of the case $d = 3$ does not use these lemmas.

Lemma 23.183 (The isomorphism from G_a^{vc} to G_v^{ab}; KT p. 686)**.** *Let G be a multigraph, and let v, a, b and c be distinct vertices of G such that the edges of G at v are va and vb, and those at a are va and ac. Let $\rho\colon V(G) \setminus \{a\} \to V(G) \setminus \{v\}$ be the bijection with $\rho(v) = a$ and $\rho(u) = u$ for $u \neq v$. Then ρ is an isomorphism from G_a^{vc} to G_v^{ab} that maps the edge vb to the new edge ab, the new edge vc to the edge ac, and every other edge to itself.*

Proof. The edges of G_a^{vc} are vb, the new edge vc, and the edges of G that meet neither v nor a. Those of G_v^{ab} are ac, the new edge ab, and the same edges of G (Definition 21.10). The bijection ρ fixes the ends of an edge that meets neither v nor a. It maps the ends v, b of vb to a, b, and the ends v, c of vc to a, c. □

Formalization note 23.184. In the formalization the edges of a multigraph are labels (Formalization note 21.11). The pinned theorem takes labels $e_0 \neq e_1$ outside $E(G)$ for the new edges ab of G_v^{ab} and vc of G_a^{vc}. It replaces ρ by the transposition of a and v on all bodies, and the map on edges by the permutation of the labels that exchanges the label of vb with e_0 and that of ac with e_1. It states that an edge joins x and y in G_a^{vc} if and only if its image joins the images of x and y in G_v^{ab}.

Lemma 23.185 (A realization of G_a^{vc}; KT eq. (6.31))**.** *Let G, v, a, b and c be as in Lemma 23.183, and let q be normals. Let q_ρ be the normals with $q_\rho(v) = q(a)$, $q_\rho(a) = q(v)$, and $q_\rho(u) = q(u)$ for every other body u. Suppose that the normals q are in general position and that every infinitesimal motion of (G_v^{ab}, q) is constant on $V(G_v^{ab})$. Then the normals q_ρ are in general position, and every infinitesimal motion of (G_a^{vc}, q_ρ) is constant on $V(G_a^{vc})$. In particular, if G_v^{ab} and G_a^{vc} are 0-dof-graphs and G_v^{ab} has a generic full-rank realization, then so has G_a^{vc}.*

Proof. The normals q_ρ are those of q, permuted, so they are in general position (Definition 23.9). By Lemma 23.183, an edge of G_a^{vc} with ends u and w corresponds to an edge of G_v^{ab} with ends $\rho(u)$ and $\rho(w)$. The two edges have the same hinge $C(q(\rho(u)), q(\rho(w)))$, which is KT eq. (6.31).

For a screw assignment S, let S_ρ agree with S except that $S_\rho(a) = S(v)$ and $S_\rho(v) = S(a)$. Then $S_\rho(\rho(u)) - S_\rho(\rho(w)) = S(u) - S(w)$ for all vertices u, w of G_a^{vc}. So if S is an infinitesimal motion of (G_a^{vc}, q_ρ), then S_ρ is one of (G_v^{ab}, q). Then S_ρ is constant on $V(G_v^{ab})$, which is the image of $V(G_a^{vc})$ under ρ, and so S is constant on $V(G_a^{vc})$. The last claim follows, as a realization of a 0-dof-graph is at the target rank exactly when its infinitesimal motions are constant on the vertices (Remark 23.23). □

Formalization note 23.186. The first pinned theorem is the first claim. It replaces ρ by the transposition of a and v, so that q_ρ is q composed with it. It also transports the endpoint selector of (G_v^{ab}, q) (Formalization note 23.6) along this transposition and the permutation of the labels of Lemma 23.183. It assumes that the selector assigns each edge of G_v^{ab} its ends, and concludes the same for the transported selector and G_a^{vc}. The second pinned theorem is the last claim.

Lemma 23.187 (The rows of the realization of G_a^{vc}; KT eq. (6.31))**.** *Let G, v, a, b, c, q and q_ρ be as in Lemma 23.185. For a screw assignment S, let S_ρ agree with S except that $S_\rho(a) = S(v)$ and $S_\rho(v) = S(a)$. Then the rows of $R(G_a^{vc}, q_\rho)$ are the forms $S \mapsto f(S_\rho)$ for the rows f of $R(G_v^{ab}, q)$, where the rows at an edge with ends u and w are all the forms $S \mapsto x(S(u) - S(w))$ for the linear forms x on K^D that vanish on its hinge. In particular, the two matrices have the same rank.*

Proof. Let uw be an edge of G_a^{vc}. As in the proof of Lemma 23.185, the edge $\rho(u)\rho(w)$ of G_v^{ab} has the same hinge, and every edge of G_v^{ab} arises in this way. For a form

x that vanishes on this hinge, the row $f\colon S \mapsto x(S(\rho(u)) - S(\rho(w)))$ of $R(G_v^{ab}, q)$ satisfies $f(S_\rho) = x(S(u) - S(w))$. This proves the first claim. The map $S \mapsto S_\rho$ is a linear automorphism, so $f \mapsto (S \mapsto f(S_\rho))$ is injective, and the two row spaces have the same dimension. □

Formalization note 23.188. The pinned theorem is the first claim, for the transported endpoint selector of Lemma 23.185 and for every endpoint selector of (G_v^{ab}, q).

We now turn to the third framework. The last row of its block M_3 comes from the rows of $R(G_v^{ab}, q)$ at ac (KT eq. (6.42)), and it is $-r$ (KT eq. (6.44), Lemma 23.157). Like Lemma 23.167, the next lemma is a case of the count of Lemma 23.193.

Lemma 23.189 (The third framework; KT eqs. (6.31)–(6.41))**.** *Under the hypotheses of Lemma 23.145, suppose that the edges of G at a are va and ac, where v, a, b and c are distinct. Let r be the linear form of Lemma 23.163. Let L'' be a $(d-2)$-dimensional affine subspace of the panel $\Pi(c)$ of (G_v^{ab}, q), and let (G, p_3) be the body-hinge framework with the hinge $q(ac)$ at va, the hinge $q(ab)$ at vb, the hinge L'' at ac, and the hinge $q(e)$ at every other edge e (KT eq. (6.33)): the framework (G, p_1) of Definition 23.147 for the normals q_ρ of Lemma 23.185, with a, c and v in the roles of v, a and b, and a normal of a hyperplane that meets $\Pi(c)$ in L''. If r does not vanish on $C(L'')$, that is, if the top-left $D \times D$ block M_3 of KT eq. (6.41) is nonsingular, then* $\operatorname{rank} R(G, p_3) \geq D(|V(G)| - 1)$.

Proof. We follow Katoh and Tanigawa [KT11, pp. 686–689]. Let (G_a^{vc}, q_ρ) be the panel-hinge framework of Lemma 23.185. In it the panel of v is $\Pi(a)$, the hinge of vb is $q(ab)$, the hinge of the new edge vc is $q(ac)$, and the other edges keep their hinges (KT eq. (6.31)). Katoh and Tanigawa obtain (G, p_3) from (G_a^{vc}, q_ρ) as (G, p_1) from (G_v^{ab}, q), with a, c and v in the roles of v, a and b. So va gets the hinge $q(ac)$ of vc, and ac gets L'' (KT eqs. (6.32) and (6.33)).

Write the rows of $R(G_v^{ab}, q)$ at ab and at ac as the forms $S \mapsto r_j(S(a) - S(b))$ and $S \mapsto s_j(S(a) - S(c))$, as in Lemma 23.157. The rows of $R(G, p_3)$ at vb and at va are then the forms $S \mapsto r_j(S(v) - S(b))$ and $S \mapsto s_j(S(v) - S(a))$. The edges of G other than va, vb and ac have the same rows in both matrices (KT eq. (6.34)). Katoh and Tanigawa add to the row $(vb)i^*$ the other terms of KT eq. (6.24), with the rows at vb and va in place of those at ab and ac (KT eq. (6.40)). Call the result w_3. For a screw assignment S, let S' agree with S except that $S'(a) = S(v)$. The edges of G_v other than ac do not meet a, so KT eq. (6.24) at S' reads

$$\sum_j \lambda_j\, r_j(S(v) - S(b)) + \sum_j \lambda_{(ac)j}\, s_j(S(v) - S(c)) + \sum_{e \neq ac} \sum_j \lambda_{ej}\, R(G_v^{ab}, q; e_j)(S) \;=\; 0,$$

where e runs over the edges of G_v. Subtracting this from $w_3(S)$ leaves

$$\sum_j \lambda_{(ac)j}\, s_j(S(c) - S(a)) \;=\; r(S(a) - S(c)),$$

by KT eq. (6.44) (Lemma 23.157). So w_3 is the form $S \mapsto r(S(a) - S(c))$.

Let $r'_1, \dots, r'_{D-1}$ be linearly independent forms that vanish on $C(L'')$, so that the rows of ac are the forms $S \mapsto r'_i(S(a) - S(c))$. Katoh and Tanigawa add the columns of a to those of c (KT eqs. (6.35) and (6.36)). By Lemma 23.149, with a and c in

place of v and a, this turns the rows of ac and w_3 into the forms $S \mapsto r'_i(S(a))$ and $S \mapsto r(S(a))$. These make up the top-left block M_3 of KT eq. (6.41), with the rows $r'_1, \dots, r'_{D-1}$ and r, up to sign. Katoh and Tanigawa write its last row as $\sum_j \lambda_{(ac)j} s_j$, which is $-r$. The block is nonsingular exactly when $r(C(L'')) \neq 0$ (Lemma 23.165).

Now suppose that $r(C(L'')) \neq 0$. Then the r'_i and r span the linear forms on K^D. So for every linear form x on K^D, the form $S \mapsto x(S(a) - S(c))$ lies in the row space of $R(G, p_3)$. Hence so does every row of $R(G_a^{vc}, q_\rho)$. Its rows at vb and at the edges of G that meet neither v nor a are rows of $R(G, p_3)$. Its rows at vc are the forms

$$s_j(S(v) - S(c)) \;=\; s_j(S(v) - S(a)) + s_j(S(a) - S(c)),$$

sums of a row of va and a form of this kind. By Lemma 23.187, $R(G_a^{vc}, q_\rho)$ has the rank $D(|V(G)| - 2)$ of $R(G_v^{ab}, q)$, and we choose $D(|V(G)| - 2)$ linearly independent rows of it. They vanish on the screw assignments that vanish off a, as a is not a vertex of G_a^{vc}. On these screw assignments the rows of ac and w_3 are the forms $S \mapsto r'_i(S(a))$ and $S \mapsto r(S(a))$, which are linearly independent. By Lemma 23.159, the $(D-1) + 1 + D(|V(G)| - 2) = D(|V(G)| - 1)$ forms are linearly independent. They lie in the row space of $R(G, p_3)$, which gives the bound. Katoh and Tanigawa read the same bound off KT eq. (6.41), as D plus the rank of the block below M_3, which KT eq. (6.23) gives. □

Formalization note 23.190. The pinned theorem is that of Lemma 23.193, and the lemma is that lemma at (G, p_3), with a, c and v in the roles of v, a and b, the normals q_ρ, the form $-r$, and $G - a$ in place of G_v. Its hypotheses hold by the proof above. The form $-r$ vanishes on $C(q(c), q(a))$, as r is a combination of the forms s_j, which vanish on the hinge $q(ac)$ (KT eq. (6.44), Lemma 23.157). By the display at S' and KT eq. (6.44), $r(S(v) - S(c))$ is $r(S(v) - S(b))$ plus the terms of the edges of G_v other than ac. So the form $S \mapsto r(S(v) - S(c))$ lies in the row space of $R(G - a, q_\rho)$, whose edge vb has the hinge $q(ab)$, on which r vanishes, and whose other edges are those edges, with their hinges. The $D(|V(G)| - 2)$ linearly independent rows of $R(G_a^{vc}, q_\rho)$ chosen there are rows of $R(G - a, q_\rho)$ or forms $S \mapsto x(S(v) - S(c))$ at vc, with x vanishing on $C(q(a), q(c))$. The formal proof of Lemma 23.195 applies it so, with the span form of KT eq. (6.44) (the note after Lemma 23.157).

Lemma 23.191 (KT Claim 6.12: r does not vanish on all six joins)**.** *Let $d = 3$. Under the hypotheses of Lemma 23.145, let r be the linear form of Lemma 23.163, and let $\bar{p}_1, \dots, \bar{p}_4 \in K^4$ be linearly independent. Then $r(\bar{p}_i \vee \bar{p}_j) \neq 0$ for some $1 \leq i < j \leq 4$.*

Proof. We follow Katoh and Tanigawa [KT11, pp. 690–691]. The form r is not zero (Lemma 23.179). For $d = 3$, the screw space K^D is $\bigwedge^2 K^4$, and the six joins $\bar{p}_i \vee \bar{p}_j$ span it (Lemma 23.169, from KT Lemma 2.1). If r vanished on all of them, then r would be zero (Lemma 23.171).

This gives KT Claim 6.12: for some choice of the lines $L \subset \Pi(a)$, $L' \subset \Pi(b)$ and $L'' \subset \Pi(c)$, one of M_1, M_2 and M_3 is nonsingular. Indeed, let c be as at the start of this subsubsection, with $q(a)$, $q(b)$ and $q(c)$ linearly independent. Let $\bar{p}_1, \dots, \bar{p}_4$ be the vectors of Lemma 23.177 for these normals. Any two of them are orthogonal to a common normal $q(u)$, with $u \in \{a, b, c\}$. Let $\bar{p}_i \vee \bar{p}_j$ be a join on which r does not

vanish, and let n' be a vector orthogonal to $\bar{p}_i$ and $\bar{p}_j$ and linearly independent of $q(u)$. Then $\bar{p}_i \vee \bar{p}_j$ is a multiple of $C(q(u), n')$ (Lemma 22.16), so r does not vanish on $C(q(u), n')$. This is a nonzero multiple of the extensor of the line in which $\Pi(u)$ meets the plane with normal n' (the remarks after Definition 23.1). So the block M_1, M_2 or M_3 for this line is nonsingular, by Lemma 23.165 and, for M_3, KT eq. (6.44).

The lines are chosen after r, and the proof needs this freedom. The extensors of three fixed lines span at most three of the six dimensions of $\bigwedge^2 K^4$, so a nonzero r can vanish on all three. □

Formalization note 23.192. The pinned theorem holds for every d and every nonzero linear form r on $\bigwedge^{d-1} K^{d+1}$, with the D extensors E_{ij} of Lemma 23.169 in place of the joins. The general case uses it in this form (Lemma 23.200). The step from a join to a line in the second paragraph is `BodyHingeFramework.exists_complementIso_ne_zero_of_homogeneousIncidence`, which applies Lemma 22.16.

The formal proofs of KT Lemmas 6.10 and 6.13 obtain the rank of every framework from one theorem: the count of KT eq. (6.29) at (G, p_1), in Katoh and Tanigawa's form, with the bottom rows of KT eq. (6.29) linearly independent outside the columns of v. It uses the block criterion of Lemma 23.165 and the restrictions of rows in the proof of Lemma 23.157. At the three frameworks of KT Lemma 6.10 it gives Lemma 23.167, Lemma 23.181 and Lemma 23.189, which pin it.

Lemma 23.193 (The rank of $R(G, p_1)$ from the bottom rows of KT eq. (6.29); from the proof of KT Lemma 6.10)**.** *Let G be a multigraph with a vertex v of degree two, whose edges va and vb join v to distinct vertices $a, b \neq v$, and let $G_v = G - v$. Let q be normals, let $n' \in K^{d+1}$ be linearly independent of $q(a)$, and let (G, p_1) be the framework of Definition 23.147 for n'. Let r be a linear form on $\bigwedge^{d-1} K^{d+1}$ that vanishes on $C(q(a), q(b))$, such that the form $S \mapsto r(S(a) - S(b))$ lies in the row space of $R(G_v, q)$. Suppose that some $D(|V(G_v)| - 1)$ linearly independent forms are each a row of $R(G_v, q)$ or a form $S \mapsto r'(S(a) - S(b))$, for a linear form r' that vanishes on $C(q(a), q(b))$. If $r(C(q(a), n')) \neq 0$, then*

$$\operatorname{rank} R(G, p_1) \;\geq\; D\,(|V(G)| - 1).$$

Proof. We follow Katoh and Tanigawa [KT11, pp. 684–686], in the form of their count: the rows of $R(G, p_1)$ in the bottom-right block of KT eq. (6.29) are linearly independent outside the columns of v. The edges of G_v are the edges of G without the end v, with the same ends (Lemma 21.12). So they keep their hinges in (G, p_1) (Definition 23.147), and the row space of $R(G_v, q)$ lies in that of $R(G, p_1)$. The hinge of vb is $C(q(a), q(b))$, so for a linear form r' that vanishes on it, $S \mapsto r'(S(v) - S(b))$ is a row of $R(G, p_1)$ at vb.

Let Φ be the column operation of Lemma 23.149, and let P be the map on screw assignments that sets the value at v to 0 and keeps the others. Composing a form f with Φ and then with P keeps its entries outside the columns of v after the column operation, and drops those in the columns of v. For each of the given forms w we choose a form $\tilde{w}$ in the row space of $R(G, p_1)$ with $\tilde{w} \circ \Phi \circ P = w$. If w is a row of $R(G_v, q)$, we take $\tilde{w} = w$: it does not read $S(v)$, as no edge of G_v has the end v (as in the proof of Lemma 23.157), and $\Phi(P(S))$ agrees with S off v. If w is

the form $S \mapsto r'(S(a) - S(b))$, we take the row $\tilde{w}$: $S \mapsto r'(S(v) - S(b))$ at vb: as $\Phi(P(S))(v) = S(a)$, we get $\tilde{w}(\Phi(P(S))) = w(S)$. This is KT eq. (6.16): outside the columns of v, $R(G, p_1)$ has the rows of $R(G_v^{ab}, q)$. The forms $\tilde{w} \circ \Phi \circ P$, which are the given forms, are linearly independent.

The hinge of va has the extensor $C(q(a), n') \neq 0$, so some $D-1$ rows of $R(G, p_1)$ at va are linearly independent (Lemma 23.122). They are the forms $S \mapsto r_i'(S(v)-S(a))$ for a basis $r_1', \dots, r_{D-1}'$ of the linear forms that vanish on $C(q(a), n')$. Let w_r be the form $S \mapsto r(S(v)-S(a))$, the row of KT eq. (6.27). After the column operation, the rows of va and w_r are the forms $S \mapsto r_i'(S(v))$ and $S \mapsto r(S(v))$, the top-left block of KT eq. (6.29), and they vanish where $S(v) = 0$. As $r(C(q(a), n')) \neq 0$, r is not a combination of the r_i', so these forms are linearly independent (Lemma 23.165). Now suppose that a combination of the rows of va, w_r and the forms $\tilde{w}$ is zero. Composed with Φ and then with P, the rows of va and w_r vanish, as $P(S)(v) = 0$. What remains is a combination of the given forms, so its coefficients are 0. The rest of the combination, composed with Φ, is a combination of the forms $S \mapsto r_i'(S(v))$ and $S \mapsto r(S(v))$, so its coefficients are 0 too. Hence the $(D-1)+1+D(|V(G_v)|-1)$ forms are linearly independent.

Each lies in the row space of $R(G, p_1)$. For the rows of va and the forms $\tilde{w}$ this was shown above. The form w_r is the row $S \mapsto r(S(v) - S(b))$ at vb minus the form $S \mapsto r(S(a) - S(b))$ (Lemma 23.161), which lies in the row space of $R(G_v, q)$. As $|V(G)| = |V(G_v)| + 1$, there are $D(|V(G)| - 1)$ of them, and $\operatorname{rank} R(G, p_1) \geq D(|V(G)| - 1)$. □

Formalization note 23.194. The pinned theorem holds for any multigraph G_v without v whose edges are edges of G, which has a and b among its vertices and one vertex fewer than G, with an endpoint selector that assigns the edges of G and of G_v their ends. It needs neither that v has degree two nor that $a \neq b$. It assumes that the endpoint selector (Formalization note 23.6) orients va and vb away from v. In the formal proof of Case III, r comes from Lemma 23.154, and the forms are chosen among the rows of $R(G_v, q)$ and the forms r_j with $j \neq i^*$ of that lemma, which span the row space of $R(G_v^{ab}, q)$, as in KT eq. (6.23).

We now prove the case $d = 3$ of Case III. The lemma assumes a generic full-rank realization of G_v^{ab}, as the induction hypothesis gives. It also assumes the bound of KT eq. (6.22), as Lemma 23.140 gives.

Lemma 23.195 (The case $d = 3$ of Case III; KT Lemma 6.10)**.** *Let $d = 3$, so that $D = 6$, and let K be an infinite field. Let G be a simple 0-dof-graph, and let v, a, b and c be distinct vertices of G such that the edges of G at v are va and vb, and those at a are va and ac. Suppose that G_v^{ab} is a 0-dof-graph with a generic full-rank realization (Definition 23.21), and that there is a nonzero polynomial P in the normals with*

$$\operatorname{rank} R(G_v, q) \;\geq\; D\,(|V(G_v)| - 1) - (D - 2)$$

wherever $P(q) \neq 0$, as in Lemma 23.140. Then G has a generic full-rank realization, of rank $D(|V(G)| - 1)$.

Proof. We follow Katoh and Tanigawa's proof of KT Lemma 6.10 (pp. 682–691). We choose the normals off the zeros of a product of polynomials fixed before the choice, as explained after Lemma 23.100. The first factor comes from the generic full-rank realization of G_v^{ab} (Lemma 23.120). The second is P, the third puts the normals in general position, and the fourth makes $q(a)$, $q(b)$ and $q(c)$ linearly independent. As K is infinite, the product has a non-root q. The first factor gives $\operatorname{rank} R(G_v^{ab}, q) \geq D(|V(G)| - 2)$. This is also an upper bound (Lemma 20.16), so equality holds, which is KT eq. (6.18). The second factor gives KT eq. (6.22), $\operatorname{rank} R(G_v, q) \geq D(|V(G_v)| - 1) - (D - 2)$.

So the hypotheses of Lemma 23.145 hold, where $k' \leq D - 2$ is the difference between $D(|V(G_v)|-1)$ and $\operatorname{rank} R(G_v, q)$. Let r be the linear form of Lemma 23.163. It is not zero, it vanishes on $C(q(a), q(b))$, and the form $S \mapsto r(S(a) - S(b))$ lies in the row space of $R(G_v, q)$ (KT eqs. (6.24) and (6.25), Lemma 23.152). Let $\bar{p}_1, \dots, \bar{p}_4$ be the vectors of Lemma 23.177. By the proof of KT Claim 6.12 (Lemma 23.191), there are $u \in \{a, b, c\}$ and a vector n', linearly independent of $q(u)$, such that r does not vanish on $C(q(u), n')$. This is a nonzero multiple of the extensor of the line L in which $\Pi(u)$ meets the plane with normal n'.

If $u = a$, then $\operatorname{rank} R(G, p_1) \geq D(|V(G)|-1)$ for this L (Lemma 23.167). If $u = b$, the same holds for (G, p_2) with $L' = L$ (Lemma 23.181). If $u = c$, it holds for (G, p_3) with $L'' = L$ (Lemma 23.189).

Let $u = a$. Katoh and Tanigawa rotate the panel of v slightly (KT Lemma 5.2, KT p. 691), and we rotate it about L. For $t \in K$, let F_t be the body-hinge framework on G whose hinges have the extensors $C(q(a), n')$ at va, $C(q(a) + t\,n', q(b))$ at vb, and those of (G_v^{ab}, q) at the edges of G_v (Definition 23.147). Then F_0 has the rank of (G, p_1). Let q_t agree with q except that $q_t(v) = q(a) + t\,n'$, whose panel contains L. For $t \neq 0$, the extensor of the hinge of va in (G, q_t) is $C(q(a) + t\,n', q(a)) = -t\,C(q(a), n')$. The two frameworks then have the same hinge spans, so (G, q_t) has the rank of F_t (Lemma 19.22, applied both ways). The rows of F_t can be chosen polynomial in t, as in the proof of Lemma 23.91. So the rank $D(|V(G)|-1)$ at $t = 0$ persists for all but finitely many t, as in the proof of Lemma 23.120: a minor that does not vanish at $t = 0$ is a nonzero polynomial in t. For all but at most one t, the normals $q(a) + t\,n'$ and $q(b)$ are linearly independent. As K is infinite, some $t \neq 0$ gives a panel-hinge framework (G, q_t) with nondegenerate hinges and rank at least $D(|V(G)| - 1)$. For $u = b$ we rotate the panel of v from $\Pi(b)$ about L. For $u = c$ the panel of v is $\Pi(a)$, and we rotate the panel of a from $\Pi(c)$ about L.

Finally, as in the proof of Lemma 23.135, there are normals in general position at which the rank is at least $D(|V(G)| - 1)$ (Lemma 23.120). The rank there is at most $D(|V(G)| - 1)$ (Lemma 20.16). So G has a generic full-rank realization of this rank. □

Formalization note 23.196. The formal proof treats the three cases by one theorem: the rank bound for (G, p_1) (Lemma 23.193), followed by the rotation. It takes r and the bottom rows from Lemma 23.154. For $u = b$ it applies this theorem with a and b exchanged. For $u = c$ it applies it with a, c and v in the roles of v, a and b, to the normals q_ρ of Lemma 23.185 and the form $-r$, with $G - a$ in place of G_v. There it uses the span form of KT eq. (6.44) (the note after Lemma 23.157). At these three

roles the theorem is that of Lemma 23.167, Lemma 23.181 and Lemma 23.189. The pinned theorem also assumes that the label of the new edge ab lies outside $E(G)$ (Remark 23.40). It asks for P for every endpoint selector that assigns each edge of G_v^{ab} its ends (Formalization note 23.6).

23.10.4. *The case of general d: KT Lemma 6.13 (KT §6.4.2).* Katoh and Tanigawa prove Case III for every d as KT Lemma 6.13 [KT11, pp. 692–698]. Let G be a minimal 0-dof-graph with at least three vertices and no proper rigid subgraph, so that G is simple (Lemma 21.57). If $|V(G)| = 3$, then G is a triangle, and Katoh and Tanigawa realize it by KT Lemma 5.4, as the cycle on three vertices (KT Lemma 6.7(i), p. 677). Their proof for $|V(G)| \geq 4$ splits off a vertex v of degree two, with neighbours a and b, and uses a generic realization of G_v^{ab}, whose panels are in general position (KT pp. 682 and 691). The induction hypothesis gives one when G_v^{ab} is simple, which holds for $|V(G)| \geq 4$ (KT Lemma 6.7(ii), Lemma 21.58). For the triangle, G_v^{ab} has the two vertices a and b and two edges that join them. It is not simple, and the induction hypothesis gives it only a nondegenerate-hinge panel realization, whose two panels may coincide, as in KT Lemma 5.3.

If $|V(G)| \geq 4$, then by KT Lemma 4.6 either G is a cycle on at most d vertices, which KT Lemma 5.4 realizes, or G has a chain $v_0v_1 \cdots v_d$ of length d (Lemma 21.68). For $d \leq 3$ the first alternative does not arise, as G has at least four vertices. For the second, Katoh and Tanigawa note that "the proof strategy is exactly the same as $d = 3$" (p. 692). They split off once, at v_1 along v_0v_2. The induction hypothesis gives a generic full-rank realization of $G_1 = G_{v_1}^{v_0v_2}$, and KT Claim 6.11 gives the form r of KT eq. (6.52) at its new edge v_0v_2, as for $d = 3$. There are d frameworks, one for each of the panels of $v_0, v_2, v_3, \ldots, v_d$. The first two put v_1 back, as for $d = 3$. For $2 \leq i \leq d-1$, the i-th puts v_i back into the graph $G_i = G_{v_i}^{v_{i-1}v_{i+1}}$. An isomorphism from G_i to G_1 renames the bodies, so the realization of G_1 serves for every G_i, and the induction hypothesis is used once (KT eqs. (6.54) and (6.55)). For the i-th framework, r is read from the linear relation of KT eq. (6.52) at the edge v_iv_{i+1} of G_1, not at v_0v_2. It appears there up to sign, because $v_2, \ldots, v_{d-1}$ have degree two (KT eq. (6.66)). Finally, the span of KT eq. (6.67), for $d+1$ points in K^{d+1}, shows that one of the d frameworks has a nonsingular block, as for $d = 3$.

We first identify the triangle. Like Katoh and Tanigawa, we then realize it as the cycle on three vertices, by Lemma 23.80 with $m = 3$ (in the proof of Lemma 23.208).

Lemma 23.197 (A minimal 0-dof-graph on three vertices is a triangle; from the proof of KT Lemma 6.7(i))**.** *Let $d \geq 2$, and let G be a simple minimal 0-dof-graph with three vertices. Let $e_a \neq e_b$ be edges of G that join a vertex v to vertices a and b. Then $a \neq b$, $V(G) = \{v, a, b\}$, and an edge of G joins a and b.*

Proof. As G is simple, it has no loops, so $a \neq v$ and $b \neq v$. Two distinct edges of G do not join the same two vertices, so $a \neq b$. The three distinct vertices v, a and b are then all the vertices of G.

As G is a 0-dof-graph, $\operatorname{rank} M(\tilde{G}) = D(|V(G)| - 1) = 2D$ (KT eq. (2.4), Theorem 20.12 and Lemma 21.4). The ground set of $M(\tilde{G})$ has $D-1$ copies of each edge of G (Definition 17.1), so $(D-1)\,|E(G)| \geq 2D > 2(D-1)$. Hence G has an edge f

other than e_a and e_b. Its ends are two distinct vertices among v, a and b. As G is simple, f does not join v and a, which e_a joins, nor v and b. So f joins a and b. □

The triangle can also be realized with three chosen normals, as in the proof of Lemma 23.80 for $m = 3$. The proof of Lemma 23.208 does not use this realization.

Lemma 23.198 (Realizing a triangle; KT Lemma 6.7(i))**.** *Let $d \geq 2$, let K be an infinite field, and let G be a simple minimal 0-dof-graph with three vertices. Let $e_a \neq e_b$ be edges of G that join a vertex v to vertices a and b. Then G has a generic full-rank realization (Definition 23.21), of rank $2D$.*

Proof. By Lemma 23.197, $V(G) = \{v, a, b\}$, and an edge f joins a and b. Let n_0, n_1 and n_2 be the normals of Lemma 23.61, and give v, a and b the normals n_0, n_1 and n_2. Up to sign, the hinges of e_a, f and e_b are $C(n_0, n_1)$, $C(n_1, n_2)$ and $C(n_2, n_0)$ (Definition 23.5), and these are linearly independent. By Lemma 23.60, every infinitesimal motion is constant on $\{v, a, b\} = V(G)$. As $\operatorname{def}(\tilde{G}) = 0$, the realization is at the target rank $2D$ (Remark 23.23). The joins $n_0 \vee n_1$, $n_1 \vee n_2$ and $n_2 \vee n_0$ are nonzero, so any two of the three normals are linearly independent, and the normals are in general position (Definition 23.9). Hence the realization is a generic full-rank realization of G. □

Formalization note 23.199. As in the note after Lemma 23.80, the formal framework also has bodies outside $V(G)$. The pinned theorem gives them the normal 0, and then chooses normals in general position for all the bodies, at which the rank is still $2D$.

Write $u_0 = v_0$ and $u_i = v_{i+1}$ for $1 \leq i \leq d-1$, so that the panels $\Pi(u_0), \dots, \Pi(u_{d-1})$ are Katoh and Tanigawa's $\Pi_0, \dots, \Pi_{d-1}$ (KT p. 698). The next lemma realizes G_1, gives the form r, and chooses one of these panels. The lemma after it builds the framework for this panel.

Lemma 23.200 (The form r and the choice of a panel; KT eqs. (6.65)–(6.67))**.** *Let $d \geq 2$, let K be an infinite field, and let G be a simple minimal 0-dof-graph with $|V(G)| \geq 3$. Let $v_0 v_1 \cdots v_d$ be a chain of length d in G (Definition 21.62), let $G_1 = G_{v_1}^{v_0 v_2}$, and let $u_0 = v_0$ and $u_i = v_{i+1}$ for $1 \leq i \leq d-1$. Suppose that G_1 is a 0-dof-graph with a generic full-rank realization, and that for every k', every minimal k'-dof-graph with at least one and fewer than $|V(G)|$ vertices has a nondegenerate-hinge panel realization, and a generic full-rank realization if it is simple. Then there are normals q in general position (Definition 23.9), a nonzero linear form r on $\bigwedge^{d-1} K^{d+1}$, an index $0 \leq i \leq d-1$ and a vector $n' \in K^{d+1}$ with the following properties.*

1. *The matrix $R(G_1, q)$ has $D(|V(G)| - 2)$ linearly independent rows (Definition 19.7; KT eq. (6.46)).*
2. *$r(C(q(v_0), q(v_2))) = 0$, and the form $S \mapsto r(S(v_0) - S(v_2))$ lies in the row space of $R(G - v_1, q)$ (KT eq. (6.52)).*
3. *The vector n' is linearly independent of $q(u_i)$, and $r(C(q(u_i), n')) \neq 0$.*

Proof. We follow Katoh and Tanigawa [KT11, pp. 692–698]. As $d \geq 2$, the vertex v_1 is interior, so its only edges join it to v_0 and to v_2 (Definition 21.62). The vertices $u_0, \dots, u_{d-1}$ are distinct.

We choose the normals q off the zeros of a product of four nonzero polynomials, fixed before the choice, as explained after Lemma 23.100 and in the remark on KT footnote 6 after Lemma 23.140. The first factor is the polynomial that Lemma 23.120 gives for the generic full-rank realization of G_1. The second is the polynomial P of Lemma 23.140 for G and its vertex v_1. The third puts the normals in general position. The fourth is a $d \times d$ minor of the matrix with the rows $q(u_0), \dots, q(u_{d-1})$, which is a nonzero polynomial as the u_l are distinct. As K is infinite, the product has a non-root q.

The first factor gives rank $R(G_1, q) \geq D(|V(G)|-2)$, so $R(G_1, q)$ has $D(|V(G)|-2)$ linearly independent rows. This is also an upper bound (Lemma 20.16), so equality holds, which is KT eq. (6.46). The second factor gives KT eq. (6.22) for $G - v_1$, and the third makes the hinge of the new edge v_0v_2 nondegenerate. So the hypotheses of KT Claim 6.11 (Lemma 23.145) hold, with v_1, v_0 and v_2 in the roles of v, a and b. Let $r = \sum_j \lambda_j r_j$ be the linear form of Lemma 23.163, which is Katoh and Tanigawa's r (KT p. 698). It is not zero, it vanishes on $C(q(v_0), q(v_2))$, and the form $S \mapsto r(S(v_0) - S(v_2))$ lies in the row space of $R(G - v_1, q)$ (KT eqs. (6.51) and (6.52), Lemma 23.152).

Katoh and Tanigawa take a point p_d on all the panels $\Pi(u_0), \dots, \Pi(u_{d-1})$, and for $0 \leq l \leq d-1$ a point p_l on all of them but $\Pi(u_l)$ (KT p. 698). As for $d = 3$, we take them in homogeneous coordinates. The normals $q(u_0), \dots, q(u_{d-1})$ are linearly independent, by the fourth factor. So there are linearly independent $\bar{p}_0, \dots, \bar{p}_d \in K^{d+1}$ such that $\bar{p}_d$ is orthogonal to every $q(u_{l'})$, and, for $0 \leq l \leq d-1$, $\bar{p}_l$ is orthogonal to $q(u_{l'})$ exactly when $l' \neq l$. The proof of Lemma 23.177 gives them, with d normals in place of three (the note after it).

For $0 \leq l < m \leq d$, let E_{lm} be the join of the $d-1$ vectors $\bar{p}_{l'}$ with $l' \neq l, m$. These D extensors span $\bigwedge^{d-1} K^{d+1}$, by KT Lemma 2.1 (Lemma 23.169; KT eq. (6.67)). As $r \neq 0$, it does not vanish on some E_{lm} (Lemma 23.171). This is KT Claim 6.12 in the form for every d (the note after Lemma 23.191).

Let $i = l$. As $l < m \leq d$, the index i is at most $d-1$, and the $d-1$ vectors $\bar{p}_{l'}$ with $l' \neq l, m$ are orthogonal to $q(u_i)$. They are linearly independent, so their span has an orthogonal complement of dimension 2, which contains $q(u_i)$. Let n' be a vector of this complement that is linearly independent of $q(u_i)$. Then the $d-1$ vectors are orthogonal to $q(u_i)$ and to n', so E_{lm} is a nonzero multiple of $C(q(u_i), n')$ (the remarks after Definition 23.1, and Lemma 22.9, of which Lemma 22.16 is the case $d = 3$). Hence $r(C(q(u_i), n')) \neq 0$. □

Formalization note 23.201. The pinned theorem also gives the form $S \mapsto r(S(v_0) - S(v_2))$ as a combination of the rows of the edges of $G - v_1$, grouped by edge, and r as $\sum_j \lambda_j r_j$ for forms r_j that give the rows of v_0v_2, as Lemma 23.154 gives them. Each of the $D(|V(G)|-2)$ rows of the first property is a row of $R(G - v_1, q)$ or a row of the new edge v_0v_2. The theorem also gives an endpoint selector that assigns each

edge of G_1 its ends (Formalization note 23.6). The formal proof of Lemma 23.206 uses all of this but the decomposition of r.

For $2 \le i \le d-1$, Katoh and Tanigawa build the i-th framework from the graph $G_i = G_{v_i}^{v_{i-1}v_{i+1}}$, obtained by splitting off at v_i along $v_{i-1}v_{i+1}$ (KT pp. 693–694). Their isomorphism ρ_i from G_i to G_1 maps v_l to v_{l+1} for $1 \le l \le i-1$ and fixes the other vertices (KT eq. (6.54)). It maps the edge v_0v_1 of G_i to the new edge v_0v_2 of G_1, and the new edge $v_{i-1}v_{i+1}$ of G_i to the edge v_iv_{i+1} of G_1. Let σ_i be the permutation of the bodies that agrees with ρ_i and maps v_i to v_1, the cycle $v_1 \mapsto v_2 \mapsto \cdots \mapsto v_i \mapsto v_1$, and let $q_i = q \circ \sigma_i$. Then (G_i, q_i) is (G_1, q) with its bodies renamed, "exactly the same framework" (KT eq. (6.55)). The next lemma renames the rows of $R(G_1, q)$, and the lemma after it carries the form r of Lemma 23.200 to the new edge of G_i.

Lemma 23.202 (The rows of (G_i, q_i); KT eqs. (6.55) and (6.62))**.** *Let G be a multigraph with a chain $v_0v_1 \cdots v_d$ of length d (Definition 21.62), let $2 \le i \le d-1$, and let q be normals. Let σ_i and q_i be as above. Let f be a row of $R(G - v_1, q)$, or a form $S \mapsto x(S(v_0) - S(v_2))$ for a linear form x on K^D that vanishes on $C(q(v_0), q(v_2))$. Then the form $S \mapsto f(S \circ \sigma_i^{-1})$ is a row of $R(G - v_i, q_i)$, or a form $S \mapsto x'(S(v_{i-1}) - S(v_{i+1}))$ for a linear form x' on K^D that vanishes on $C(q_i(v_{i-1}), q_i(v_{i+1}))$.*

Proof. We follow Katoh and Tanigawa [KT11, pp. 694–696]. Let u and w be the ends of an edge of $G - v_1$, or let $u = v_0$ and $w = v_2$, and let f be the form $S \mapsto x(S(u) - S(w))$, where x vanishes on $C(q(u), q(w))$. Then $f(S \circ \sigma_i^{-1})$ is the form $S \mapsto x(S(\sigma_i^{-1}(u)) - S(\sigma_i^{-1}(w)))$, and x vanishes on $C(q_i(\sigma_i^{-1}(u)), q_i(\sigma_i^{-1}(w))) = C(q(u), q(w))$. So it is enough that $\sigma_i^{-1}(u)$ and $\sigma_i^{-1}(w)$ are the ends of an edge of $G - v_i$, or are v_{i-1} and v_{i+1}.

The vertices $v_2, \dots, v_i$ are interior vertices of the chain, so their only edges in G are chain edges (Definition 21.62). For an edge v_lv_{l+1} of $G - v_1$ with $2 \le l \le i-1$, σ_i^{-1} maps its ends to those of the edge $v_{l-1}v_l$ of $G - v_i$. It maps the ends of v_iv_{i+1} to v_{i-1} and v_{i+1}, and v_0 and v_2 to the ends of the edge v_0v_1 of $G - v_i$. Every other edge of $G - v_1$ has no end among $v_1, \dots, v_i$, so it is an edge of $G - v_i$, and σ_i^{-1} fixes its ends. The edges of G_1 are those of $G - v_1$ and the new edge v_0v_2 (Definition 21.10), and likewise for G_i. So renaming takes the rows of $R(G_1, q)$ to those of $R(G_i, q_i)$, which is Katoh and Tanigawa's row correspondence (KT eq. (6.62)), with the column correspondence given by ρ_i. □

Formalization note 23.203. The pinned theorem takes $R(G - v_i, q_i)$ with the endpoint selector of $(G - v_1, q)$ (Formalization note 23.6) renamed along σ_i and along a permutation of the edge labels that extends ρ_i. It assumes that the selector of $(G - v_1, q)$ assigns each of its edges its ends, and the new edge v_0v_2 the ends v_2 and v_0, in this order, and that the renamed selector assigns each edge of $G - v_i$ its ends. That the renamed selector assigns each edge of G_i its ends, which gives the latter, is a separate step of the formal proof of Lemma 23.206, which applies the lemma to the bottom rows of KT eq. (6.64).

Lemma 23.204 (The form r at the edge v_iv_{i+1}; KT eq. (6.66))**.** *Let $d \ge 3$, let G be a multigraph with a chain $v_0v_1 \cdots v_d$ of length d (Definition 21.62), and let q*

be normals. Let r be a linear form on $\bigwedge^{d-1} K^{d+1}$ that vanishes on $C(q(v_0), q(v_2))$, such that the form $S \mapsto r(S(v_0) - S(v_2))$ lies in the row space of $R(G - v_1, q)$. Let $2 \le i \le d-1$, and let σ_i and q_i be as above. Then r vanishes on $C(q_i(v_{i-1}), q_i(v_{i+1}))$, the hinge of the new edge $v_{i-1}v_{i+1}$ of (G_i, q_i), and the form $S \mapsto r(S(v_{i-1}) - S(v_{i+1}))$ lies in the row space of $R(G - v_i, q_i)$.

Proof. We follow Katoh and Tanigawa [KT11, pp. 696–698]. Write the form $S \mapsto r(S(v_2) - S(v_0))$ as a combination of the rows of $R(G - v_1, q)$, and add up its terms at each edge: for an edge e of $G - v_1$ with ends x and y, they add up to a form $S \mapsto \psi_e(S(x) - S(y))$, where ψ_e vanishes on the hinge $q(e)$. Let $\psi_{v_0v_2} = r$ for the new edge of G_1, with ends v_0 and v_2. Then the forms $S \mapsto \psi_e(S(x) - S(y))$, for the edges e of G_1 with ends x and y, sum to zero, as in KT eq. (6.52). Let S vanish off a vertex u. Then the term at e takes the value $\pm\psi_e(S(u))$ at S if u is an end of e, and 0 otherwise, as in the proof of Lemma 23.157. As $S(u)$ can be any element of K^D, the forms ψ_e of the edges e at u, with signs given by the orientations, sum to zero.

The edges of $G - v_1$ are the edges of G without the end v_1 (Lemma 21.12). For $2 \le l \le d-1$, the vertex v_l is interior, so its only edges in G are $v_{l-1}v_l$ and v_lv_{l+1} (Definition 21.62). So its edges in G_1 are the edge $v_{l-1}v_l$, or the new edge v_0v_2 when $l = 2$, and the edge v_lv_{l+1}. Hence $\psi_{v_lv_{l+1}}$ is $\pm$ the form of the other edge, and by induction on l,

$$\psi_{v_lv_{l+1}} \;=\; \pm r \qquad (2 \le l \le d-1),$$

which is KT eq. (6.66). For $d = 3$ and $l = 2$ this is KT eq. (6.44) (Lemma 23.157).

This has two consequences at $l = i$. First, $\psi_{v_iv_{i+1}}$ vanishes on the hinge of v_iv_{i+1} in (G_1, q), which is $C(q(v_i), q(v_{i+1}))$, and so does r. As $\sigma_i(v_{i-1}) = v_i$ and $\sigma_i(v_{i+1}) = v_{i+1}$, this hinge is $C(q_i(v_{i-1}), q_i(v_{i+1}))$. Second, the term of the relation at v_iv_{i+1} is the form $S \mapsto \pm r(S(v_i) - S(v_{i+1}))$, so this form lies in the span of the rows of $R(G_1, q)$ at its other edges. The term at v_0v_2 is such a row, as r vanishes on $C(q(v_0), q(v_2))$. Renamed by σ_i, the rows at these edges are rows of $R(G - v_i, q_i)$, as in the proof of Lemma 23.202, and the form becomes $S \mapsto \pm r(S(v_{i-1}) - S(v_{i+1}))$. Katoh and Tanigawa make this step with the row correspondence of KT eq. (6.62) and the row operations of KT eq. (6.63). □

Formalization note 23.205. The two pinned theorems are the two claims. They take the form $S \mapsto r(S(v_0) - S(v_2))$ with a combination of rows, from which the proof reads the forms ψ_e: the first at edges of $G - v_1$, and the second at edges of G of which only v_2v_3 meets v_2, with the membership in the row space of $R(G - v_1, q)$. The first assumes that the endpoint selector (Formalization note 23.6) assigns each edge of $G - v_1$ and the edge v_iv_{i+1} its ends, and the second that it assigns every edge of G its ends and the new edge v_0v_2 the ends v_0 and v_2. The second takes $R(G - v_i, q_i)$ with this selector. In the formal proof of Lemma 23.206, the combination is the one that Lemma 23.200 gives (the note after it).

Lemma 23.206 (Case III for a chain of length d; from the proof of KT Lemma 6.13)**.** *Under the hypotheses of Lemma 23.200, G has a generic full-rank realization (Definition 23.21), of rank $D(|V(G)| - 1)$.*

Proof. We follow Katoh and Tanigawa [KT11, pp. 692–698]. Let q, $r = \sum_j \lambda_j r_j$, i and n' be as in Lemma 23.200 and its proof, with the linear relation of KT eq. (6.52) among the rows of $R(G_1, q)$. The extensor $C(q(u_i), n')$ is a nonzero multiple of the extensor of the $(d-2)$-dimensional affine subspace L_i in which $\Pi(u_i)$ meets the hyperplane with normal n' (the remarks after Definition 23.1). So $r(C(L_i)) \neq 0$. Katoh and Tanigawa's i-th framework (G, p_i) has the hinge L_i, and its rank is at least $D(|V(G)| - 1)$ when r does not vanish on $C(L_i)$, that is, when its block M_i is nonsingular (KT p. 697). We show this first for $i \leq 1$ and then for $i \geq 2$.

Let $i = 0$ or $i = 1$. Katoh and Tanigawa's (G, p_0) is the framework of KT eq. (6.12) with v_1, v_0 and v_2 in the roles of v, a and b, and with $L = L_0$. Their (G, p_1) is the framework of KT eq. (6.19) for the same roles, with $L' = L_1$ (KT eqs. (6.47) and (6.48)). By Lemma 23.167 and Lemma 23.181, their ranks are at least $D(|V(G)| - 1)$ when r does not vanish on $C(L_0)$, respectively $C(L_1)$ (KT eqs. (6.50) and (6.53)).

Now let $2 \leq i \leq d-1$, so that $d \geq 3$. Katoh and Tanigawa split off at v_i instead, along $v_{i-1}v_{i+1}$, and rename the bodies of (G_1, q) by their isomorphism ρ_i from $G_i = G_{v_i}^{v_{i-1}v_{i+1}}$ to G_1 (KT eqs. (6.54) and (6.55)). Let σ_i and $q_i = q \circ \sigma_i$ be as before Lemma 23.202. Then (G_i, q_i) is (G_1, q) with its bodies renamed. By that lemma, renaming takes the rows of $R(G_1, q)$ to rows of $R(G_i, q_i)$, and it keeps linearly independent forms linearly independent. So $\operatorname{rank} R(G_i, q_i) \geq D(|V(G)|-2)$, the rank of $R(G_1, q)$.

By the relation of KT eq. (6.52), the form $S \mapsto r(S(v_0) - S(v_2))$ lies in the row space of $R(G - v_1, q)$, and r vanishes on $C(q(v_0), q(v_2))$. So by Lemma 23.204, which is KT eq. (6.66), r vanishes on the hinge of the new edge $v_{i-1}v_{i+1}$ of (G_i, q_i), and the form $S \mapsto r(S(v_{i-1}) - S(v_{i+1}))$ lies in the row space of $R(G - v_i, q_i)$. The new row of the i-th framework thus comes from the coefficients of the relation at the edge v_iv_{i+1} of G_1, an edge that depends on i.

Katoh and Tanigawa's (G, p_i) is the framework of KT eq. (6.12) built from (G_i, q_i), with v_i, v_{i+1} and v_{i-1} in the roles of v, a and b, and with $L = L_i$, which lies in the panel $\Pi(v_{i+1})$ of v_{i+1} in (G_i, q_i) (KT eqs. (6.57) and (6.58)). The proof of Lemma 23.167 applies to it with $-r$ in place of r. There we used only that r vanishes on the hinge of the new edge ab, that $S \mapsto r(S(a) - S(b))$ lies in the row space of $R(G_v, q)$, and that $R(G_v^{ab}, q)$ has rank at least $D(|V(G)| - 2)$. Here these hold by Lemma 23.204 and by the bound on rank $R(G_i, q_i)$. So $\operatorname{rank} R(G, p_i) \geq D(|V(G)|-1)$ when r does not vanish on $C(L_i)$ (KT eq. (6.64) and p. 697).

In each case the vertex put back, v_1 or v_i, has the panel $\Pi(u_i)$ in (G, p_i), which contains L_i (KT pp. 692 and 694). As in the proof of Lemma 23.195, we rotate this panel about L_i, where Katoh and Tanigawa apply KT Lemma 5.2 (KT p. 694). This gives a panel-hinge framework on G with nondegenerate hinges and rank at least $D(|V(G)| - 1)$. Finally, as in the proof of Lemma 23.135, there are normals in general position at which the rank is at least $D(|V(G)| - 1)$ (Lemma 23.120). The rank there is at most $D(|V(G)| - 1)$ (Lemma 20.16). So G has a generic full-rank realization of this rank. □

Formalization note 23.207. As for Lemma 23.195, the formal proof treats every framework by one theorem: the rank bound for the framework of KT eq. (6.12)

(Lemma 23.193), followed by the rotation. For $i = 0$ it applies this theorem with v_1, v_0 and v_2 in the roles of v, a and b, as Lemma 23.167 does, and for $i = 1$ with v_0 and v_2 exchanged, as Lemma 23.181 does. For $i \geq 2$ it applies it with v_i, v_{i+1} and v_{i-1} in the roles of v, a and b, to the normals q_i and the form $-r$, with $G - v_i$ in place of G_v. It takes the bottom rows from Lemma 23.202, and the two facts it needs on $-r$ from Lemma 23.204. So it does not use the count of KT eq. (6.12) one row short of the target rank, from a realization of G_v^{ab} whose motions are constant on its vertex set (KT p. 680, Lemma 23.128). For $d = 3$ and $i = 2$, the isomorphism ρ_2 is Katoh and Tanigawa's isomorphism ρ from G_a^{vc} to G_v^{ab} (KT p. 686, Lemma 23.183), and (G_2, q_2) is their realization (G_a^{vc}, q_ρ) of KT eq. (6.31). Its normals and its rows are those of Lemma 23.185 and Lemma 23.187. The formal proof does not use these three lemmas either: it renames by the permutation σ_i, as the two lemmas before this one do.

Lemma 23.208 (Case III: no proper rigid subgraph and $k = 0$; KT Lemma 6.13)**.** *Let $d \geq 2$, let K be an infinite field, and let G be a simple minimal 0-dof-graph with $|V(G)| \geq 3$ and no proper rigid subgraph. Suppose that for every k', every minimal k'-dof-graph with at least one and fewer than $|V(G)|$ vertices has a nondegenerate-hinge panel realization, and a generic full-rank realization if it is simple. Then G has a generic full-rank realization (Definition 23.21), of rank $D(|V(G)| - 1)$.*

Proof. We follow Katoh and Tanigawa's proof of KT Lemma 6.13 [KT11, p. 692], with the graphs on three vertices treated first, as in KT Lemma 6.7(i) (p. 677).

Let $|V(G)| = 3$. We need a single vertex v of degree two, and it exists for every $d \geq 2$. Indeed, some vertex v has degree at most two (Lemma 21.49). As a 0-dof-graph, G is 2-edge-connected (KT Lemma 3.1, Lemma 20.18), so v has degree two. Let e_a and e_b be the edges of v, which join it to vertices a and b. By Lemma 23.197, $V(G) = \{v, a, b\}$, and an edge f joins a and b. As G is simple, e_a, e_b and f are its only edges, so G is a cycle on three vertices (Definition 21.64). As $3 \leq d + 1$, Lemma 23.80 gives a generic full-rank realization of G.

Let $|V(G)| \geq 4$. By Lemma 21.68, G is a cycle on at most d vertices, or G has a chain $v_0 v_1 \cdots v_d$ of length d such that $G_1 = G_{v_1}^{v_0 v_2}$ is a simple minimal 0-dof-graph with at least two and fewer than $|V(G)|$ vertices. In the first case, which arises only for $d \geq 4$, Lemma 23.80 gives the realization, as $|V(G)| \leq d$. In the second case, the induction hypothesis gives G_1 a generic full-rank realization, and Lemma 23.206 gives one of G. For $d = 3$ the chain is $b\,v\,a\,c$ in the notation of Lemma 23.195, whose three frameworks are the d frameworks of this case. □

Formalization note 23.209. The pinned theorem also assumes that some edge label lies outside $E(G)$ (Remark 23.40). The chain uses it for the new edge $v_0 v_2$ (the note after Definition 21.62).

24. The generic bar-joint rigidity matroid

Jordán [Jor16] (§1.3.1, p. 39) defines the d-dimensional rigidity matroid $\mathcal{R}_d(G)$ of a graph G as the rigidity matroid of a generic framework of G, one whose coordinates satisfy no algebraic dependency (p. 36), and writes $r_d(G)$ for its rank. At $d = 3$,

r_3 is the rank function r of KT Corollary 5.7: for a graph G of minimum degree at least two,

$$r(G^2) \;=\; 3|V| - 6 - \mathrm{def}(\tilde{G}),$$

where the square G^2 joins every two vertices at distance one or two in G (Definition 25.1), and $\mathrm{def}(\tilde{G})$ is the deficiency of Definition 20.3 at $D = 6$. Jackson and Jordán [JJ08a] proved the inequality $\leq$ (their Theorem 4.1), and showed that equality is equivalent to the molecular conjecture (their Theorem 4.3). We prove the formula as Corollary 26.8. In this chapter we construct the matroid and its rank function, in every dimension d.

In the plane we needed no notion of genericity (Section 10.4). There we found one framework at which every $(2, 3)$-sparse edge set is row-independent, using Pollaczek-Geiringer's and Laman's characterization of these edge sets as the ones that are row-independent at some framework. No combinatorial characterization of this kind is known in dimension three or more (Jordán, p. 39; KT §7), and none is needed here. We work with placements that are generic for row independence: every edge set that is row-independent at some placement is row-independent at them. Such placements exist in every dimension, by the argument of the planar case (Section 24.1). We define the matroid $\mathcal{R}_d(V)$ on the edges of the complete graph K_V as the linear rigidity matroid at one of them. Its independent sets are the edge sets that are row-independent at some placement, so it does not depend on the placement, and in the plane it is the planar rigidity matroid. Finally, for a graph H on V, the rank of $E(H)$ in $\mathcal{R}_d(V)$ is the rank of the rigidity map of H at any generic placement (Section 24.2).

Throughout, V is a finite set and $d \in \mathbb{N}$. As in Section 6, a placement of V in $\mathbb{R}^d$ is a framework on V in $\mathbb{R}^d$, and $R(G, p)$ is the rigidity map. The rigidity rows r_e and row-independent edge sets are defined in Section 9.

24.1. **Generic placements.** Jackson and Jordán [JJ08a] (§2) call a framework generic when its rigidity matrix and all of its submatrices have the largest rank among all placements of the graph. The entries of the rigidity matrix are polynomials with integer coefficients in the coordinates (Jordán, p. 39), so a placement whose coordinates are algebraically independent over $\mathbb{Q}$ is generic in this sense. We ask this of the rigidity matrix of K_V and its submatrices formed by sets of rows: every set of edges of K_V that is row-independent at some placement must be row-independent at the generic one. This is all that the construction of the matroid uses, and placements with this property exist by an argument that needs no algebraic independence.

Definition 24.1 (Generic placement for row independence)**.** Let p be a placement of V in $\mathbb{R}^d$. We say that p is *generic for row independence* if every set $I \subseteq E(K_V)$ that is row-independent at some placement of V in $\mathbb{R}^d$ is row-independent at p (Definition 9.3, at $G = K_V$).

Lemma 24.2 (Existence of generic placements)**.** *Let V be finite and $d \in \mathbb{N}$. Then there is a placement of V in $\mathbb{R}^d$ that is generic for row independence.*

Proof. Let $\mathcal{F}$ be the family of sets $I \subseteq E(K_V)$ that are row-independent at some placement. It is finite, since $E(K_V)$ is, and a placement at which every member

of $\mathcal{F}$ is row-independent is generic. We find one as in the proof of Lemma 10.18. There each $(2,3)$-sparse edge set is row-independent at some framework by Pollaczek-Geiringer's and Laman's characterization (Theorem 10.10); here each member of $\mathcal{F}$ is row-independent at some placement by definition.

We show, by induction on $|\mathcal{F}'|$, that for every $\mathcal{F}' \subseteq \mathcal{F}$ there is a placement at which every member of $\mathcal{F}'$ is row-independent. For $\mathcal{F}' = \emptyset$ any placement will do. Otherwise let $I_0 \in \mathcal{F}'$, let p_0 be a placement at which every member of $\mathcal{F}' \setminus \{I_0\}$ is row-independent, and let q be a placement at which I_0 is row-independent. For $t \in \mathbb{R}$, let $p^t = p_0 + t\,(q - p_0)$. The rigidity row of an edge uv at a placement p' is $x \mapsto \langle p'_u - p'_v,\, x_u - x_v\rangle$ (Lemma 6.4), which depends linearly on p'.

Let I be a member of $\mathcal{F}' \setminus \{I_0\}$, or I_0, and let t_0 be 0 or 1 accordingly. A set of edges is row-independent at a placement exactly when its rigidity rows there are linearly independent (Lemma 9.6). Fix a basis of $((\mathbb{R}^d)^V)^*$. The coordinate vectors of the rows at p^t of the edges in I are the rows of a matrix $A + t\,B$, where A and B do not depend on t. These rows are linearly independent at t_0, so some $|I| \times |I|$ minor of $A + t_0 B$ is nonzero. The same minor of $A + t\,B$ is a polynomial in t that is nonzero at t_0, so it has finitely many roots. At every other t, the rows at p^t of the edges in I are linearly independent. Choose t outside the roots of these finitely many polynomials, one for each member of $\mathcal{F}'$. Then every member of $\mathcal{F}'$ is row-independent at p^t. □

24.2. **The matroid and its rank function.** In the plane we defined $\mathcal{R}(V)$ by counting, and showed that the linear rigidity matroid at one framework equals it (Theorem 10.19). In general dimension there is no count to start from, and we take the linear rigidity matroid at a generic placement as the definition.

Definition 24.3 (The generic rigidity matroid $\mathcal{R}_d(V)$)**.** Fix a placement p of V in $\mathbb{R}^d$ that is generic for row independence (Lemma 24.2). The *generic rigidity matroid* $\mathcal{R}_d(V)$ is the linear rigidity matroid $\mathcal{L}_p(V)$ at p (Definition 10.16), a matroid on $E(K_V)$.

Lemma 24.4 (Independence in $\mathcal{R}_d(V)$)**.** *Let $I \subseteq E(K_V)$. Then I is independent in $\mathcal{R}_d(V)$ if and only if I is row-independent at some placement of V in $\mathbb{R}^d$.*

Proof. Let p be the placement in the definition of $\mathcal{R}_d(V)$. A set is independent in $\mathcal{R}_d(V) = \mathcal{L}_p(V)$ if and only if it is row-independent at p (Lemma 10.17). If I is row-independent at p, then it is row-independent at some placement, namely p. Conversely, if I is row-independent at some placement, then it is row-independent at p, since p is generic (Definition 24.1). □

Lemma 24.5 (Placement independence)**.** *Let p be a placement of V in $\mathbb{R}^d$ that is generic for row independence. Then $\mathcal{L}_p(V) = \mathcal{R}_d(V)$.*

Proof. Both matroids have ground set $E(K_V)$, so it suffices to show that they have the same independent sets. A set $I \subseteq E(K_V)$ is independent in $\mathcal{L}_p(V)$ if and only if it is row-independent at p (Lemma 10.17). Since p is generic, this holds if and only if I is row-independent at some placement, that is, if and only if I is independent in $\mathcal{R}_d(V)$ (Lemma 24.4). □

Lemma 24.6 (Dimension two: the planar rigidity matroid)**.** *The generic rigidity matroid $\mathcal{R}_2(V)$ is the planar rigidity matroid $\mathcal{R}(V)$, the $(2,3)$-count matroid on V (Definition 10.12).*

Proof. Both matroids have ground set $E(K_V)$. A set $I \subseteq E(K_V)$ is independent in $\mathcal{R}_2(V)$ if and only if it is row-independent at some placement in $\mathbb{R}^2$ (Lemma 24.4). The same holds in $\mathcal{R}(V)$ (Theorem 10.13), so the two matroids are equal. □

Jordán's $r_d(G)$ is the rank of the rigidity matroid of a generic framework of G (p. 39). The rows of $R(G,p)$ are the rows of $R(K_V,p)$ indexed by $E(G)$, so this matroid is the restriction of the rigidity matroid of K_V to $E(G)$. We therefore define the rank of a graph on V through $\mathcal{R}_d(V)$.

Definition 24.7 (Generic rank)**.** Let H be a graph on V. The *generic rank* $r_d(H)$ of H in dimension d is the rank of $E(H)$ in $\mathcal{R}_d(V)$.

Lemma 24.8 (Row-space form of the generic rank)**.** *Let p be a placement of V in $\mathbb{R}^d$ that is generic for row independence, and let H be a graph on V. Then*

$$r_d(H) \;=\; \dim \operatorname{span}\{\, r_e \mid e \in E(H)\,\},$$

where r_e is the rigidity row of K_V at p indexed by e (Definition 9.4).

Proof. The matroid $\mathcal{R}_d(V)$ is $\mathcal{L}_p(V)$ (Lemma 24.5). It is represented by the extended rigidity rows $\tilde{r}_e$ at p (Definition 10.16), and $\tilde{r}_e = r_e$ for every edge e of K_V (Definition 10.14). In a represented matroid, the vectors of a set span a space whose dimension is the rank of the set (Lemma 16.7). For the set $E(H)$, this is the claim. □

Lemma 24.9 (The rank of the rigidity map at a generic placement)**.** *Let p be a placement of V in $\mathbb{R}^d$ that is generic for row independence, and let H be a graph on V. Then* $\operatorname{rank} R(H,p) = r_d(H)$.

Proof. The map $R(H,p)$ has the same rank as its transpose, whose image is spanned by the rigidity rows of H at p (Lemma 9.5). For an edge uv of H, the rigidity row of H at uv is $x \mapsto \langle p_u - p_v,\, x_u - x_v\rangle$ (Lemma 6.4), which is also the row of K_V at uv. So the rank of $R(H,p)$ is the dimension of the span in Lemma 24.8, which is $r_d(H)$. □

25. Projective duality and the molecule modelling equivalence

Let $G = (V,E)$ be a graph of minimum degree at least two. KT Corollary 5.7 is the rank formula

$$r(G^2) \;=\; 3|V| - 6 - \operatorname{def}(\tilde{G}),$$

where r is the rank function of the generic bar-joint rigidity matroid in $\mathbb{R}^3$ (Section 24), G^2 is the square of G, and $\operatorname{def}(\tilde{G})$ is the deficiency of Definition 20.3 at $D = 6$. Jackson and Jordán conjectured the formula and proved it equivalent to the molecular conjecture [JJ08a, Conjecture 2.2 and Theorem 4.3]. Its proof relates three models, all in $\mathbb{R}^3$, of a molecule whose atoms and bonds form the graph G:

$$\begin{array}{c}\text{bar-joint}\\ \text{framework on } G^2\end{array} \iff \begin{array}{c}\text{molecular}\\ \text{framework on } G\end{array} \iff \begin{array}{c}\text{panel-hinge}\\ \text{framework on } G\end{array}$$

A molecular framework is a body-hinge framework in which the hinges at each body pass through one point, the centre of the body. The second equivalence is projective duality. A polarity of projective three-space sends points to planes and lines to lines, so it turns a molecular framework into a panel-hinge framework, and infinitesimal rigidity is invariant under it [CW82, §3.6]. The first equivalence is Whiteley's: a graph of minimum degree at least two has an infinitesimally rigid molecular realization if and only if its square has an infinitesimally rigid bar-joint realization. Jackson and Jordán [JJ08a, §2.1] and Katoh and Tanigawa [KT11, pp. 650 and 671] cite it from Whiteley's manuscript *The equivalence of molecular rigidity models as geometric and generic graphs* (2004).

Whiteley's equivalence, in the form that Jackson and Jordán and Katoh and Tanigawa state, concerns rigid realizations. In this form it gives the rank formula only when $\operatorname{def}(\tilde{G}) = 0$, where the formula says that G^2 is rigid. Jackson and Jordán deduce the general formula from this case [JJ08a, §§3–4]. Their upper bound on $r(G^2)$ is the value of an independent cover of G^2 built from a partition of V of largest deficiency (their Theorem 4.1). It uses results on such covers, and on the maximal subgraphs of deficiency zero, from two further papers of theirs. Their lower bound is an induction on the deficiency, which attaches to G a path of length five between two of these maximal subgraphs (their Theorem 4.3).

We avoid this development. We prove both equivalences as equalities between the dimensions of the spaces of infinitesimal motions, at a single placement. KT Theorem 5.6 (Theorem 23.50) realizes G as a panel-hinge framework of rank $6(|V| - 1) - \operatorname{def}(\tilde{G})$. The two equivalences turn this realization into a bar-joint framework on G^2 of rank $3|V| - 6 - \operatorname{def}(\tilde{G})$, as Katoh and Tanigawa do for the second (KT p. 671). The reverse inequality comes from the first equivalence at a generic placement, together with the lower bound $6 + \operatorname{def}(\tilde{G})$ on the dimension of the motions of a molecular framework with nondegenerate hinges (Lemma 20.16). Section 26 derives the formula in this way.

The chapter proceeds in five steps. Section 25.1 defines the square of a graph and molecular frameworks. Section 25.2 proves the projective invariance of the motion spaces, and shows that the polar of a molecular framework is a panel-hinge framework. Section 25.3 describes a screw by its field of velocities. Section 25.4 defines general position up to order four, and gives a molecular realization at the rank of KT Theorem 5.6 whose centres are in general position. Section 25.5 proves the first equivalence at such centres. Throughout, $d = 3$ and $D = 6$, the setting of Katoh and Tanigawa's connection with molecules (KT §1.2). We use the extensors and the meet of Section 18 and Section 22, the body-hinge and panel-hinge frameworks of Section 19 and Section 23, and the bar-joint frameworks of Section 6 and Section 24.

25.1. **The square graph and molecular frameworks.** Katoh and Tanigawa describe two models of a molecule whose atoms are joined by covalent bonds [KT11, §1.2]. Let G be the graph of the bonds. In the first model the molecule is a bar-joint framework on the square of G, which joins two atoms when they are at distance one or two in G (Definition 25.1). In the second it is a body-hinge framework on G, with the atoms as bodies and the bonds as hinges, in which all the hinges at an atom

pass through its centre (Definition 25.3). In the square, an atom and its neighbours induce a complete subgraph (Lemma 25.2), which acts as a rigid body [JJ08a, §2.1].

Definition 25.1 (Square of a graph)**.** The *square* of a graph $G = (V, E)$ is the graph $G^2 = (V, E^2)$ with

$$E^2 = E \cup \{\, uv \mid u \neq v,\ \exists w \in V \setminus \{u, v\} \text{ with } uw, wv \in E \,\},$$

that is, G together with an edge joining every two vertices at distance two in G.

Lemma 25.2 (Closed neighbourhoods are cliques of the square)**.** *Let G be a graph. For every vertex v of G, the closed neighbourhood $N[v] = \{v\} \cup N(v)$ induces a complete subgraph of G^2. Moreover, every edge of G^2 has both ends in $N[w]$ for some vertex w of G.*

Proof. Let $x \neq y$ be vertices of $N[v]$. If one of them is v, then xy is an edge of G. Otherwise v is a common neighbour of x and y. In both cases $xy \in E^2$. Now let uw be an edge of G^2. If it is an edge of G, then u and w lie in $N[u]$. Otherwise u and w have a common neighbour v, and they lie in $N[v]$. □

Katoh and Tanigawa call a body-hinge framework *hinge-concurrent* if all the hinges at each body pass through one point, the centre of the body [KT11, p. 650]. A molecule is modelled by such a framework, since the bonds at an atom pass through its centre. Following Whiteley [Whi99, Definition 4], we give the framework by the centres of its bodies.

Definition 25.3 (Molecular (hinge-concurrent) framework)**.** Let $G = (V, E)$ be a multigraph, and let $c \colon V \to \mathbb{R}^3$ assign a *centre* $c(v)$ to each body v. The *molecular framework* on (G, c) is the body-hinge framework on G (Definition 19.3) whose supporting extensor at an edge e with ends u and v is

$$C(c(u), c(v)) \;=\; \bar{c}(u) \vee \bar{c}(v)$$

(Definition 18.8), where $\bar{c}(v) = (c(v), 1)$.

Let e be an edge with ends u and v. If $c(u) \neq c(v)$, the extensor at e is a nonzero extensor of the line through $c(u)$ and $c(v)$ (the remarks after Definition 18.8), and this line is the hinge at e. So when adjacent bodies have distinct centres, every hinge at a body v passes through $c(v)$, and the molecular framework is hinge-concurrent. Conversely, let a body-hinge framework on G be hinge-concurrent with centres c, where $c(u) \neq c(v)$ at every edge uv. The hinge at uv passes through $c(u)$ and $c(v)$, so it is the line through them, and the framework is the molecular framework on (G, c). Jackson and Jordán call these frameworks molecular frameworks [JJ08a, §2.1], and Katoh and Tanigawa use their name [KT11, p. 671].

If $c(u) = c(v)$, the extensor at e is zero. The hinge at e is then degenerate, and it welds u and v together (the remarks after Definition 23.5). Whiteley calls a molecular structure non-degenerate if distinct bodies have distinct centres [Whi99, Definition 4]. Where nondegenerate hinges are needed below, the centres are assumed distinct or in general position (Definition 25.16).

Formalization note 25.4. The formal molecular framework records an ordered pair of bodies for every edge label, as a formal panel-hinge framework does (Formalization note 23.6). Its extensor at a label is that of the centres of the pair.

25.2. **Projective invariance and duality.** A projective transformation of space acts linearly on homogeneous coordinates, and through them on the screw space. A collineation sends points to points, and a correlation sends points to planes. Crapo and Whiteley [CW82, §3.6] showed that both take the infinitesimal motions of a body-hinge framework to those of the transformed framework. So infinitesimal rigidity is a projective invariant. We prove this for every linear automorphism of the screw space (Theorem 25.7). We then apply it to a polarity, which turns a molecular framework into a panel-hinge framework (Lemma 25.11).

In every dimension d, an invertible linear map of K^{d+1} induces a linear automorphism of the screw space $\bigwedge^{d-1} K^{d+1}$. For $d = 3$, Crapo and Whiteley let a collineation act on screws in this way [CW82, §3.6].

Definition 25.5 (The screw map of a linear map)**.** Let K be a field, let $d \geq 1$, and let g be an invertible linear map of K^{d+1}. The *screw map* of g is the linear automorphism Λ_g of the screw space $\bigwedge^{d-1} K^{d+1}$ (Definition 19.1) with

$$\Lambda_g(v_1 \vee \cdots \vee v_{d-1}) \;=\; gv_1 \vee \cdots \vee gv_{d-1} \qquad (v_1, \dots, v_{d-1} \in K^{d+1}).$$

The map Λ_g is the $(d-1)$-th exterior power of g, and its inverse is $\Lambda_{g^{-1}}$. The extensors span $\bigwedge^{d-1} K^{d+1}$ (the remarks after Definition 18.3), so the formula determines Λ_g.

Formalization note 25.6. The first pin is Λ_g, and the second is its formula.

Theorem 25.7 (Projective invariance of infinitesimal rigidity; Crapo–Whiteley)**.** *Let K be a field, let (G, p) be a body-hinge framework in K^3 (Definition 19.3), and let Λ be a linear automorphism of the screw space $\bigwedge^2 K^4$. Let $(G, \Lambda p)$ be the body-hinge framework on G whose supporting extensor at each edge e is $\Lambda\, C(p(e))$.*

1. *The map $S \mapsto \Lambda \circ S$ is an isomorphism from $Z(G, p)$ onto $Z(G, \Lambda p)$. In particular the two motion spaces have the same dimension, the two rigidity matrices have the same rank, and $(G, \Lambda p)$ is infinitesimally rigid if and only if (G, p) is.*
2. *The hinge at an edge is nondegenerate in $(G, \Lambda p)$ if and only if it is nondegenerate in (G, p).*
3. *The space $Z(G, p)$ does not change when the supporting extensor at each edge is multiplied by a nonzero scalar.*

Proof. (1) Let e be an edge with ends u and v. As Λ is linear and injective, a screw w lies in $\operatorname{span} C(p(e))$ if and only if Λw lies in $\operatorname{span} \Lambda C(p(e))$. Taking $w = S(u) - S(v)$ shows that S satisfies the hinge constraint of (G, p) at e if and only if $\Lambda \circ S$ satisfies that of $(G, \Lambda p)$. The map $S \mapsto \Lambda \circ S$ is a linear automorphism of the space of screw assignments, with inverse $S \mapsto \Lambda^{-1} \circ S$, so it maps $Z(G, p)$ onto $Z(G, \Lambda p)$. Moreover $\Lambda \circ S$ is constant if and only if S is, so the map takes the trivial motions of one framework onto those of the other (Definition 19.15), and infinitesimal rigidity transfers. The ranks agree because the rank of the rigidity matrix is $D|V| - \dim Z$ (the remark after Definition 19.7).

(2) A linear automorphism sends nonzero vectors to nonzero vectors. (3) A nonzero multiple of $C(p(e))$ spans the same line as $C(p(e))$, so no hinge constraint changes.

This is the argument of Crapo and Whiteley [CW82, §3.6]. A collineation "is expressed by a non-singular linear transformation on the projective coordinates of the points". The linear map that it induces on the space of screws "will transform an instantaneous motion into an instantaneous motion", and it has an inverse. □

Formalization note 25.8. The pinned results hold for body-hinge frameworks in K^d for every d. The first pin is the framework $(G, \Lambda p)$. The next four prove (1): the isomorphism, the equality of the dimensions, that $(G, \Lambda p)$ satisfies the rank hypothesis (Definition 23.21) at a target exactly when (G, p) does, and rigidity. The last two prove (2) and (3).

For a general Λ, the screw $\Lambda\, C(p(e))$ need not be the extensor of a line. The hinge constraint at e is then $S(u) - S(v) \in \operatorname{span} \Lambda\, C(p(e))$, as in the formal definition (the note after Definition 19.3).

Over $\mathbb{R}$, every projective transformation of real projective space induces such a map Λ, and Theorem 25.7 is then Crapo and Whiteley's projective invariance of infinitesimal rigidity [CW82, §3.6]; see also Whiteley [Whi99, Theorem 6]. A collineation is given by a nonsingular linear map g of $\mathbb{R}^4$ on homogeneous coordinates. It induces the screw map Λ_g (Definition 25.5), with $\Lambda_g(v_0 \vee v_1) = gv_0 \vee gv_1$, which sends the extensor of the line through two points to that of the line through their images. A correlation sends points to planes by a nonsingular linear map, and it also induces a linear automorphism of the screw space that sends lines to lines [CW82, §3.6]. The polarity of the next lemma is a correlation. The proof uses only that Λ is linear and invertible, so we state the theorem over every field. The pencil chapters apply it over an arbitrary field (Section 29.3). For the rows of the rigidity matrix, Lemma 28.35 proves the invariance of the rank directly.

For $d = 3$ and $j = 2$, the complement isomorphism $*$ (Definition 22.5) is a linear automorphism of the screw space $\bigwedge^2 K^4$, over every field K. It sends a nonzero extensor of a two-dimensional subspace U of K^4 to a nonzero extensor of $U^\perp$ (the remarks after Definition 22.7). So it is the map that the polarity $x \mapsto x^\perp$ of projective space induces on lines: the line through two points goes to the line in which their polar planes meet. We call $*$ the *polarity*. The next lemma gives its value on joins.

Lemma 25.9 (The polarity on joins)**.** *Let K be a field, and let $*$ be the polarity of $\bigwedge^2 K^4$. For all $v_0, v_1 \in K^4$, $*(v_0 \vee v_1) = C(v_0, v_1)$. In particular, for points $a, b \in \mathbb{R}^3$, the polarity of $\bigwedge^2 \mathbb{R}^4$ sends the extensor $C(a, b) = \bar{a} \vee \bar{b}$ to $C(\bar{a}, \bar{b})$.*

Proof. The first statement holds by the definition of $C(v_0, v_1)$ (Definition 23.1). The second is the first at $K = \mathbb{R}$, $v_0 = \bar{a}$ and $v_1 = \bar{b}$, as $C(a, b) = \bar{a} \vee \bar{b}$ (Definition 18.8). □

Formalization note 25.10. The pins are the polarity and the two statements, in this order.

The polar plane of a point $c \in \mathbb{R}^3$ is $\{x \in \mathbb{R}^3 : x \cdot c + 1 = 0\}$, since $(x, 1) \cdot (c, 1) = x \cdot c + 1$, and c is the *pole* of this plane. Its normal is $\bar{c} = (c, 1)$, and it is the panel

that Katoh and Tanigawa attach to the vector c [KT11, §5.1]. They observe that the polar of a hinge-concurrent framework is a panel-hinge framework. Conversely, the polar of a nonparallel panel-hinge framework, in which any two panels meet in a line, is hinge-concurrent, with the poles of the panels as centres. By projective invariance, the two frameworks are infinitesimally rigid together [KT11, pp. 650 and 671]. Whiteley states this correspondence for molecular structures [Whi99, Corollary 12]. The next lemma proves it in the form that we use.

Lemma 25.11 (Projective dual of a panel-hinge framework)**.** *Let $*$ be the polarity of $\bigwedge^2 \mathbb{R}^4$.*

(1) Let G be a multigraph on V, and let $c\colon V \to \mathbb{R}^3$. The panel-hinge framework on G with normals $\bar{c}(v)$ (Definition 23.5) is obtained from the molecular framework on (G, c) (Definition 25.3) by applying $*$ to the supporting extensor at every edge. Hence the two frameworks have motion spaces of the same dimension and rigidity matrices of the same rank. One of them is infinitesimally rigid if and only if the other is, and the hinge at an edge is nondegenerate in one if and only if it is nondegenerate in the other.

(2) The conclusions of (1) also hold for the panel-hinge framework on G with normals $\lambda_v \bar{c}(v)$, for any nonzero scalars λ_v. Its motion space is that of the panel-hinge framework with normals $\bar{c}(v)$.

Proof. For (1), both frameworks have the multigraph G. At an edge with ends u and v, the molecular framework has the supporting extensor $\bar{c}(u) \vee \bar{c}(v)$. Its image under $*$ is $C(\bar{c}(u), \bar{c}(v))$ (Lemma 25.9), the supporting extensor of the panel-hinge framework at that edge. As $*$ is a linear automorphism of $\bigwedge^2 \mathbb{R}^4$ (Definition 22.5), Theorem 25.7 gives the rest. For (2), the join is bilinear, so $C(\lambda_u \bar{c}(u), \lambda_v \bar{c}(v)) = \lambda_u \lambda_v \, C(\bar{c}(u), \bar{c}(v))$. The two panel-hinge frameworks differ by nonzero multiples of their supporting extensors, so they have the same motion space by Theorem 25.7(3). □

Formalization note 25.12. The first pin is the identity of the two frameworks in (1), and the next four are its consequences: the dimensions of the motion spaces, the rank hypothesis (Definition 23.21), rigidity, and the nondegeneracy of the hinges. The last two pins are the first two of Lemma 30.38, which hold for any normals over any field, and give (2) at the normals $\bar{c}(v)$. The first scales the hinge at an edge with ends u and v by $\lambda_u \lambda_v \neq 0$, so it is nondegenerate exactly when the hinge at the normals $\bar{c}(v)$ is. The second is the equality of the motion spaces, from which the other conclusions of (1) follow by the first five pins. The proof of Theorem 25.20 rescales the normals it uses directly, by the argument above.

25.3. **Screw velocity fields.** The square-graph dictionary (Theorem 25.22) compares the infinitesimal motions of the two models of a molecule. A motion of the molecular framework assigns a screw to each body, and a motion of the bar-joint framework assigns a velocity to each atom. A screw is an infinitesimal rigid motion of space. It moves a point x with velocity $\omega \times x + t$, where ω is its angular velocity and t is the velocity of the origin. This subsection defines this velocity field and proves the facts about it that the dictionary uses.

Definition 25.13 (Velocity field of a screw)**.** For $v \in \mathbb{R}^4$, let $v' \in \mathbb{R}^3$ be its first three coordinates and v_4 its last. The *rotation part* $\omega_S \in \mathbb{R}^3$ and the *translation part* $t_S \in \mathbb{R}^3$ of a screw $S \in \bigwedge^2 \mathbb{R}^4$ are the linear functions of S with

$$\omega_{v\vee w} = v_4\, w' - w_4\, v', \qquad t_{v\vee w} = v' \times w' \qquad (v, w \in \mathbb{R}^4).$$

The *velocity field* of S is

$$\mathrm{vel}_S(x) = \omega_S \times x + t_S \qquad (x \in \mathbb{R}^3).$$

The two formulas are bilinear and alternating in v and w, so they define linear maps $S \mapsto \omega_S$ and $S \mapsto t_S$ on $\bigwedge^2 \mathbb{R}^4$. Write $S = \sum_{i<j} S_{ij}\, e_i \vee e_j$ in the standard basis. Then $\omega_S = -(S_{14}, S_{24}, S_{34})$ and $t_S = (S_{23}, -S_{13}, S_{12})$, so up to sign and order ω_S and t_S are the six coordinates of S. In particular S is determined by ω_S and t_S. For two points $a, b \in \mathbb{R}^3$, the extensor $\bar{a} \vee \bar{b}$ of the line through them has $\omega = b - a$ and $t = a \times b$, and

$$\mathrm{vel}_{\bar{a}\vee\bar{b}}(x) \;=\; (b-a) \times x + a \times b \;=\; (b-a) \times (x-a).$$

This is the velocity field of the rotation about the line through a and b with angular velocity $b - a$. For every screw S and all $x, y \in \mathbb{R}^3$,

$$\mathrm{vel}_S(x) - \mathrm{vel}_S(y) \;=\; \omega_S \times (x - y).$$

Lemma 25.14 (Screws whose velocity field vanishes at two points)**.** *Let $a \neq b$ be points of $\mathbb{R}^3$, and let S be a screw. Then $\mathrm{vel}_S(a) = \mathrm{vel}_S(b) = 0$ if and only if S is a multiple of $\bar{a} \vee \bar{b}$. Moreover $\langle x - y,\, \mathrm{vel}_S(x) - \mathrm{vel}_S(y)\rangle = 0$ for all $x, y \in \mathbb{R}^3$, that is, every velocity field is an infinitesimal isometry.*

Proof. The second statement holds because $\mathrm{vel}_S(x) - \mathrm{vel}_S(y) = \omega_S \times (x - y)$ is orthogonal to $x - y$. The velocity field of $\bar{a} \vee \bar{b}$ is $(b-a) \times (x-a)$, which vanishes at a and at b, and so do the velocity fields of its multiples. Conversely, suppose that $\mathrm{vel}_S(a) = \mathrm{vel}_S(b) = 0$. Then $\omega_S \times (b-a) = \mathrm{vel}_S(b) - \mathrm{vel}_S(a) = 0$, and since $b - a \neq 0$, $\omega_S = \lambda\,(b-a)$ for some $\lambda \in \mathbb{R}$. From $\mathrm{vel}_S(a) = 0$ we get $t_S = -\omega_S \times a = \lambda\, a \times b$. So S and $\lambda\,(\bar{a} \vee \bar{b})$ have the same rotation and translation parts, and they are equal. □

Lemma 25.15 (A screw is determined by point velocities)**.**

1. *A screw whose velocity field vanishes at three non-collinear points of $\mathbb{R}^3$ is zero.*
2. *Let $(p_i)_{i\in I}$ be a family of points of $\mathbb{R}^3$, and let $i_0, i_1, i_2 \in I$ be such that $p_{i_0}, p_{i_1}, p_{i_2}$ are not collinear and, for every $j \in I$ other than i_0, i_1 and i_2, the points $p_j, p_{i_0}, p_{i_1}, p_{i_2}$ are affinely independent. Let $(x_i)_{i\in I}$ be vectors of $\mathbb{R}^3$ with $\langle p_i - p_j,\, x_i - x_j\rangle = 0$ for all $i, j \in I$. Then there is a unique screw S with $\mathrm{vel}_S(p_i) = x_i$ for every $i \in I$.*

Proof. (1) Let vel_S vanish at non-collinear points q_0, q_1, q_2. Then $\omega_S \times (q_i - q_0) = \mathrm{vel}_S(q_i) - \mathrm{vel}_S(q_0) = 0$ for $i = 1, 2$, so ω_S is a multiple of both $q_1 - q_0$ and $q_2 - q_0$. These vectors are linearly independent, so $\omega_S = 0$, and then $t_S = \mathrm{vel}_S(q_0) = 0$. Hence $S = 0$ (the remarks after Definition 25.13).

(2) Uniqueness follows from (1): two such screws differ by a screw whose velocity field vanishes at the non-collinear points $p_{i_0}, p_{i_1}, p_{i_2}$.

For existence, let $q_k = p_{i_k}$, with prescribed velocity $y_k = x_{i_k}$, for $k = 0, 1, 2$, and let $e_i = q_i - q_0$ and $d_i = y_i - y_0$ for $i = 1, 2$. The hypothesis on the pairs among q_0, q_1, q_2 reads $e_1 \cdot d_1 = e_2 \cdot d_2 = 0$ and $e_1 \cdot d_2 + e_2 \cdot d_1 = 0$. We first find $\omega \in \mathbb{R}^3$ with $\omega \times e_1 = d_1$ and $\omega \times e_2 = d_2$. By the vector triple product, $\omega_0 = (e_1 \cdot e_1)^{-1} (e_1 \times d_1)$ satisfies $\omega_0 \times e_1 = d_1$. The vector $u = d_2 - \omega_0 \times e_2$ is orthogonal to e_2, and the third of these equations makes it orthogonal to e_1 as well. So $u = \mu (e_1 \times e_2)$ for some $\mu \in \mathbb{R}$, and $\omega = \omega_0 + \mu e_1$ satisfies both equations. Let S be the screw with $\omega_S = \omega$ and $t_S = y_0 - \omega \times q_0$. Then $\mathrm{vel}_S(q_0) = y_0$, and $\mathrm{vel}_S(q_i) = y_0 + \omega \times e_i = y_i$ for $i = 1, 2$.

Let $j \in I$ be other than i_0, i_1 and i_2, let $w = p_j$, with prescribed velocity $y_w = x_j$, and let $r = \mathrm{vel}_S(w) - y_w$. For $k = 0, 1, 2$, the vector $w - q_k$ is orthogonal to $\mathrm{vel}_S(w) - \mathrm{vel}_S(q_k)$ (Lemma 25.14) and to $y_w - y_k$ (by hypothesis). As $\mathrm{vel}_S(q_k) = y_k$, it is orthogonal to r. By hypothesis the points w, q_0, q_1, q_2 are affinely independent, so the vectors $w - q_0$, $w - q_1$, $w - q_2$ span $\mathbb{R}^3$, and $r = 0$.

The hypothesis of affine independence cannot be dropped: four coplanar points, three of them not collinear, admit the out-of-plane flex of a flat complete quadrilateral, which is not the restriction of a screw. Indeed, every assignment of velocities perpendicular to the plane satisfies $\langle p_i - p_j, x_i - x_j \rangle = 0$, and these assignments form a space of dimension four. But on the plane, the perpendicular component of a velocity field is an affine function, which is determined by its values at three non-collinear points. □

25.4. **Molecular realizations in general position.** The square-graph dictionary (Theorem 25.22) needs the centres of a molecular framework to be in general position up to order four. This subsection defines the condition and constructs placements that satisfy it: first together with genericity for row independence, then as the centres of a molecular realization at the rank of KT Theorem 5.6.

Definition 25.16 (General position up to order four)**.** A placement $p \colon V \to \mathbb{R}^3$ (Definition 6.1) is *in general position up to order four* if, for every set $S \subseteq V$ of at most four vertices, the points $p(v)$ with $v \in S$ are affinely independent.

Such a placement is injective, and no three of its points are collinear. For every finite V there is one. Let $t \colon V \to \mathbb{R}$ be injective, and place each vertex v at the point (t_v, t_v^2, t_v^3) of the moment curve. For four vertices with parameters $t_1, \dots, t_4$, the matrix with rows $(t_i, t_i^2, t_i^3, 1)$ has the Vandermonde determinant $\pm \prod_{i<j} (t_j - t_i) \neq 0$, so the four points are affinely independent (Lemma 18.2). Fewer than four points of the curve lie among four of its points at distinct parameters, and a subfamily of an affinely independent family is affinely independent.

Lemma 25.17 (Generic placements in general position)**.** *Let V be finite. Then there is a placement $p \colon V \to \mathbb{R}^3$ that is generic for row independence (Definition 24.1) and in general position up to order four.*

Proof. Let p_0 be a placement that is generic for row independence (Lemma 24.2), let q be a placement in general position up to order four, and let $p^t = p_0 + t\,(q - p_0)$ for $t \in \mathbb{R}$.

Let $I \subseteq E(K_V)$ be a set of edges that is row-independent at some placement. Then it is row-independent at p_0. Its rigidity rows at p^t are affine in t, so it is row-independent at p^t for all but finitely many t, as in the proof of Lemma 24.2. Let $S \subseteq V$ have at most four vertices, and fix $v_0 \in S$. The points $p^t(v)$ with $v \in S$ are affinely independent exactly when the vectors $p^t(v) - p^t(v_0)$ with $v \in S \setminus \{v_0\}$ are linearly independent. These vectors are affine in t and linearly independent at $t = 1$, so in the same way they are linearly independent for all but finitely many t.

There are finitely many sets I and S. Choose t outside the finitely many exceptional values of all of them. Then p^t is in general position up to order four, and it is generic for row independence, since every edge set that is row-independent at some placement is row-independent at p^t. □

KT Theorem 5.6 realizes a simple graph as a panel-hinge framework at the target rank (Definition 23.21). The realization can be taken with nonparallel panels and nondegenerate hinges (Theorem 23.48). We need more. A normal $n \in \mathbb{R}^4$ with a nonzero last coordinate is a nonzero multiple of $\bar{c} = (c, 1)$ for a unique point $c \in \mathbb{R}^3$, the pole of its panel (Section 25.2). We ask that the poles be in general position up to order four. Up to a rescaling of its normals, the realization is then the image under the polarity of a molecular framework whose centres are the poles (Lemma 25.11).

Lemma 25.18 (KT Theorem 5.6 at $d = 3$ with poles in general position)**.** *Let $G = (V, E)$ be a simple graph with $V \neq \emptyset$. Then G has a panel-hinge realization (G, p) in $\mathbb{R}^3$ whose normals are $n_v = \lambda_v \bar{c}(v)$, for nonzero scalars λ_v and points $c(v) \in \mathbb{R}^3$ in general position up to order four (Definition 25.16), whose panels are nonparallel (Definition 23.10), and which has*

$$\dim Z(G, p) \;=\; D + \operatorname{def}(\tilde{G}),$$

where $D = 6$.

Proof. If $|V| = 1$, then G has no edge, as it is simple, and $\operatorname{def}(\tilde{G}) = 0$. Every realization of G then has $\dim Z(G, p) = D$, the dimension of the screw space, and any normal $\bar{c}(v)$ will do. Its panels are nonparallel, as it has no two distinct bodies.

Let $|V| \geq 2$, and let $N = D(|V| - 1) - \operatorname{def}(\tilde{G})$. KT Theorem 5.6 at $d = 3$ (Theorem 23.52) holds with nondegenerate hinges (Theorem 23.48). So G has a panel-hinge realization with normals q_0 whose hinges are all nondegenerate and whose rigidity matrix has rank N. We move the normals into general position without lowering the rank. The rank is lower semicontinuous on the normals at which every hinge is nondegenerate (Lemma 23.112). We use this in the form of Lemma 23.120, which keeps the minor of its proof: a polynomial Q_1 in the coordinates of the normals, an $N \times N$ minor of a matrix of rows of the rigidity matrix, such that $Q_1(q_0) \neq 0$ and $\operatorname{rank} R(G, q) \geq N$ at every q with $Q_1(q) \neq 0$.

A second polynomial Q_2 tests general position. Regard the normals as the rows of a $|V| \times 4$ matrix of variables. Let Q_2 be the product of the last coordinates of all the normals and of the minors on the first j columns of the submatrices formed by j distinct rows, for $j = 2, 3, 4$. At normals with $Q_2 \neq 0$, each normal has a nonzero last coordinate, so that $n_v = \lambda_v \bar{c}(v)$ with $\lambda_v \neq 0$. Any j of the normals, for $2 \leq j \leq 4$, have a nonzero $j \times j$ minor, so they are linearly independent, and so are

the vectors $\bar{c}(v)$, which differ from them by nonzero factors. Hence any four or fewer of the points $c(v)$ are affinely independent (Lemma 18.2). The polynomial Q_2 is not zero. At the normals $(1, t_v, t_v^2, t_v^3)$ of the moment curve, with distinct nonzero t_v, the last coordinates are $t_v^3 \neq 0$, and each of the minors is a Vandermonde determinant, which is nonzero.

A third polynomial Q_3 tests nonparallel panels: the product, over the ordered pairs (u, v) of distinct bodies, of the 2×2 minor on the first two columns of the rows n_u and n_v. Where it is not zero, the first three coordinates of n_u and of n_v are linearly independent, so the panels are nonparallel (Definition 23.10). At the same moment-curve normals each factor is $t_v - t_u \neq 0$ (Lemma 23.12), so Q_3 is not zero.

As $\mathbb{R}$ is infinite, the nonzero polynomial $Q_1 Q_2 Q_3$ has a non-root q, and at q the poles are in general position up to order four and the panels are nonparallel. In particular any two of the normals are linearly independent, so every hinge of (G, q) is nondegenerate (Lemma 23.11). The deficiency bound for nondegenerate hinges (Lemma 20.16) gives $\operatorname{rank} R(G, q) \leq N$, and $Q_1(q) \neq 0$ gives $\operatorname{rank} R(G, q) \geq N$. So $\dim Z(G, q) = D|V| - N = D + \operatorname{def}(\tilde{G})$.

Katoh and Tanigawa need only a nonparallel realization, one whose panels meet pairwise in lines, and they obtain it by a projective transformation (KT pp. 670–671). The square-graph dictionary needs the poles in general position up to order four. The argument above provides both. □

Formalization note 25.19. The pinned theorem assumes that G spans the set W of bodies, and nothing of the edge labels (Remark 23.40). It states the general position of the poles for the normals: each has a nonzero last coordinate, and any four or fewer of them are linearly independent. It also states that every hinge at an edge of G is nondegenerate. Its endpoint selector assigns each edge of G its two ends, as every panel-hinge framework's does (Formalization note 23.6).

Katoh and Tanigawa deduce from KT Theorem 5.6 that every simple graph has a molecular realization at the target rank (KT p. 671). Taking the poles of Lemma 25.18 as centres, we obtain such a realization with centres in general position up to order four.

Theorem 25.20 (Molecular realizations at the target rank; KT p. 671)**.** *Let $G = (V, E)$ be a simple graph with $V \neq \emptyset$. Then there is a placement $c \colon V \to \mathbb{R}^3$ in general position up to order four (Definition 25.16) such that the space $Z_{\mathrm{mol}}(G, c)$ of infinitesimal motions of the molecular framework on (G, c) (Definition 25.3) has dimension*

$$\dim Z_{\mathrm{mol}}(G, c) \;=\; D + \operatorname{def}(\tilde{G}),$$

where $D = 6$. In particular, if $\operatorname{def}(\tilde{G}) = 0$, then this molecular framework is infinitesimally rigid.

Proof. Let (G, p) be the realization of Lemma 25.18, with normals $n_v = \lambda_v \bar{c}(v)$, where each $\lambda_v \neq 0$ and the points $c(v)$ are in general position up to order four. Up to the factors λ_v, these are the normals $\bar{c}(v)$ of the image of the molecular framework on (G, c) under the polarity (Lemma 25.11(1)). The factors multiply the supporting extensor at an edge with ends u and v by $\lambda_u \lambda_v \neq 0$, which does not change the

motion space (Theorem 25.7(3)). So by Lemma 25.11(2),

$$\dim Z_{\mathrm{mol}}(G,c) \;=\; \dim Z(G,p) \;=\; D + \mathrm{def}(\tilde{G}).$$

If $\mathrm{def}(\tilde{G}) = 0$, then $\dim Z_{\mathrm{mol}}(G,c) = D$. As $V \neq \emptyset$, the trivial motions form a D-dimensional subspace of $Z_{\mathrm{mol}}(G,c)$ (Lemma 19.18), so every infinitesimal motion is trivial. □

Formalization note 25.21. The pinned theorem assumes that G spans the set W of bodies, and nothing of the edge labels (Remark 23.40). It also returns the endpoint selector of the molecular framework, and states that it assigns each edge of G its two ends (Formalization note 23.6). This is the form in which Theorem 25.22 takes the framework. The rescaling in the proof is proved directly, from the equality of the spans of the supporting extensors, by the inclusion of motion spaces of Lemma 19.22 in both directions. The pinned theorem does not state the case $\mathrm{def}(\tilde{G}) = 0$, which the last paragraph of the proof derives. Its formal step is the last pin of Lemma 19.18: a framework on a nonempty set of bodies whose motion space has dimension D is infinitesimally rigid.

25.5. **The square-graph dictionary.** We prove Whiteley's equivalence for the motion spaces at one placement c in general position up to order four. A screw assignment S of the molecular framework on (G,c) gives each atom v the velocity of its own screw $S(v)$ at its centre $c(v)$. We show that this map is an isomorphism from the motions of the molecular framework onto those of the bar-joint framework on G^2 at c. Its inverse recovers the screw of a body v from the velocities at the centres of the closed neighbourhood $N[v]$, which is a clique of G^2 (Lemma 25.2). Neither Jackson and Jordán nor Katoh and Tanigawa say which general position the equivalence needs. The proof needs any four centres of a closed neighbourhood to be affinely independent, because four coplanar points admit the out-of-plane flex of a flat complete quadrilateral, which is not the field of velocities of a screw (the end of the proof of Lemma 25.15). It also needs every atom to have at least two bonds. Whiteley notes that, except in how atoms with only one bond or none are handled, the hinge model of a molecule is equivalent to the bond-bending networks used in modelling glasses [Whi99, p. 27].

Theorem 25.22 (Molecular frameworks and bar-joint frameworks on the square; Whiteley)**.** *Let $G = (V,E)$ be a graph of minimum degree at least two, and let $c\colon V \to \mathbb{R}^3$ be a placement in general position up to order four. Then the space $Z_{\mathrm{mol}}(G,c)$ of infinitesimal motions of the molecular framework on (G,c) (Definition 25.3) and the kernel of the rigidity map $R(G^2,c)$ of the square of G (Definition 25.1) are isomorphic. So*

$$\dim Z_{\mathrm{mol}}(G,c) \;=\; \dim\ker R(G^2,c),$$

and $\operatorname{rank} R(G^2,c) = 3|V| - \dim Z_{\mathrm{mol}}(G,c)$*. Moreover, G has an infinitesimally rigid molecular realization with nondegenerate hinges, that is, with $c(u) \neq c(v)$ at every edge uv, if and only if G^2 has an infinitesimally rigid bar-joint realization in $\mathbb{R}^3$.*

Proof. The map. For a screw assignment S, let $\Phi(S)$ be the velocity assignment $v \mapsto \mathrm{vel}_{S(v)}(c(v))$ (Definition 25.13). It is linear in S. Let $S \in Z_{\mathrm{mol}}(G,c)$, and let u

be a neighbour of a vertex v. By the hinge constraint at uv, the screw $S(u) - S(v)$ is a multiple of $\bar{c}(u) \vee \bar{c}(v)$, whose velocity field vanishes at $c(u)$ (Lemma 25.14). So $\Phi(S)(u) = \mathrm{vel}_{S(v)}(c(u))$: on the closed neighbourhood $N[v]$, the velocities $\Phi(S)$ are those of the single screw $S(v)$.

Motions go to motions. Every edge xy of G^2 has both ends in $N[w]$ for some vertex w (Lemma 25.2). Hence $\Phi(S)(x) - \Phi(S)(y) = \mathrm{vel}_{S(w)}(c(x)) - \mathrm{vel}_{S(w)}(c(y))$, which is orthogonal to $c(x) - c(y)$ (Lemma 25.14). So $\Phi(S)$ lies in $\ker R(G^2, c)$ (Lemma 6.4).

Injectivity. Let $\Phi(S) = 0$, and let v be a vertex. The velocity field of $S(v)$ vanishes at the centres of $N[v]$. There are at least three of them, as v has at least two neighbours, and they are not collinear, as c is in general position. So $S(v) = 0$ (Lemma 25.15(1)).

Surjectivity. Let $x \in \ker R(G^2, c)$, and let v be a vertex. The closed neighbourhood $N[v]$ is a clique of G^2 (Lemma 25.2), so $\langle c(a) - c(b),\, x(a) - x(b)\rangle = 0$ for all $a, b \in N[v]$. The centres of $N[v]$ include three non-collinear points, and any four of them are affinely independent. By Lemma 25.15(2) there is a unique screw $S(v)$ with $\mathrm{vel}_{S(v)}(c(a)) = x(a)$ for every $a \in N[v]$. In particular $\Phi(S)(v) = x(v)$. Let uv be an edge of G. Both u and v lie in $N[u]$ and in $N[v]$, so the velocity fields of $S(u)$ and $S(v)$ agree at $c(u)$ and at $c(v)$. As $c(u) \neq c(v)$, the screw $S(u) - S(v)$ is a multiple of $\bar{c}(u) \vee \bar{c}(v)$ (Lemma 25.14), which is the hinge constraint at uv. So $S \in Z_{\mathrm{mol}}(G, c)$, and $\Phi(S) = x$.

Hence Φ restricts to an isomorphism from $Z_{\mathrm{mol}}(G, c)$ onto $\ker R(G^2, c)$. The map $R(G^2, c)$ is defined on a space of dimension $3|V|$ (Lemma 6.2), so its rank is $3|V| - \dim Z_{\mathrm{mol}}(G, c)$.

Realizations. Both sides of the equivalence hold when $V = \emptyset$, so let $V \neq \emptyset$. Suppose first that the molecular framework on (G, c_0) is infinitesimally rigid for some c_0 with $c_0(u) \neq c_0(v)$ at every edge uv. Its motions are the trivial ones, so $\dim Z_{\mathrm{mol}}(G, c_0) = D$ (Lemma 19.18). Its hinges are nondegenerate, so $\dim Z_{\mathrm{mol}}(G, c_0) \geq D + \mathrm{def}(\tilde{G})$ (Lemma 20.15), and $\mathrm{def}(\tilde{G}) = 0$. By Theorem 25.20 there is a placement c' in general position up to order four with $\dim Z_{\mathrm{mol}}(G, c') = D = 6$. By the first part, $\dim \ker R(G^2, c') = 6$, so c' is infinitesimally rigid for G^2 (Definition 6.5).

Conversely, let p be infinitesimally rigid for G^2, so that $\mathrm{rank}\, R(G^2, p) \geq 3|V| - 6$. Let c' be a placement that is generic for row independence and in general position up to order four (Lemma 25.17). The rank of $R(G^2, p)$ is at most the generic rank of G^2 (Lemma 26.1), which is the rank of $R(G^2, c')$ (Lemma 24.9), so $\mathrm{rank}\, R(G^2, c') \geq 3|V| - 6$. By the first part, $\dim Z_{\mathrm{mol}}(G, c') \leq 6 = D$. The trivial motions form a D-dimensional subspace of $Z_{\mathrm{mol}}(G, c')$ (Lemma 19.18), so the molecular framework on (G, c') is infinitesimally rigid. Its hinges are nondegenerate, as c' is injective. □

Formalization note 25.23. The first pinned theorem states the equality of the dimensions. It builds the molecular framework on a multigraph on V in which two distinct vertices are joined by an edge exactly when they are adjacent in G, with an endpoint selector that assigns each edge its two ends (Formalization note 23.6). Edges with the same ends impose the same hinge constraint, and a loop imposes none, so the

motions are those of the molecular framework on (G, c). The second pinned theorem states the equivalence of realizations in the last sentence of the theorem, for a simple graph G, on the multigraph of Lemma 26.3: a molecular realization takes an endpoint selector that assigns each edge its two ends, and centres c with $c(u) \neq c(v)$ at every edge uv of G. Its proof is the one above, the case $V = \emptyset$ included.

26. The molecule rank formula (KT Corollary 5.7)

Let $G = (V, E)$ be a graph of minimum degree at least two. In this chapter we prove KT Corollary 5.7, the rank formula

$$r(G^2) \;=\; 3|V| - 6 - \mathrm{def}(\tilde{G}),$$

where $r = r_3$ is the generic rank in $\mathbb{R}^3$ (Definition 24.7) and G^2 is the square of G (Definition 25.1). The deficiency $\mathrm{def}(\tilde{G})$ is that of Definition 20.3 at $D = 6$, with $\tilde{G} = 5 \cdot G$. Jackson and Jordán conjectured the formula and proved the inequality $\leq$ [JJ08a, Conjecture 2.2 and Theorem 4.1]. Their Conjecture 2.1 is the bar-joint form of the molecular conjecture of Tay and Whiteley [TW84], and they proved that the formula is equivalent to it (their Theorem 4.3). Katoh and Tanigawa deduce the formula from their Theorem 5.6, through Whiteley's equivalence between molecular and bar-joint realizations and Jackson and Jordán's Theorem 4.3 (KT p. 671).

We derive both inequalities from KT Theorem 5.6, through the equalities between dimensions of motion spaces that Section 25 proves. The opening of that chapter outlines the argument and compares it with Jackson and Jordán's. Section 26.1 shows that the rank of a graph at any placement is at most its generic rank. Section 26.2 passes from the simple graph G to a multigraph, on which the formalization states the deficiency and KT Theorem 5.6. Section 26.3 proves the two inequalities and the formula, and derives from it when the square of a graph is generically rigid.

26.1. The generic rank bounds the rank at every placement.

At a placement that is generic for row independence, the rank of the rigidity map of a graph equals its generic rank (Lemma 24.9). The lower bound compares the two at the centres of a molecular realization, which need not be generic.

Lemma 26.1 (The rank at a placement is at most the generic rank)**.** *Let H be a graph on a finite set V, and let p be a placement of V in $\mathbb{R}^d$. Then*

$$\dim \mathrm{span}\{\, r_e \mid e \in E(H) \,\} \;\leq\; r_d(H),$$

where r_e is the rigidity row of K_V at p indexed by e (Definition 9.4). Equivalently, $\mathrm{rank}\, R(H, p) \leq r_d(H)$.

Proof. The linear rigidity matroid $\mathcal{L}_p(V)$ is represented by the extended rigidity rows $\tilde{r}_e$ at p (Definition 10.16), and $\tilde{r}_e = r_e$ for every edge e of K_V. In a represented matroid, the vectors of a set span a space whose dimension is the rank of the set (Lemma 16.7). So the span has dimension equal to the rank of $E(H)$ in $\mathcal{L}_p(V)$. A set that is independent in $\mathcal{L}_p(V)$ is row-independent at p (Lemma 10.17). It is therefore independent in $\mathcal{R}_d(V)$ (Lemma 24.4). Hence the rank of $E(H)$ in $\mathcal{L}_p(V)$ is at most its rank $r_d(H)$ in $\mathcal{R}_d(V)$.

The rank of $R(H,p)$ is the dimension of the same span. Indeed, $R(H,p)$ has the same rank as its transpose, whose image is spanned by the rigidity rows of H at p (Lemma 9.5). For an edge uv of H, the row of H at uv is $x \mapsto \langle p_u - p_v,\, x_u - x_v\rangle$ (Lemma 6.4), which is also the row r_{uv} of K_V. □

Formalization note 26.2. The first pin states the bound for the span, and the second the bound for rank $R(H,p)$, which the proof of Lemma 26.5 uses.

26.2. **Simple graphs as multigraphs.** Katoh and Tanigawa state their Theorem 5.6 for multigraphs, and the deficiency is defined for a multigraph (Definition 20.3). We regard the simple graph G as a multigraph. In the formalization this step is the following lemma.

Lemma 26.3 (A simple graph as a multigraph)**.** *Let G be a simple graph on a finite set V. Then there is a simple multigraph G' with vertex set V, in which two distinct vertices are joined by an edge if and only if they are adjacent in G.*

Proof. Take as labels the $|V|(|V|+1)/2$ unordered pairs of elements of V. For each edge uv of G, let G' have one edge joining u and v, labelled by the pair $\{u,v\}$. Distinct edges of G' have distinct labels, hence distinct pairs of ends, and no edge is a loop, as G has none. So G' is simple, and its edges join exactly the adjacent pairs of G. □

Formalization note 26.4. In the formalization, simple graphs and multigraphs are different objects. The square and the generic rank are defined for simple graphs, and the deficiency and the frameworks of Section 25 for multigraphs, whose edges are drawn from a fixed set of labels (Formalization note 21.11). The first pin is G', with the unordered pairs of V as its labels, and the next three are its simplicity, its vertex set and its edges. The pinned forms of the three results below state the deficiency of G', and apply the results of Section 25 to G'.

A partition of V separates as many edges of G' as of G, since the edges of G' join exactly the adjacent pairs of G. So G' and G have the same deficiency. It is the deficiency $\operatorname{def}(G)$ of Jackson and Jordán [JJ08a, §2], the largest value of $6(|P|-1)-5\,e_G(P)$ over the partitions P of V, where $e_G(P)$ is the number of edges of G that join different parts of P. We write $\operatorname{def}(\tilde{G})$ for it, as Katoh and Tanigawa do.

26.3. **The two inequalities and the formula.** Both inequalities pass from molecular frameworks on G to bar-joint frameworks on G^2 by the square-graph dictionary (Theorem 25.22). The dictionary applies at placements in general position up to order four, and it needs every vertex of G to have degree at least two.

Lemma 26.5 (The lower bound)**.** *Let G be a simple graph of minimum degree at least two on a finite, nonempty vertex set V. Then $3|V| - 6 - \operatorname{def}(\tilde{G}) \le r(G^2)$.*

Proof. Regard G as a multigraph, as in Lemma 26.3. By Theorem 25.20 there is a placement $c\colon V \to \mathbb{R}^3$ in general position up to order four such that the molecular framework on (G,c) has $\dim Z_{\mathrm{mol}}(G,c) = 6+\operatorname{def}(\tilde{G})$. By the square-graph dictionary

(Theorem 25.22),

$$\operatorname{rank} R(G^2, c) \;=\; 3|V| - \dim Z_{\mathrm{mol}}(G, c) \;=\; 3|V| - 6 - \operatorname{def}(\tilde{G}).$$

The generic rank $r(G^2)$ is at least this rank (Lemma 26.1). □

Lemma 26.6 (The upper bound; Jackson and Jordán)**.** *Let G be a simple graph of minimum degree at least two on a finite, nonempty vertex set V. Then $r(G^2) \leq 3|V| - 6 - \operatorname{def}(\tilde{G})$.*

Proof. Regard G as a multigraph, as in Lemma 26.3. Let $p\colon V \to \mathbb{R}^3$ be a placement that is generic for row independence and in general position up to order four (Lemma 25.17). As p is generic, the rank of $R(G^2, p)$ is $r(G^2)$ (Lemma 24.9). So the square-graph dictionary (Theorem 25.22) gives

$$r(G^2) \;=\; 3|V| - \dim Z_{\mathrm{mol}}(G, p).$$

As p is in general position, the centres $p(v)$ are distinct, so the hinges of the molecular framework on (G, p) are nondegenerate. Hence $\dim Z_{\mathrm{mol}}(G, p) \geq 6 + \operatorname{def}(\tilde{G})$ (Lemma 20.15), and the bound follows. □

Formalization note 26.7. The pinned form of the dictionary takes the molecular framework with an endpoint selector that assigns each edge its two ends (Formalization note 23.6). The pinned lemma builds one. It assigns each edge of the multigraph of Lemma 26.3 its two ends, and every other label one fixed pair of adjacent vertices of G. So every label has two distinct ends, and as p is injective, a nonzero hinge.

Corollary 26.8 (The molecule rank formula; Jackson and Jordán, KT Corollary 5.7)**.** *Let G be a simple graph of minimum degree at least two on a finite, nonempty vertex set V. Then*

$$r(G^2) \;=\; 3|V| - 6 - \operatorname{def}(\tilde{G}),$$

where r is the rank function of the generic rigidity matroid $\mathcal{R}_3(V)$ (Definition 24.7), and $\operatorname{def}(\tilde{G})$ is the deficiency of Definition 20.3 at $D = 6$, with $\tilde{G} = 5 \cdot G$.

Proof. Combine Lemma 26.5 and Lemma 26.6. □

The formula decides when the square of a graph is rigid. As $\operatorname{def}(\tilde{G}) \geq 0$, it gives $r(G^2) \leq 3|V| - 6$, with equality if and only if $\operatorname{def}(\tilde{G}) = 0$, that is, if and only if $5 \cdot G$ contains six edge-disjoint spanning trees (Lemma 20.14).

A placement p is infinitesimally rigid for G^2 if and only if $\operatorname{rank} R(G^2, p) \geq 3|V| - 6$ (Definition 6.5). This rank is at most $r(G^2)$ (Lemma 26.1). At a placement that is generic for row independence it equals $r(G^2)$ (Lemma 24.9). So G^2 is generically rigid in $\mathbb{R}^3$ (Definition 6.7) if and only if $5 \cdot G$ contains six edge-disjoint spanning trees. This is Jackson and Jordán's Conjecture 2.1 [JJ08a]. Katoh and Tanigawa note it as a corollary of their work (KT §7). When $5 \cdot G$ contains six edge-disjoint spanning trees, every placement that is generic for row independence is infinitesimally rigid for G^2 (Corollary 28.3).

27. Independence and rank of molecular graphs

This chapter proves two results of Jackson and Jordán [JJ08a] on squares of graphs. We cite their paper as JJ, as we cite Katoh and Tanigawa's as KT. The first result is a conjecture of Jacobs [Jac98]: the square G^2 of a graph G is independent in the three-dimensional generic rigidity matroid if and only if G^2 satisfies JJ's three-dimensional Laman condition (Theorem 27.39). JJ state it as their Conjecture 5.1. They note that Jacobs states it as a result, but that his proof assumes an observation for which no proof was known (JJ §5). JJ derive the conjecture (their Theorem 5.4) from their Conjecture 2.2, the rank formula

$$r(G^2) \;=\; 3|V| - 6 - \mathrm{def}(\tilde{G})$$

for graphs $G = (V, E)$ of minimum degree at least two. This formula is KT Corollary 5.7 (Corollary 26.8), so the conjecture holds.

The second result is the degree-one rank formula, JJ's Lemma 4.2. It concerns connected graphs with vertices of degree one, to which the rank formula does not apply. For a tree G it gives $r(G^2)$ explicitly (Theorem 27.43). For any other connected graph G, it gives $r(G^2)$ through the rank of the square of the core of G, its maximal subgraph of minimum degree at least two (Theorem 27.44). JJ prove it to show that the rank formula, applied to the core, determines $r(G^2)$ for every graph.

The hard direction of Jacobs' conjecture rests on JJ's counting bound (their Theorem 5.3): if G has minimum degree at least two and G^2 satisfies the Laman condition, then $|E(G^2)| \le 3|V| - 6 - \mathrm{def}(\tilde{G})$. By the rank formula the right-hand side is $r(G^2)$, so $E(G^2)$ is independent. Section 27.1 defines the Laman condition, and shows that if G^2 satisfies it, then G has maximum degree at most three. Section 27.2 studies the partitions of V that attain the deficiency. Section 27.3 proves the counting bound by counting the edges of G^2 against one of them. Section 27.4 proves the easy direction of the conjecture, for every graph. Section 27.5 proves the 0-extension lemma in the form we use: adding a vertex joined to at most three vertices raises the rank by the number of new edges. Section 27.6 proves the conjecture, deleting vertices of degree one by this lemma until the minimum degree is at least two. Section 27.7 proves the degree-one rank formula in the same way.

Throughout, V is a finite set, and a graph on V is a simple graph with vertex set V. We write $r = r_3$ for the rank function of the generic rigidity matroid $\mathcal{R}_3(V)$ (Definition 24.7). We call a graph H on V independent when $E(H)$ is independent in $\mathcal{R}_3(V)$, that is, row-independent at some placement of V in $\mathbb{R}^3$ (Lemma 24.4). For a partition P of V, let $d_G(P)$ be the number of edges of G that join different parts of P. The value of P is

$$\mathrm{def}_{\tilde{G}}(P) \;=\; 6(|P| - 1) - 5\, d_G(P),$$

and the deficiency $\mathrm{def}(\tilde{G})$ is the largest value over the partitions of V. This is Definition 20.3 at $D = 6$, for the multigraph $\tilde{G} = 5 \cdot G$.

27.1. **The three-dimensional Laman condition.**

This subsection defines JJ's counting condition for graphs in three dimensions (JJ §5) and proves its first consequence for squares (JJ Lemma 5.2): if G^2 satisfies the condition, then every vertex

of G has degree at most three. Following JJ, we call a graph that satisfies the condition Laman. The condition is the three-dimensional analogue of $(2,3)$-sparsity (Definition 3.1). Unlike the planar Laman graphs of Definition 4.1, it asks for no equality on the number of edges.

Definition 27.1 (The three-dimensional Laman condition)**.** A graph H on V is *Laman* (in the three-dimensional counting sense of [JJ08a, §5]) if for every $X \subseteq V$ with $|X| \geq 3$,

$$\left|E_H[X]\right| \;\leq\; 3|X| - 6.$$

Formalization note 27.2. The condition is not the (k,ℓ)-sparsity of Definition 3.1 at $(k,\ell) = (3,6)$. That predicate bounds every set X with $\ell \leq k|X|$, and at $(3,6)$ these include the sets with $|X| = 2$. For such a set the bound $3|X| - 6 = 0$ fails as soon as X spans an edge, so no graph with an edge is $(3,6)$-sparse. The Laman condition bounds only the sets with $|X| \geq 3$, and K_2, for instance, satisfies it. At $(2,3)$ the two guards agree, as $3 \leq 2|X|$ means $|X| \geq 2$, and the planar Laman graphs of Definition 4.1 are defined through $(2,3)$-sparsity. So the condition is formalized as a predicate of its own, which bounds only the sets with $|X| \geq 3$.

Lemma 27.3 (Monotonicity)**.** *A subgraph of a Laman graph is Laman.*

Proof. Let H be a subgraph of a Laman graph H'. Every edge of H inside a set $X \subseteq V$ is an edge of H' inside X, so $\left|E_H[X]\right| \leq \left|E_{H'}[X]\right| \leq 3|X| - 6$ whenever $|X| \geq 3$. □

Lemma 27.4 (Edge count of a clique)**.** *Let H be a graph on V, and let $X \subseteq V$ be a clique of H, that is, a set of pairwise adjacent vertices. Then $\left|E_H[X]\right| = \binom{|X|}{2}$.*

Proof. The edges of H inside X are exactly the two-element subsets of X. □

Lemma 27.5 (Maximum degree three; JJ Lemma 5.2)**.** *Let G be a graph such that G^2 is Laman. Then every vertex of G has degree at most three.*

Proof. Let v be a vertex of G. Its closed neighborhood $N_G[v]$ is a clique of G^2 on $d_G(v) + 1$ vertices (Lemma 25.2). If $d_G(v) \geq 4$, five of these vertices form a clique of G^2, which spans $\binom{5}{2} = 10 > 3 \cdot 5 - 6$ edges of G^2 (Lemma 27.4). Then G^2 is not Laman. So $d_G(v) \leq 3$. □

27.2. **Tight partitions.** The counting bound of Section 27.3 is proved against a partition of V that attains the deficiency, a tight partition (Definition 20.5). This subsection proves the facts about such partitions that the count uses. They come from JJ Lemma 3.2, which JJ take from their paper on the partition of a graph into its maximal subgraphs of deficiency zero. We prove its part (a), a consequence of its part (b), and two consequences of part (a) that JJ draw in the proofs of their Theorems 3.4 and 5.3: two parts are joined by at most one edge, and two nonadjacent vertices in distinct parts have at most one common neighbor. Apart from the existence of these partitions (Lemma 20.6), all of this follows from one identity, which gives the change in the value $\mathrm{def}_{\tilde{G}}(P)$ when several parts of P are merged into one (Lemma 27.7). The arguments are the same for every D. So we state the lemmas for the D-deficiency of Definition 20.3 at every D, and each lemma names the bound on D that it needs. The chapter applies them at $D = 6$.

Definition 27.6 (Edges between the members of a family of parts)**.** Let G be a multigraph, let P be a partition of $V(G)$, and let $Q \subseteq P$. We write $e_G(Q)$ for the number of edges of G that join distinct members of Q.

Lemma 27.7 (Deficiency under coarsening)**.** *Let G be a multigraph, let P be a partition of $V(G)$, and let $Q \subseteq P$ be nonempty. Let P' be the partition obtained from P by merging the members of Q into a single part. Then*

$$\operatorname{def}_{\tilde{G}}(P') \;=\; \operatorname{def}_{\tilde{G}}(P) \;-\; (D\,(|Q|-1) - (D-1)\,e_G(Q)),$$

where $e_G(Q)$ is the number of edges of G that join distinct members of Q (Definition 27.6).

Proof. The partition P' has $|Q|-1$ fewer parts than P. An edge joins different parts of P' exactly when it joins different parts of P and does not join two members of Q, so $d_G(P') = d_G(P) - e_G(Q)$. The identity follows from the definition of the value (Definition 20.3); when $|Q| = 1$, it says that $P' = P$ has the value of P. Read from right to left, it gives the change in value when a part is split into several: if a part of P' is split into the members of Q, the value rises by $D(|Q|-1) - (D-1)\,e_G(Q)$. □

JJ state their Lemma 3.2(a), with $D = 6$, for the subgraph H of G induced by the union of the members of Q: as a partition of $V(H)$, Q has $\operatorname{def}_{\tilde{H}}(Q) \geq 0$. An edge of H joins two members of Q exactly when it is an edge of G that does, so this is the following inequality.

Lemma 27.8 (Subfamilies of a tight partition; JJ Lemma 3.2(a))**.** *Let G be a multigraph, let P be a tight partition of $V(G)$, and let $Q \subseteq P$ with $|Q| \geq 2$. Then*

$$(D-1)\,e_G(Q) \;\leq\; D\,(|Q|-1),$$

where $e_G(Q)$ is the number of edges of G that join distinct members of Q.

Proof. Merging the members of Q changes the value of P by $(D-1)\,e_G(Q) - D(|Q|-1)$ (Lemma 27.7). The merged partition has value at most $\operatorname{def}(\tilde{G})$, which is the value of P because P is tight. So the change is at most zero. □

JJ's Lemma 3.2(b) says that every part of a tight partition induces a subgraph of deficiency zero. This chapter needs only the following consequence, which JJ draw in the proof of their Theorem 5.3, and we prove it directly.

Lemma 27.9 (Parts of a tight partition; a consequence of JJ Lemma 3.2(b))**.** *Let $D \geq 2$, let G be a simple graph, and let P be a tight partition of $V(G)$. Then every part of P is a singleton or has at least three vertices. In the latter case every vertex of the part has at least two neighbors in the part.*

Proof. Let $A \in P$ with $|A| \geq 2$, let $v \in A$, and let m be the number of neighbors of v in A. Splitting v off A as a part of its own raises the value of P by $D - (D-1)\,m$ (Lemma 27.7, read from right to left with $Q = \{A \setminus \{v\}, \{v\}\}$). As P is tight, the value does not rise, so $D \leq (D-1)\,m$. Since $D \geq 2$, this forces $m \geq 2$. So A contains v and at least two neighbors of v. □

Lemma 27.10 (Edge multiplicity between parts of a tight partition)**.** *Let $D \geq 3$, let G be a simple graph, and let P be a tight partition of $V(G)$. Then two distinct parts of P are joined by at most one edge of G.*

Proof. Let $A \neq B$ be parts of P. For $Q = \{A, B\}$, Lemma 27.8 gives $(D-1)\, e_G(Q) \leq D$. If $e_G(Q) \geq 2$, then $2(D-1) \leq D$, that is, $D \leq 2$, against $D \geq 3$. □

Lemma 27.11 (The common neighbor of two vertices in distinct parts)**.** *Let $D \geq 5$, let G be a simple graph, and let P be a tight partition of $V(G)$. Then two nonadjacent vertices in distinct parts of P have at most one common neighbor.*

Proof. Let u and w lie in distinct parts of P, and suppose that they have two common neighbors $v \neq v'$. The vertices u, v, w, v' are distinct, as $u \neq w$ and adjacent vertices are distinct. So uv, vw, wv' and $v'u$ are four distinct edges of G, and they form a closed walk.

Let Q be the set of parts of P that meet $\{u, v, w, v'\}$, so that $2 \leq |Q| \leq 4$. The walk meets at least two parts, so it enters every member of Q from another member at least once. These entries use distinct edges, each joining two members of Q, so $e_G(Q) \geq |Q|$. Now Lemma 27.8 gives $(D-1)\,|Q| \leq D(|Q|-1)$, that is, $D \leq |Q| \leq 4$, a contradiction. When $|Q| = 2$, two of the four edges join the same two parts, which Lemma 27.10 excludes directly. □

Formalization note 27.12. As in Formalization note 20.4, a partition is a labeling of the vertices, and a tight partition is a labeling whose value is the deficiency. Merging the members of Q gives all their vertices one of their labels. Three of the formal statements assume less than their lemmas. Lemma 20.6 does not need $V(G)$ nonempty: the empty set has one partition, with no parts and value $-D$. Lemma 27.10 holds for every multigraph, with parallel edges counted. Lemma 27.11 does not assume the two vertices nonadjacent, and its proof above does not use this.

27.3. **The counting bound.** This subsection proves JJ Theorem 5.3: if G has minimum degree at least two and G^2 is Laman, then $|E(G^2)| \leq 3|V| - 6 - \mathrm{def}(\tilde{G})$ (Theorem 27.26). We follow JJ's proof, which counts the edges of G^2 part by part against a tight partition of V. Inside a part A with $|A| \geq 3$, the Laman condition bounds them by $3|A| - 6$. The endpoints of an edge of G^2 that joins two parts and is not an edge of G have a unique common neighbor, and the edge is counted at the part that contains it. A part A with $|A| \geq 3$ receives $2\, d_G(A)$ of these edges (JJ eq. (6)), where $d_G(A)$ is the number of edges of G with exactly one endpoint in A. A singleton part $\{v\}$ receives $2\, d_G(v) - 3$ of them (JJ eqs. (5) and (7)). Summing over the parts gives the bound.

For the rest of this subsection, let G be a simple graph of minimum degree at least two whose square is Laman, and let P be a tight partition of V at $D = 6$ (Definition 20.5). Then every vertex of G has degree two or three (Lemma 27.5). Every part of P is a singleton or has at least three vertices, and in the latter case each of its vertices has at least two neighbors in the part (Lemma 27.9). As in Definition 20.3, $d_G(P)$ is the number of edges of G that join distinct parts.

Let uw be an edge of G^2 whose endpoints lie in distinct parts. Following JJ, we call uw a *cross edge* if it is not an edge of G. Then u and w are at distance two in G

(Definition 25.1), so they have a common neighbor v. It is unique (Lemma 27.11), and we call it the common neighbor of the cross edge. At least one of the edges uv and vw joins two parts, as u and w lie in distinct parts. JJ call the cross edge *normal* if exactly one of them does, and *special* if both do. So uw is normal if v lies in the part of u or in that of w, and special otherwise.

Lemma 27.13 (Classification of the edges of the square)**.** *Let G and P be as above. Then $E(G^2)$ is the disjoint union of four classes: the edges inside a part, the $d_G(P)$ edges of G that join distinct parts, the normal cross edges, and the special cross edges. Moreover, the common neighbor of a special cross edge forms a singleton part. The common neighbor of a normal cross edge lies in a part of at least three vertices, together with exactly one endpoint of the edge.*

Proof. An edge of G^2 lies inside a part or joins two parts. In the latter case it is an edge of G or a cross edge, and a cross edge is normal or special. The first three classes are disjoint by definition. The last two are disjoint because the common neighbor of a cross edge is unique (Lemma 27.11).

Let uw be a special cross edge with common neighbor v. Then u and w are two neighbors of v outside its part. If the part of v were not a singleton, v would also have two neighbors inside it (Lemma 27.9), and so degree at least four. This contradicts the degree bound (Lemma 27.5). Now let uw be a normal cross edge with common neighbor v. As u and w lie in distinct parts, v shares its part with exactly one of them. This part contains two adjacent vertices, so it is not a singleton, and it has at least three vertices (Lemma 27.9). □

Formalization note 27.14. The first pin is the union of the edges inside a part, the edges of G between parts and the cross edges, and the fifth splits the cross edges into the normal and the special ones. The second, third, fourth and sixth are the disjointness of the classes, and the seventh counts the second class. The last two are the claims on the common neighbor of a special and of a normal cross edge.

We first count the special cross edges.

Lemma 27.15 (Special cross edges at a singleton part; JJ eq. (5), (7))**.** *Let G and P be as above, and let $\{v\}$ be a singleton part of P. Then the neighborhood $N_G(v)$ is a clique of G^2, all of whose edges are special cross edges with common neighbor v, and*

$$\left|E_{G^2}[N_G(v)]\right| \;=\; 2\,d_G(v) - 3.$$

Proof. The neighborhood $N_G(v)$ is a clique of G^2 (Lemma 25.2). Let $u \neq w$ be neighbors of v. Neither lies in the part $\{v\}$. They do not lie in one part A, as then vu and vw would be two edges between A and $\{v\}$ (Lemma 27.10). They are not adjacent in G. Otherwise the triangle uvw would put three edges between the three parts that contain u, v and w. Then $3(D-1) \leq 2D$ (Lemma 27.8), that is, $D \leq 3$, while $D = 6$. So uw is a cross edge with common neighbor v. It is special, as the part $\{v\}$ contains neither u nor w.

Finally, v has degree two or three (Lemma 27.5). So the clique has $\binom{d_G(v)}{2} = 2\,d_G(v) - 3$ edges (Lemma 27.4). □

JJ also state the converse.

Lemma 27.16 (Special cross edges arise from a unique singleton part)**.** *Let G and P be as above. Then every special cross edge is an edge of G^2 inside $N_G(v)$ for exactly one singleton part $\{v\}$ of P.*

Proof. Let uw be a special cross edge with common neighbor v. Then $\{v\}$ is a part of P (Lemma 27.13), and $u, w \in N_G(v)$. If also $u, w \in N_G(v')$ for a singleton part $\{v'\}$, then v' is a common neighbor of u and w, so $v' = v$ (Lemma 27.11). □

We now count the normal cross edges, following JJ's proof of their eq. (6). Let uw be a normal cross edge with common neighbor v. Exactly one of the edges uv and vw lies inside a part, and JJ root uw at that part, which is the part of v. We root a special cross edge in the same way, at the part of its common neighbor, which is a singleton part (Lemma 27.13). JJ do not root the special cross edges: they count them in the neighborhoods of the singleton parts.

Definition 27.17 (Rooted cross edges)**.** Let G and P be as above, and let A be a part of P. A normal or special cross edge is *rooted* at A if its common neighbor lies in A.

Formalization note 27.18. The first pin is the set of normal cross edges rooted at a part, and the second that of special ones. Formally, as for the classes of Lemma 27.13, a cross edge is rooted at the part of any common neighbor of its endpoints. As P is tight, there is only one.

Lemma 27.19 (The root of a normal cross edge)**.** *Let G and P be as above. Then every normal cross edge is rooted at exactly one part, and this part has at least three vertices.*

Proof. The common neighbor of a normal cross edge is unique (Lemma 27.11), so the edge is rooted at exactly one part. This part has at least three vertices by Lemma 27.13. □

Let A be a part with $|A| \geq 3$. We show in the next three lemmas that each edge vw of G with $v \in A$ and $w \notin A$ gives exactly two normal cross edges rooted at A: the edges uw, for the two neighbors u of v in A.

Lemma 27.20 (Two neighbors in the part)**.** *Let G and P be as above, let A be a part of P with $|A| \geq 3$, and let $v \in A$ have a neighbor outside A. Then v has exactly two neighbors in A.*

Proof. By Lemma 27.9, v has at least two neighbors in A. It has at most three neighbors (Lemma 27.5), and one of them lies outside A. So exactly two lie in A. □

Lemma 27.21 (A normal cross edge from two edges at a vertex)**.** *Let G and P be as above, let v be a vertex of G, and let u and w be neighbors of v with u in the part of v and w outside it. Then uw is a normal cross edge rooted at the part of v, and v is the only common neighbor of u and w.*

Proof. The vertices u and w lie in distinct parts, and both are neighbors of v. So uw is an edge of G^2 (Lemma 25.2) that joins two parts. It is not an edge of G. Otherwise vw and uw would be two edges between the part of v and that of w (Lemma 27.10). So uw is a cross edge, and v is its common neighbor (Lemma 27.11). As v lies in the part of u, the cross edge is normal and rooted at the part of v. □

Lemma 27.22 (Two normal cross edges for each edge leaving a part)**.** *Let G and P be as above, let A be a part of P with $|A| \geq 3$, and let vw be an edge of G with $v \in A$ and $w \notin A$. Then the edges uw with $u \in N_G(v) \cap A$, which are normal cross edges rooted at A (Lemma 27.21), number exactly two.*

Proof. The map $u \mapsto uw$ is injective on $N_G(v) \cap A$, as no vertex of A is w. So there are as many edges uw as neighbors of v in A, that is, two (Lemma 27.20). Each is a normal cross edge rooted at A (Lemma 27.21). □

It remains to show that every normal cross edge rooted at A arises in this way, and from only one edge of G. We count these edges of G by the degree of A, in the sense of JJ §3.

Definition 27.23 (The degree of a part)**.** Let P be a partition of V, and let A be a part of P. We write $d_G(A)$ for the number of edges xy of G with $x \in A$ and $y \notin A$, that is, with exactly one endpoint in A. The ordered pairs (x, y) with $x \in A$, $y \notin A$ and $xy \in E(G)$ are the *pairs leaving* A.

For a singleton part $\{v\}$, every edge at v has exactly one endpoint in $\{v\}$, so $d_G(\{v\})$ is the degree $d_G(v)$ of v.

Lemma 27.24 (The normal cross edges rooted at a part, by the edges leaving it)**.** *Let G and P be as above, and let A be a part of P. Then $(v, w) \mapsto vw$ is a bijection from the pairs (v, w) leaving A (Definition 27.23) onto the $d_G(A)$ edges of G with exactly one endpoint in A. The normal cross edges rooted at A are the union, over these pairs (v, w), of the sets $\{uw : u \in N_G(v) \cap A\}$, and the sets of two distinct pairs are disjoint.*

Proof. An edge of G with exactly one endpoint in A is vw for exactly one of these pairs, the one with $v \in A$. By Lemma 27.21, every edge uw with $u \in N_G(v) \cap A$ is a normal cross edge rooted at A. Conversely, let e be a normal cross edge rooted at A, with common neighbor $v \in A$. Exactly one endpoint u of e lies in A (Lemma 27.13). Let w be the other. Then (v, w) is one of the pairs, and $e = uw$ with $u \in N_G(v) \cap A$.

Finally, let uw, with $u \in A$, lie in the sets of two pairs (v, w) and (v', w'). Then $w' = w$, as w is the endpoint outside A. So v and v' are common neighbors of u and w, and $v = v'$ (Lemma 27.11). □

Lemma 27.25 (Normal cross edges rooted at a part; JJ eq. (6))**.** *Let G and P be as above, and let A be a part of P with $|A| \geq 3$. Then the normal cross edges rooted at A (Definition 27.17) number exactly $2\, d_G(A)$.*

Proof. By Lemma 27.24, the normal cross edges rooted at A are the disjoint union of $d_G(A)$ sets, one for each edge of G with exactly one endpoint in A. Each of these sets has exactly two elements (Lemma 27.22). □

Theorem 27.26 (The counting bound for Laman squares; JJ Theorem 5.3)**.** *Let G be a simple graph of minimum degree at least two on a finite nonempty vertex set V, and suppose G^2 is Laman. Then*

$$|E(G^2)| \;\le\; 3|V| - 6 - \operatorname{def}(\tilde{G}).$$

Proof. Let P be a tight partition of V (Lemma 20.6), and let t be its number of parts. Let C_n and C_s be the numbers of normal and of special cross edges. By Lemma 27.13,

$$|E(G^2)| \;=\; \sum_{A\in P} \big|E_{G^2}[A]\big| + d_G(P) + C_n + C_s.$$

Each part has one vertex or at least three (Lemma 27.9). A singleton part spans no edge, and a part A with $|A| \ge 3$ spans at most $3|A|-6$ edges of G^2, as G^2 is Laman.

We count each cross edge at its root (Definition 27.17), the part that contains its common neighbor, which is unique (Lemma 27.11). No normal cross edge is rooted at a part A unless $|A| \ge 3$ (Lemma 27.19). So C_n is the sum of $2\,d_G(A)$ over the parts A with $|A| \ge 3$ (Lemma 27.25), which is JJ eq. (6).

The special cross edges are rooted at singleton parts (Lemma 27.13). Those rooted at $\{v\}$ have common neighbor v, so they lie inside $N_G(v)$. Conversely, the edges of G^2 inside $N_G(v)$ are special cross edges with common neighbor v, and there are $2\,d_G(v) - 3$ of them (Lemma 27.15). So C_s is the sum of $2\,d_G(v) - 3$ over the singleton parts $\{v\}$, which is JJ eqs. (5) and (7).

For a singleton part $A = \{v\}$ we have $d_G(A) = d_G(v)$, so $2\,d_G(v) - 3 = (3|A| - 6) + 2\,d_G(A)$. Hence, for each part A, the edges of G^2 inside A and the cross edges counted at A number at most $3|A| - 6 + 2\,d_G(A)$, and

$$|E(G^2)| \;\le\; \sum_{A\in P} \big(3|A| - 6 + 2\,d_G(A)\big) + d_G(P) \;=\; 3|V| - 6t + 5\,d_G(P).$$

Here $\sum_A |A| = |V|$, and $\sum_A d_G(A) = 2\,d_G(P)$, as an edge that joins two parts has one endpoint in each. The right-hand side is $3|V| - 6 - \operatorname{def}_{\tilde{G}}(P)$ (Definition 20.3, at $D = 6$), and $\operatorname{def}_{\tilde{G}}(P) = \operatorname{def}(\tilde{G})$, as P is tight. □

Formalization note 27.27. Formally, the classes of Lemma 27.13 are defined for every partition, and a cross edge is normal or special according to some common neighbor of its endpoints. These two classes are disjoint because the partition is tight, which makes the common neighbor unique. The formal statements of this subsection assume of G and P only what their proofs use. So only those whose proofs use the degree bound assume that G^2 is Laman, and among the lemmas only the count of Lemma 27.15 assumes the minimum degree, at v. The theorem does not assume V nonempty: for $V = \emptyset$ both sides of its bound are 0, as the empty partition has value -6. The last claim of Lemma 27.19 is one of the formal statements of Lemma 27.13, and the bijection of Lemma 27.24 is stated formally as the equality of the two counts.

27.4. **Independence implies the Laman condition.** This subsection proves the easy direction of Jacobs' conjecture: if $E(H)$ is independent in $\mathcal{R}_3(V)$, then H is Laman (Corollary 27.29). It holds for every graph H on V, square or not. JJ (§5) derive it from their eq. (1), the bound $\operatorname{rank} R(H,p) \le 3|V| - 6$ for frameworks on

at least three vertices, applied to the subgraph induced on each set of at least three vertices. Our proof is that of the plane (Lemma 9.12), one dimension up. The rank bound we use needs the points to affinely span the space (Lemma 9.10). So we first prove the count at a placement at which every set of at least four points affinely spans $\mathbb{R}^3$, and then choose such a placement that is also generic.

Lemma 27.28 (The Laman condition from row independence, dimension three)**.** *Let p be a placement of V in $\mathbb{R}^3$ such that, for every set S of at least four vertices, the points $p(v)$, $v \in S$, affinely span $\mathbb{R}^3$. Let H be a graph on V whose edge set is row-independent at p. Then H is Laman.*

Proof. Let $X \subseteq V$ with $|X| \geq 3$. If $|X| = 3$, then X spans at most $3 = 3|X| - 6$ edges. Let $|X| \geq 4$, let $H[X]$ be the subgraph of H induced on X, and let $p|_X$ be the restriction of p to X. The row of an edge of $H[X]$ at $p|_X$, composed with the restriction $(\mathbb{R}^3)^V \to (\mathbb{R}^3)^X$, is the row of H at p indexed by the same edge. The rows of H at p are linearly independent, so the rows of $H[X]$ at $p|_X$ are too, and their number is the rank of $R(H[X], p|_X)$, as in the proof of Lemma 9.12:

$$\big|E_H[X]\big| \;=\; \operatorname{rank} R(H[X], p|_X).$$

The points $p(v)$, $v \in X$, affinely span $\mathbb{R}^3$, so this rank is at most $3|X|-6$ (Lemma 9.10, at $d = 3$). □

Corollary 27.29 (Independent edge sets are Laman)**.** *Let H be a graph on V with $E(H)$ independent in $\mathcal{R}_3(V)$. Then H is Laman.*

Proof. Choose a placement p of V in $\mathbb{R}^3$ that is generic for row independence and in general position up to order four (Lemma 25.17). As $E(H)$ is independent, it is row-independent at some placement (Lemma 24.4), and hence at p, since p is generic. A set of at least four vertices contains four whose points are affinely independent (Definition 25.16), and four affinely independent points affinely span $\mathbb{R}^3$. So Lemma 27.28 applies at p. □

27.5. **Zero-extensions in dimension three.** The reduction of Jacobs' conjecture to graphs of minimum degree at least two (Section 27.6) and the degree-one rank formula (Section 27.7) both use the 0-extension lemma. JJ state it as their Lemma 3.3, citing Whiteley's Vertex 3-Addition Lemma 9.1.3 [Whi96]: if a graph is obtained from H by adding a new vertex joined to $s \in \{1, 2, 3\}$ distinct vertices of H, then it is independent if and only if H is. They call this operation a 0-*extension*. Whiteley's lemma is stated at a fixed placement, for a new vertex joined to three vertices: if the point of the new vertex and the points of its three neighbors are not coplanar, then the new framework is independent if and only if the old one is, and its rank is larger by three.

Since $\mathcal{R}_3(V)$ is defined on a fixed vertex set, we keep V fixed. For a vertex v of a graph H on V, let $E_H(v)$ be the set of edges of H at v, and let $H - E_H(v)$ be the graph obtained from H by deleting them. Then v is isolated in $H - E_H(v)$, and if $1 \leq d_H(v) \leq 3$, then H is obtained from $H - E_H(v)$ by a 0-extension at v. We prove the lemma at a fixed placement (Lemma 27.30) and derive two corollaries for the generic rank. If $d_H(v) \leq 3$, then the rank of H exceeds that of $H - E_H(v)$ by $d_H(v)$,

and H is independent if and only if $H - E_H(v)$ is (Corollary 27.31). At a vertex of any degree, the rank grows by $\min(3, d_H(v))$ if the neighborhood $N_H(v)$ is a clique of H (Corollary 27.32).

The clique hypothesis cannot be dropped. The star $K_{1,4}$ with center v has rank four, not $0 + \min(3, 4) = 3$: at a generic placement, each of its four rows is the only row with nonzero entries in the columns of its leaf. JJ invoke the same hypothesis in the proof of their Theorem 4.3, where they apply their Lemma 3.3 at vertices whose degree may exceed three: "the neighbour sets … induce complete (and hence rigid) subgraphs". Both applications below, the degree-one rank formulas of Section 27.7, satisfy it. There the vertex removed has degree one in G, and its neighbors in G^2 lie in a closed neighborhood of G, which is a clique of G^2 (Lemma 27.35).

The subsection ends with the restriction of the generic rigidity matroid to a subset of the vertices, and with four facts about squares and induced subgraphs. The reduction of Jacobs' conjecture uses them all.

Lemma 27.30 (Row-independence lift for a vertex of degree at most three)**.** *Let H be a graph on V, let v be a vertex of H of degree $s \leq 3$ with neighbors $u_1, \dots, u_s$, and let $H' = H - E_H(v)$. Suppose that $E(H')$ is row-independent at a placement p' of V in $\mathbb{R}^3$ at which the points $p'(u_1), \dots, p'(u_s)$ are affinely independent. Then there is a placement p that agrees with p' away from v and at which $E(H)$ is row-independent.*

Proof. The points $p'(u_1), \dots, p'(u_s)$ are affinely independent and $s \leq 3$, so their affine hull is a proper affine subspace of $\mathbb{R}^3$. Choose a point q outside it, and let p place v at q and agree with p' elsewhere. The vectors $q - p'(u_i)$ are linearly independent. Indeed, let $\sum_i c_i(q - p'(u_i)) = 0$. If $\sum_i c_i \neq 0$, then q is an affine combination of the points $p'(u_i)$, which it is not. So $\sum_i c_i = 0$, the relation reads $\sum_i c_i\, p'(u_i) = 0$, and every c_i is zero by affine independence.

The edges of H are the edges of H' and the edges vu_i. Neither end of an edge of H' is v, so its row at p is its row at p' (Lemma 6.4). Suppose that a linear combination of the rows of $E(H)$ at p vanishes, with coefficient c_i at the row of vu_i. For $\alpha \in \mathbb{R}^3$, let x^α give v the velocity α and every other vertex the velocity zero, as in the proof of Lemma 10.2. The rows of the edges of H' vanish at x^α, and the row of vu_i takes the value $\langle q - p'(u_i), \alpha \rangle$. Evaluating the relation at every x^α gives $\sum_i c_i(q - p'(u_i)) = 0$, so every c_i is zero. What remains is a vanishing combination of the rows of $E(H')$ at p', and its coefficients are zero, as $E(H')$ is row-independent at p'. □

The hypothesis that $p'(u_1), \dots, p'(u_s)$ are affinely independent cannot be dropped either. On the six vertices v, u_1, u_2, a, b, c, let H consist of the complete graph on u_1, u_2, a, b, c minus the edge u_1u_2, together with the edges vu_1 and vu_2, and let $H' = H - E_H(v)$. Let p' place a, b and c at three non-collinear points, and both u_1 and u_2 at one point off their plane. Then $E(H')$ is row-independent at p', but $E(H)$ is row-dependent at every placement that agrees with p' away from v.

Indeed, the rows of the three edges of H' at u_1 are the only ones with nonzero entries in the columns of u_1, and there their entries $p'(u_1) - p'(x)$, $x \in \{a, b, c\}$, are linearly independent. The same holds at u_2, and the rows of the triangle abc are independent. So the nine rows of $E(H')$ are independent, and the infinitesimal

motions of H' at p', restricted to u_1, u_2, a, b, c, form a space of dimension $15-9=6$. They are the trivial motions, as these five points affinely span $\mathbb{R}^3$. At a placement that agrees with p' away from v, the difference of the rows of vu_1 and vu_2 involves only the velocities of u_1 and u_2. It vanishes on the trivial motions, which give u_1 and u_2 the same velocity. So it vanishes on every infinitesimal motion of H' at p', and it lies in the span of the rows of $E(H')$. The corollaries below meet the hypothesis by taking p' generic for row independence and in general position up to order four (Lemma 25.17).

Corollary 27.31 (0-extension at a vertex of degree at most three; JJ Lemma 3.3)**.** *Let H be a graph on V and let v be a vertex of H with $d_H(v) \leq 3$. Then*

$$r(H) \;=\; r(H - E_H(v)) + d_H(v),$$

and $E(H)$ is independent in $\mathcal{R}_3(V)$ if and only if $E(H - E_H(v))$ is.

Proof. Let $H' = H - E_H(v)$ and $s = d_H(v)$. The edges of H are the edges of H' and the s edges at v. Adding s elements to a set raises its rank by at most s, so $r(H) \leq r(H') + s$.

For the reverse inequality, choose a placement p' of V in $\mathbb{R}^3$ that is generic for row independence and in general position up to order four (Lemma 25.17). Let J be a maximal independent subset of $E(H')$, so that $|J| = r(H')$, and let H_J be the graph whose edges are those of J and the s edges at v. Then $H_J - E_{H_J}(v)$ has edge set J. As p' is generic, J is row-independent at p' (Lemma 24.4). The neighbors of v in H_J are its $s \leq 3$ neighbors in H, and their points are affinely independent, as p' is in general position (Definition 25.16). So $E(H_J)$ is row-independent at some placement (Lemma 27.30), that is, independent in $\mathcal{R}_3(V)$. It is a subset of $E(H)$ with $r(H') + s$ elements, so $r(H) \geq r(H') + s$.

Finally, $|E(H)| = |E(H')| + s$, and a finite edge set is independent exactly when its rank equals its size. So $E(H)$ is independent if and only if $E(H')$ is. □

Corollary 27.32 (Rank of a 0-extension at a clique neighborhood)**.** *Let H be a graph on V and let v be a vertex of H whose neighborhood $N_H(v)$ is a clique of H. Then*

$$r(H) \;=\; r(H - E_H(v)) + \min(3,\, d_H(v)).$$

Proof. Let $H' = H - E_H(v)$ and $s = d_H(v)$. If $s \leq 3$, the claim is Corollary 27.31. Let $s \geq 4$, fix three neighbors u_1, u_2, u_3 of v, and let H_3 be H' together with the edges vu_1, vu_2 and vu_3. Then $H_3 - E_{H_3}(v) = H'$ and $d_{H_3}(v) = 3$, so $r(H_3) = r(H') + 3$ (Corollary 27.31). As $E(H_3) \subseteq E(H)$, this gives $r(H) \geq r(H') + 3$. This lower bound holds without the clique hypothesis.

For the upper bound we show that every further edge vw, with $w \in N_H(v) \setminus \{u_1, u_2, u_3\}$, lies in the closure of $E(H_3)$ in $\mathcal{R}_3(V)$. Then $E(H)$ lies in this closure, and $r(H) \leq r(H_3) = r(H') + 3$. As $N_H(v)$ is a clique, the five vertices v, u_1, u_2, u_3, w span a complete graph K_5 in H. Its edges other than vw lie in $E(H_3)$: the edges vu_i by construction, and the six edges among u_1, u_2, u_3, w because they are edges of H not at v.

These nine edges are independent. Indeed, starting from the graph with no edges, join u_2 to u_1, then u_3 to u_1 and u_2, then w to u_1, u_2 and u_3, and last v to u_1, u_2 and

u_3. Each step is a 0-extension at a vertex of degree at most three, which preserves independence (Corollary 27.31). All ten edges of the K_5 are dependent. Otherwise the graph they form would be Laman (Corollary 27.29). But its five vertices form a clique, which spans $\binom{5}{2} = 10 > 3 \cdot 5 - 6$ edges (Lemma 27.4). Hence vw lies in the closure of the other nine edges, and so in the closure of $E(H_3)$. □

The reduction of Jacobs' conjecture restricts a graph to its non-isolated vertices, and so changes the vertex set of the matroid. The next lemma compares the generic rigidity matroids of a set and of a subset. Its proof is the same in every dimension.

Lemma 27.33 (Independence and rank under vertex-set restriction)**.** *Let $d \in \mathbb{N}$, let $S \subseteq V$, and let H be a graph on S. Then $E(H)$ is independent in $\mathcal{R}_d(S)$ if and only if its image under the inclusion $S \hookrightarrow V$ is independent in $\mathcal{R}_d(V)$. The rank of $E(H)$ in $\mathcal{R}_d(S)$ equals the rank of its image in $\mathcal{R}_d(V)$.*

Proof. Let q be a placement of S in $\mathbb{R}^d$, and let p be a placement of V that agrees with q on S. The row at p of an edge of H is its row at q composed with the restriction $\rho\colon (\mathbb{R}^d)^V \to (\mathbb{R}^d)^S$, as in the proof of Lemma 9.12. Since ρ is surjective, the rows of the edges of H at p are linearly independent if and only if their rows at q are. Every placement of S extends to one of V, and every placement of V restricts to one of S. By Lemma 24.4, this proves the first claim.

The inclusion maps the subsets of $E(H)$ bijectively onto the subsets of its image, preserving their size. By the first claim, applied to the subgraphs of H, it also preserves independence. The rank of a set is the largest size of an independent subset, so the two ranks are equal. □

The reduction also uses four facts about squares and induced subgraphs.

Lemma 27.34 (Deleting the edges at a vertex of degree at most one commutes with squaring)**.** *Let G be a graph on V, and let v be a vertex of degree at most one in G. Then*

$$\big(G - E_G(v)\big)^2 \;=\; G^2 - E_{G^2}(v).$$

Proof. Neither side has an edge at v. Let $x \neq y$ be vertices other than v. If v were a common neighbor of x and y, it would have two neighbors. So x and y are adjacent or have a common neighbor in G if and only if they do in $G - E_G(v)$. That is, xy is an edge of G^2 if and only if it is an edge of $(G - E_G(v))^2$ (Definition 25.1). □

Lemma 27.35 (The square-neighborhood of a degree-one vertex)**.** *Let G be a graph on V, and let v be a vertex of degree one in G, with neighbor u. Then*

$$N_{G^2}(v) \;=\; N_G[u] \setminus \{v\},$$

where $N_G[u] = N_G(u) \cup \{u\}$ is the closed neighborhood of u. Moreover, $d_{G^2}(v) = d_G(u)$, and $N_{G^2}(v)$ is a clique of G^2.

Proof. Let $w \neq v$. If w is adjacent to v, then $w = u$. If w and v have a common neighbor, then it is u, so $w \in N_G(u)$. Conversely, u is adjacent to v, and every vertex of $N_G(u)$ other than v has the common neighbor u with v. This proves the identity. As $v \in N_G(u)$, the set $N_G[u] \setminus \{v\}$ has $d_G(u)$ elements. It lies in $N_G[u]$, which is a clique of G^2 (Lemma 25.2). □

Lemma 27.36 (Squaring commutes with restriction to the support)**.** *Let G be a graph on V, and let $S \subseteq V$ contain every non-isolated vertex of G. Then the square of the induced subgraph $G[S]$ is the subgraph of G^2 induced on S, and every edge of G^2 has both ends in S.*

Proof. Each end of an edge of G^2 lies on an edge of G, so it lies in S. Let $x \neq y$ lie in S. An edge xy of G is an edge of $G[S]$. A common neighbor of x and y in G lies on an edge of G, so it lies in S, and it is a common neighbor of x and y in $G[S]$. So xy is an edge of $G[S]^2$ if and only if it is an edge of G^2. □

Lemma 27.37 (The Laman condition restricts to induced subgraphs)**.** *Let G be a Laman graph on V, and let $S \subseteq V$. Then the induced subgraph $G[S]$ is Laman.*

Proof. A set $X \subseteq S$ of at least three vertices spans the same edges in $G[S]$ as in G, and so at most $3|X| - 6$ of them. □

27.6. **Jacobs' conjecture.** In the proof of their Theorem 5.4, JJ treat graphs of minimum degree at least two, and reduce the general case to them in a single sentence: if G has maximum degree at most three, then G^2 is independent if and only if the square of its maximal subgraph of minimum degree at least two is. We prove this reduction by induction on the number of edges, for the graphs whose square is Laman (Theorem 27.39). These graphs have maximum degree at most three (Lemma 27.5).

Theorem 27.38 (Jacobs' conjecture, minimum degree two)**.** *Let G be a simple graph of minimum degree at least two on a finite nonempty vertex set V. If G^2 is Laman, then $E(G^2)$ is independent in $\mathcal{R}_3(V)$.*

Proof. The rank of a finite set is at most its size, with equality if and only if the set is independent. By Theorem 27.26 and the rank formula (Corollary 26.8),

$$|E(G^2)| \;\leq\; 3|V| - 6 - \mathrm{def}(\tilde{G}) \;=\; r(G^2) \;\leq\; |E(G^2)|.$$

So $r(G^2) = |E(G^2)|$, and $E(G^2)$ is independent. □

Theorem 27.39 (Jacobs' conjecture; JJ Conjecture 5.1, Theorem 5.4)**.** *Let G be a simple graph on a finite vertex set V. Then $E(G^2)$ is independent in $\mathcal{R}_3(V)$ if and only if G^2 is Laman.*

Proof. Independence implies the Laman condition by Corollary 27.29. For the converse, suppose that G^2 is Laman. We show that $E(G^2)$ is independent, by induction on the number of edges of G.

Suppose first that G has a vertex v of degree one, with neighbor u, and let $G' = G - E_G(v)$. Then $G'^2 = G^2 - E_{G^2}(v)$ (Lemma 27.34). So G'^2 is Laman (Lemma 27.3), and $E(G'^2)$ is independent by induction, as G' has one edge fewer than G. In G^2 the vertex v has degree $d_G(u)$ (Lemma 27.35), and $d_G(u) \leq 3$ (Lemma 27.5). So $E(G^2)$ is independent because $E(G'^2)$ is (Corollary 27.31).

Otherwise every vertex of G has degree zero or at least two. If G has no edges, then G^2 has none either, and the empty set is independent. Else let S be the set of non-isolated vertices of G. Every edge of G^2 lies inside S, and the square of the induced subgraph $G[S]$ is the subgraph of G^2 induced on S (Lemma 27.36). So

$G[S]^2$ is Laman (Lemma 27.37). A vertex of S has the same neighbors in $G[S]$ as in G, so $G[S]$ has minimum degree at least two. Hence $E(G[S]^2)$ is independent in $\mathcal{R}_3(S)$ (Theorem 27.38, on the nonempty set S). Its image under the inclusion $S \hookrightarrow V$ is $E(G^2)$, which is therefore independent in $\mathcal{R}_3(V)$ (Lemma 27.33). □

27.7. **The degree-one rank formula.** The rank formula (Corollary 26.8) assumes minimum degree at least two, and it fails without this assumption. For a single edge, $r(G^2) = 1$, while $3|V| - 6 - \mathrm{def}(\tilde{G}) = -1$, as the partition of the two vertices into singletons has value one. JJ's Lemma 4.2 gives $r(G^2)$ for every connected graph G on at least two vertices. Its part (a) gives it explicitly for trees. Its part (b) gives it, for the other graphs, through the rank of the square of the core of G, its maximal subgraph of minimum degree at least two. JJ prove both parts by deleting a vertex of degree one, and cite their Lemma 3.3 for the change in rank. Here that step is Corollary 27.32, as the deleted vertex may have more than three neighbors in G^2.

We write $V_1(G)$ for the set of vertices of degree one of a graph G, and call the set of its non-isolated vertices its *support*. A graph on a subset S of V is also a graph on V, in which the vertices outside S are isolated, and its rank is the same in both readings (Lemma 27.33). So for a vertex v of G we write $G - v$ for the graph obtained by deleting v, and read its rank either way. In the definition of the core, the minimum degree of a subgraph is taken over its support.

Definition 27.40 (The core)**.** Let G be a graph on V. The *core* G^{core} of G is the graph on V whose edges are those of all subgraphs H of G in which every vertex of the support of H has degree at least two in H.

Lemma 27.41 (The core is the largest subgraph of minimum degree two)**.** *Let G be a graph on V. Then G^{core} is a subgraph of G, it contains every subgraph H of G in which every vertex of the support of H has degree at least two in H, and every vertex of its support has degree at least two in G^{core}. So G^{core} is the largest subgraph of G of minimum degree at least two.*

Proof. The first two claims hold by definition. A vertex of the support of the core lies on an edge of some subgraph H as in the definition. So it has at least two neighbors in H, and hence in the core. □

Lemma 27.42 (Graphs of minimum degree two, and vertices of degree at most one)**.** *Let G be a graph on V. If every vertex of the support of G has degree at least two, then G is its own core. If v is a vertex of degree at most one in G, then v is not in the support of G^{core}, and*

$$\big(G - E_G(v)\big)^{\mathrm{core}} \;=\; G^{\mathrm{core}}.$$

Proof. If every vertex of the support of G has degree at least two, then G is one of the subgraphs H that define the core. So G is a subgraph of its core, which is a subgraph of G (Lemma 27.41).

Let v have degree at most one. A vertex of the support of the core has degree at least two in the core, and so in G, as the core is a subgraph of G (Lemma 27.41). So v is not in the support of the core. A subgraph of G in which every vertex of the support has degree at least two has no edge at v, as v has degree at most one. So G and $G - E_G(v)$ have the same such subgraphs, and the same core. □

As JJ note, the core of a connected graph on at least two vertices has no edges if and only if the graph is a tree. Indeed, a subgraph of a tree that has an edge is a forest with an edge, so it has a vertex of degree one. A connected graph that is not a tree contains a cycle, in which every vertex has degree two. For a tree, the core has empty support, and the right-hand side of the formula of Theorem 27.44 would be $2|V| + |V_1(G)|$. This is five more than $r(G^2)$ (Theorem 27.43), so trees are treated separately.

Theorem 27.43 (The degree-one rank formula for trees; JJ Lemma 4.2(a))**.** *Let G be a tree on a finite vertex set V with $|V| \geq 2$. Then*

$$r(G^2) \;=\; 2|V| - 5 + |V_1(G)|.$$

JJ credit the formula to Franzblau [Fra00]*.*

Proof. We induct on $|V|$. If $|V| = 2$, then G is a single edge, and $G^2 = G$. Deleting the edge at one of its ends leaves no edge, so $r(G^2) = 1$ (Corollary 27.31). And $2 \cdot 2 - 5 + 2 = 1$.

Let $|V| \geq 3$, and let v be a vertex of degree one, with neighbor u. Then $d_G(u) \geq 2$, as otherwise $\{u, v\}$ would be a connected component of G. The graph $G - v$ is a tree on $|V| - 1$ vertices, and its square is $G^2 - v$ (Lemma 27.34). In G^2 the vertex v has $d_G(u)$ neighbors, and they form a clique (Lemma 27.35). So

$$r(G^2) \;=\; r((G-v)^2) + \min(3, d_G(u))$$

(Corollary 27.32).

The term $2|V|$ is two less for $G - v$ than for G. If $d_G(u) \geq 3$, then $V_1(G - v) = V_1(G) \setminus \{v\}$, so the right-hand side of the formula is three less for $G - v$. If $d_G(u) = 2$, then $V_1(G - v)$ is $V_1(G)$ with u in place of v, so the right-hand side is two less. In both cases it falls by $\min(3, d_G(u))$, as the rank does. So the formula for $G - v$, which holds by induction, gives the formula for G. □

Theorem 27.44 (The degree-one rank formula; JJ Lemma 4.2(b))**.** *Let G be a connected simple graph on a finite vertex set V with $|V| \geq 2$ that is not a tree. Then*

$$r(G^2) \;=\; r\big((G^{\text{core}})^2\big) + 2\left|V \setminus V(G^{\text{core}})\right| + |V_1(G)|,$$

where $V(G^{\text{core}})$ is the support of the core.

Proof. We induct on $|V|$. If $V_1(G)$ is empty, then every vertex of G has degree at least two, as G is connected and $|V| \geq 2$. So G is its own core (Lemma 27.42), its support is V, and both sides equal $r(G^2)$.

Otherwise let v be a vertex of degree one, with neighbor u. Then $d_G(u) \geq 2$, as otherwise G would be the single edge uv, a tree. A connected graph is a tree exactly when it has one edge fewer than vertices, so G has at least as many edges as vertices. Deleting v removes one vertex and one edge. So $G - v$ is connected, has at least two vertices, and is not a tree. As in the proof of Theorem 27.43, $r(G^2) = r((G-v)^2) + \min(3, d_G(u))$.

Deleting v does not change the core, and v is not in $V(G^{\text{core}})$ (Lemma 27.42). So $|V \setminus V(G^{\text{core}})|$ is one less for $G - v$ than for G. The set V_1 changes as in the proof of Theorem 27.43. So the right-hand side falls by $2 + 1 = 3$ if $d_G(u) \geq 3$, and by

2 if $d_G(u) = 2$. In both cases it falls by $\min(3, d_G(u))$, as the rank does, and the formula for $G - v$, which holds by induction, gives the formula for G. □

Formalization note 27.45. The two inductions are formalized on the number of edges, over a fixed vertex set. Deleting the edge at v leaves v in V as an isolated vertex, so the formal proofs do not change the vertex set of the matroid, and they do not use Lemma 27.33. They prove stronger statements, in which $|V|$ is the size of the support of G, connectivity is asked only between vertices of the support, and, for Theorem 27.44, "not a tree" is the bound $|E(G)| \geq |V(G)|$, with $V(G)$ the support of G. The core is a graph on V, and $V(G^{\text{core}})$ is its support. The formal statement of Theorem 27.44 does not assume $|V| \geq 2$, which follows from its other hypotheses, as a connected graph on one vertex is a tree. The tree formula is formalized as $r(G^2) + 5 = 2|V| + |V_1(G)|$, which avoids a subtraction of natural numbers.

28. The generic form of the realization theorems

Several theorems of the preceding chapters assert that a graph has some realization with a given property, or give the rank of a matroid. Tay's theorem (Theorem 16.27) and the theorem of Tay and Whiteley (Theorem 17.5) characterize the multigraphs that have an independent, or an independent and infinitesimally rigid, body-bar or body-hinge framework. In the formalization the body-hinge frameworks of the second theorem are bundles of bars, which need not come from hinges (the note after Definition 17.2). KT Theorem 5.6 (Theorem 23.50) gives a panel-hinge realization of a simple graph, with nonparallel panels, at the target rank of Definition 23.21. The molecule rank formula (Corollary 26.8) gives the rank of a square G^2 in the generic rigidity matroid $\mathcal{R}_3(V)$. This chapter proves generic forms of these results: they hold at every generic realization, and almost all realizations are generic. We follow the coordinate argument of Jackson and Jordán [JJ10], not Whiteley's argument on irreducible varieties [Whi88, §3.1], which the body-bar chapter recalls (Section 16).

In each class of frameworks a realization is given by parameters: the coordinates of the points of a placement, of the normals of the panels, of the endpoints of the bars, or of points that span the hinges. A fixed family of linear forms, with coefficients that are polynomials in the parameters, spans the rows of the rigidity matrix. For placements and bars the forms are the rows. For a hinge they span its rows when it is nondegenerate. A realization is generic for row independence if every subfamily that is linearly independent at some realization is linearly independent at it (Definition 24.1 is the bar-joint instance). So a rank that the rigidity matrix of a graph attains at one realization, with nondegenerate hinges where there are hinges, is attained or exceeded at every generic realization. A bound that holds at every realization then gives the rank there. The realizations that are not generic lie in the zero set of a nonzero polynomial in the parameters (Lemma 28.1 for placements). So over an infinite field generic realizations exist, and over $\mathbb{R}$ they form a dense open set, which is the precise content of "almost all realizations are generic". Jackson and Jordán instead note that realizations whose coordinates are algebraically independent over $\mathbb{Q}$ are generic [JJ10, §5].

Jackson and Jordán call a realization generic if its rigidity matrix and all its edge-induced submatrices have the largest rank over all realizations of the graph [JJ10,

§§5–7]. Where each edge has one row, as for placements (Section 24.1) and for bars, the two notions agree; for bars this needs the dimension d to be positive and the multigraph to have no loops. A hinge has $D-1$ rows, where $D = \binom{d+1}{2}$, and a degenerate hinge welds its two bodies together. For $d \geq 2$ and a graph without loops, a generic assignment of normals or of hinge points has nondegenerate hinges. It attains the largest ranks over the assignments whose hinges are nondegenerate, as Jackson and Jordán's are. For hinge points these assignments are their realizations, so a generic hinge-point assignment is generic in their sense (Lemma 28.34, where loops are allowed). The panel, body-bar and body-hinge sections each compare the two notions.

Section 28.1 shows that the placements that are not generic lie in the zero set of a nonzero polynomial. Section 28.2 proves the molecule rank formula at every generic placement. It deduces that every generic placement is infinitesimally rigid for G^2 when $5 \cdot G$ contains six edge-disjoint spanning trees, as Jackson and Jordán noted the molecular conjecture would imply. Section 28.3 proves the rank formula of KT Proposition 2.3, Jackson and Jordán's Theorem 6.1, at every generic assignment of normals to a simple graph. Its rigidity corollary contains their theorem on generic panel-and-hinge realizations (their Theorem 7.2). Section 28.4 proves Tay's theorem at every generic choice of the endpoints of the bars. Section 28.5 proves the rank formula at every generic choice of points spanning the hinges of a multigraph, with parallel edges and loops allowed, and with it the theorem of Tay and Whiteley for frameworks with hinges.

28.1. **Abundance of generic placements.** Generic placements exist in every dimension, by an argument that moves along one line at a time (Lemma 24.2). Taken as polynomials in all the coordinates of a placement, the minors of that argument show more: the placements that are not generic lie in the zero set of one nonzero polynomial.

Lemma 28.1 (Abundance of generic placements)**.** *Let V be finite and $d \in \mathbb{N}$. Then there is a nonzero real polynomial P in the $d\,|V|$ coordinates of a placement of V in $\mathbb{R}^d$ such that every placement p with $P(p) \neq 0$ is generic for row independence. In particular, the placements that are not generic lie in the zero set of a nonzero polynomial.*

Proof. Let $\mathcal{F}$ be the family of sets $I \subseteq E(K_V)$ that are row-independent at some placement. It is finite, since $E(K_V)$ is. Let $I \in \mathcal{F}$, and let q be a placement at which I is row-independent. The rigidity row of an edge uv at a placement p is $x \mapsto \langle p_u - p_v,\, x_u - x_v \rangle$ (Lemma 6.4), which depends linearly on p. Fix a basis of $((\mathbb{R}^d)^V)^*$. The coordinate vectors of the rows at p of the edges in I are the rows of a matrix $A(p)$ whose entries are linear forms in the coordinates of p. These rows are linearly independent at q (Lemma 9.6), so some $|I| \times |I|$ minor of $A(q)$ is nonzero. The same minor of $A(p)$ is a polynomial P_I in the coordinates of p. It is nonzero, as $P_I(q) \neq 0$, and I is row-independent at every placement p with $P_I(p) \neq 0$.

Let P be the product of the polynomials P_I for $I \in \mathcal{F}$. It is nonzero, as a finite product of nonzero polynomials. If $P(p) \neq 0$, then $P_I(p) \neq 0$ for every $I \in \mathcal{F}$. So every member of $\mathcal{F}$ is row-independent at p, and p is generic (Definition 24.1). □

28.2. **Every generic realization of a square.** The molecule rank formula (Corollary 26.8) gives the rank of the square G^2 in the generic rigidity matroid $\mathcal{R}_3(V)$. We show that it gives the rank of the rigidity map of G^2 at every placement that is generic for row independence, and we deduce when every such placement is infinitesimally rigid for G^2.

Theorem 28.2 (The molecule rank formula at every generic placement)**.** *Let G be a simple graph of minimum degree at least two on a finite, nonempty vertex set V, and let p be a placement of V in $\mathbb{R}^3$ that is generic for row independence. Then*

$$\operatorname{rank} R(G^2, p) \;=\; 3|V| - 6 - \operatorname{def}(\tilde{G}),$$

where $\operatorname{def}(\tilde{G})$ is the deficiency of Definition 20.3 at $D = 6$, with $\tilde{G} = 5 \cdot G$.

Proof. As p is generic, $\operatorname{rank} R(G^2, p) = r(G^2)$, the rank of G^2 in $\mathcal{R}_3(V)$ (Lemma 24.9). By the molecule rank formula (Corollary 26.8), $r(G^2) = 3|V| - 6 - \operatorname{def}(\tilde{G})$. □

Corollary 28.3 (Every generic realization of a rigid square)**.** *Let G be a simple graph of minimum degree at least two on a finite, nonempty vertex set V, and suppose that $\operatorname{def}(\tilde{G}) = 0$, where $\tilde{G} = 5 \cdot G$. Then every placement p of V in $\mathbb{R}^3$ that is generic for row independence is infinitesimally rigid for G^2.*

Proof. By Theorem 28.2, $\operatorname{rank} R(G^2, p) = 3|V| - 6$. The space of placements has dimension $3|V|$ (Lemma 6.2), so by rank–nullity $\dim\ker R(G^2, p) = 6$. This is the bound of Definition 6.5 at $d = 3$. □

Jackson and Jordán noted that the molecular conjecture (their Conjecture 1.1), together with Tay and Whiteley's theorem on body-and-hinge frameworks (their Corollary 6.3; Section 17.2), would imply that every generic bar-joint realization of G^2 in $\mathbb{R}^3$ is infinitesimally rigid whenever $5 \cdot G$ contains six edge-disjoint spanning trees [JJ10, p. 586]. Katoh and Tanigawa proved the molecular conjecture [KT11], and Theorem 23.56 is its form for simple graphs. We derive the consequence from Corollary 28.3, and so from the molecule rank formula.

Corollary 28.4 (Generic rigidity of squares from six spanning trees; Jackson–Jordán)**.** *Let G be a simple graph on a finite, nonempty vertex set V, and suppose that $5 \cdot G$ contains six edge-disjoint spanning trees. Then every placement of V in $\mathbb{R}^3$ that is generic for row independence is infinitesimally rigid for G^2.*

Proof. If $|V| = 1$, then G^2 has no edge, and every placement p is infinitesimally rigid for G^2, as $\dim\ker R(G^2, p) \le 3 \le 6$ (Lemma 6.2). Let $|V| \ge 2$, and let $\tilde{G} = 5 \cdot G$.

We first show that $\operatorname{def}(\tilde{G}) = 0$. The argument is that of the proof of KT Proposition 2.2 (Lemma 20.14). Each of the six spanning trees has $|V|-1$ edges and is a forest of $\tilde{G}$. They are edge-disjoint, so their union F has $6(|V|-1)$ edges, and F is independent in the union of six copies of the cycle matroid of $\tilde{G}$ (Lemma 15.8). This union is $M(\tilde{G})$ (Definition 20.1). As $\operatorname{def}(\tilde{G}) \ge 0$, KT eq. (2.4) gives $\operatorname{rank} M(\tilde{G}) \le 6(|V|-1)$ (Theorem 20.12). So F is a base of $M(\tilde{G})$, and KT eq. (2.4) gives $\operatorname{def}(\tilde{G}) = 0$.

Next we show that G has minimum degree at least two. Regard G as the multigraph of Lemma 26.3. As its deficiency is $\operatorname{def}(\tilde{G}) = 0$, it is a 0-dof-graph (Definition 20.8), with at least two vertices. So each of its vertices has degree at least two

(Lemma 20.23). It is simple, and its edges join exactly the adjacent pairs of G, so each vertex has the same degree in G. Now Corollary 28.3 applies. □

28.3. **Every generic panel-and-hinge realization.** As in Section 23.1, d is the dimension, $D = \binom{d+1}{2}$, and K is a field. A panel-and-hinge realization of a multigraph $G = (V, E)$ is a panel-hinge framework (Definition 23.5). It gives each body v a normal $q_v \in K^{d+1}$, and the hinge of an edge with ends u and v has the supporting extensor $C(q_u, q_v)$. So the realization parameters are the $(d+1)|V|$ coordinates of the normals. Jackson and Jordán describe the panel of v as the hyperplane $\{x \in \mathbb{R}^d : x \cdot p_v = 1\}$ for a vector $p_v \in \mathbb{R}^d$ [JJ10, §7]. Its normal is $(p_v, -1)$ (the remarks after Definition 23.1).

Our main result is Jackson and Jordán's rank formula, which Katoh and Tanigawa state for body-hinge frameworks as their Proposition 2.3 (Proposition 28.42), at every generic assignment of normals to a simple graph (Theorem 28.12). Here $\tilde{G} = (D-1)\cdot G$, and $\mathrm{def}(\tilde{G})$ is its deficiency (Definition 20.3). For a multigraph the formula can fail. At a generic assignment the two-vertex double edge has rank $D-1$, while the formula gives D (Formalization note 23.57).

Genericity is the transfer condition of Definition 24.1, applied to the forms of Lemma 23.91. These forms span the rows of the rigidity matrix at each edge. Jackson and Jordán derive their Theorem 7.2 from the realization of their Theorem 6.5, whose hinges all lie in one hyperplane. They perturb it so that each hinge is the intersection of two hyperplanes, one for each end of its edge (their Lemma 7.1). These hyperplanes are the panels of a panel-and-hinge realization. We start instead from KT Theorem 5.6 (Theorem 23.50). It gives a panel-hinge realization at the target rank $D(|V|-1) - \mathrm{def}(\tilde{G})$ (Definition 23.21) whose panels are nonparallel, so that its hinges are nondegenerate.

Definition 28.5 (Generic normal assignment)**.** Let $G = (V, E)$ be a multigraph. A *normal assignment* q of G gives each body v a vector $q_v \in K^{d+1}$, and (G, q) is the panel-hinge framework with these normals. The *annihilator-row family* of q is the family of linear forms

$$S \;\mapsto\; C_t\, x_{t'} - C_{t'}\, x_t, \qquad \text{where } x = S(u) - S(v) \text{ and } C = C(q_u, q_v),$$

on screw assignments S, indexed by an edge e of G, with ends u and v, and a pair (t, t') of indices of the standard basis of $\bigwedge^{d-1} K^{d+1}$. Here y_t is the coordinate of y at the t-th basis vector. The assignment q is *generic for row independence*, and (G, q) has *generic normals*, if every subfamily of the annihilator-row family that is linearly independent at some normal assignment is linearly independent at q.

These are the forms of Lemma 23.91 at the edges of G. A form does not depend on the order of u and v, since exchanging them changes the signs of both x and C (Definition 23.1). When the hinge of e is nondegenerate, the forms at e span the same space as the rows of $R(G, q)$ at e (Lemma 23.91).

Formalization note 28.6. The formalization indexes the family by all the edge labels, and reads the ends of each label from an endpoint selector (Formalization note 23.6), not from G. Genericity is a property of a panel-hinge framework: it asks about

the subfamilies indexed by edges of G, with the framework's own selector, which gives each edge of G its two ends. As a form does not depend on the order of the ends, two frameworks with the same graph and the same normals have generic normals together, whatever their selectors. The pinned lemma on abundance fixes one selector that gives each edge of G its two ends, and the one on existence assumes a body, for the endpoint choice of G (Formalization note 21.11). The first pin is the family, the second genericity, and the third that genericity reads only the graph and the normals.

Lemma 28.7 (Abundance of generic normal assignments)**.** *Let $G = (V, E)$ be a multigraph with V and E finite. Then there is a nonzero polynomial P in the $(d+1)|V|$ coordinates of a normal assignment such that every normal assignment q with $P(q) \neq 0$ is generic for row independence. In particular, the normal assignments that are not generic lie in the zero set of a nonzero polynomial.*

Proof. Fix the standard basis of the space of screw assignments. In its dual basis, each coordinate of a member of the annihilator-row family at an edge with ends u and v is 0 or $\pm$ a coordinate of $C(q_u, q_v) = *(q_u \vee q_v)$, where $q_u \vee q_v$ is the join (Definition 18.4). Its coordinates are the 2×2 minors $q_u(i)\, q_v(j) - q_u(j)\, q_v(i)$ (Lemma 18.6). The complement isomorphism $*$ sends each standard basis vector to $\pm$ a standard basis vector (Definition 22.5). So each coordinate of each member of the family is a polynomial of degree at most two in the coordinates of q, as in the proof of Lemma 23.91.

The argument of Lemma 28.1 now applies. The family has finitely many indices, as E is finite. Let s be a set of indices whose members are linearly independent at some normal assignment q^s. Some $|s| \times |s|$ minor of the coordinate matrix of these members is nonzero at q^s. As a polynomial P_s in the coordinates of q, it is nonzero, and the members indexed by s are linearly independent at every q with $P_s(q) \neq 0$. The product P of the polynomials P_s is nonzero, and every q with $P(q) \neq 0$ is generic. □

Lemma 28.8 (Existence of generic normal assignments)**.** *Let K be infinite, and let $G = (V, E)$ be a multigraph with V and E finite. Then some normal assignment of G is generic for row independence.*

Proof. The polynomial P of Lemma 28.7 is nonzero. Over an infinite field a nonzero polynomial has a non-root, and every non-root of P is generic. □

Lemma 28.9 (Generic normal assignments have nondegenerate hinges)**.** *Let $d \geq 2$, and let $G = (V, E)$ be a multigraph without loops. Let q be a normal assignment of G that is generic for row independence. Then every hinge of (G, q) is nondegenerate.*

Proof. Let e be an edge of G, with ends $u \neq v$. Let q^0 be the normal assignment that gives u the point $(1, 0, \dots, 0)$ of the moment curve and every other body the point $(1, 1, \dots, 1)$ (Lemma 23.12). These two vectors are linearly independent, so $C^0 = C(q^0_u, q^0_v)$ is not zero (the remark after Definition 23.1). Let t be an index with $C^0_t \neq 0$. As $d \geq 2$, the screw space has dimension $D \geq 3$, so there is an index $t' \neq t$. As $u \neq v$, there is a screw assignment S for which $S(u) - S(v)$ is the t'-th

basis vector. At S the member of the annihilator-row family of q^0 at (e, t, t') takes the value $C_t^0 \neq 0$.

So this member alone is a linearly independent family at q^0. As q is generic, the member of the family of q at (e, t, t') is not zero. If $C(q_u, q_v)$ were zero, every member of the family of q at e would be zero. So the hinge of e in (G, q) is nondegenerate. □

Jackson and Jordán call a panel-and-hinge realization generic if its rigidity matrix, and the submatrix of the rows of every set of edges, have the largest rank over all realizations of the graph. Their realizations have nondegenerate hinges [JJ10, §7]. Let $d \geq 2$, let G have no loops, and let q be generic, so that the hinges of (G, q) are nondegenerate. Then for every set F of edges, the rows of $R(G, q)$ at the edges of F have the largest rank over all normal assignments at which the hinges of these edges are nondegenerate. Indeed, at such an assignment q' the members of the annihilator-row family at F span the rows at F (Lemma 23.91). So as many of these members as the rank are linearly independent at q', and hence at q, where they lie in the span of the rows at F. The condition on the hinges is needed. A degenerate hinge welds its two bodies together (the remarks after Definition 23.5), so its edge imposes D independent linear conditions, where a nondegenerate hinge imposes $D - 1$.

Lemma 28.10 (Independent annihilator rows of a realization)**.** *Let $G = (V, E)$ be a multigraph with V and E finite, and let (G, q) be a panel-hinge framework whose hinges are all nondegenerate. Then for every $N \leq \operatorname{rank} R(G, q)$, some N members of the annihilator-row family of q are linearly independent.*

Proof. As the hinges are nondegenerate, the members of the family at each edge span the same space as the rows of $R(G, q)$ at that edge (Lemma 23.91). So the family spans the row space of $R(G, q)$, which has dimension $\operatorname{rank} R(G, q) \geq N$. Hence some N of its members are linearly independent (Lemma 23.110). □

Formalization note 28.11. The pinned lemma takes the family with any selector that gives each edge of G its two ends (Formalization note 28.6), and extracts the rows at that selector by the pinned theorem of Lemma 23.110, all at edges of G. Their hinges are the realization's, read with its own selector, which also gives each edge its two ends, in either order (Formalization note 23.6). Where the two orders differ, the hinge of the realization is $-C(q_u, q_v)$ (Definition 23.1), and the rows it extracts differ from the members of the family by a sign. Signs do not affect linear independence. The pinned lemma asks nondegenerate hinges at the edges of G only.

Theorem 28.12 (The rank formula at every generic normal assignment)**.** *Let $d \geq 2$, let K be an infinite field, and let $G = (V, E)$ be a simple graph with V finite and nonempty. Let q be a normal assignment of G that is generic for row independence. Then*

$$\operatorname{rank} R(G, q) \;=\; D(|V| - 1) - \operatorname{def}(\tilde{G}).$$

Proof. As G has no loops, the hinges of (G, q) are nondegenerate (Lemma 28.9). So $\operatorname{rank} R(G, q) \leq D(|V| - 1) - \operatorname{def}(\tilde{G})$ (Lemma 20.16).

For the reverse inequality, KT Theorem 5.6 (Theorem 23.50) gives a panel-hinge realization (G, q^0) at the target rank $D(|V| - 1) - \operatorname{def}(\tilde{G})$, whose hinges are nondegenerate, as its panels are nonparallel. By Lemma 28.10, some $D(|V| - 1) - \operatorname{def}(\tilde{G})$

members of the annihilator-row family of q^0 are linearly independent. As q is generic, the members of the family of q with the same indices are linearly independent. Each of them lies in the row space of $R(G,q)$. Indeed, a member at an edge with ends u and v is a linear form in $S(u) - S(v)$ that vanishes at $C(q_u, q_v)$, and so a linear combination of the rows of $R(G,q)$ at that edge (the remark after Definition 19.5). Hence $\operatorname{rank} R(G,q) \geq D(|V|-1) - \operatorname{def}(\tilde{G})$. □

Corollary 28.13 (Every generic panel-and-hinge realization of a rigid graph; Jackson–Jordán)**.** *Let $d \geq 2$, let K be an infinite field, and let $G = (V,E)$ be a simple graph with V finite and nonempty. Then (G,q) is infinitesimally rigid at every normal assignment q of G that is generic for row independence if and only if* $\operatorname{def}(\tilde{G}) = 0$.

Proof. Suppose first that $\operatorname{def}(\tilde{G}) = 0$, and let q be generic. By Theorem 28.12, $\operatorname{rank} R(G,q) = D(|V|-1)$, so $\dim Z(G,q) = D|V| - D(|V|-1) = D$. Hence every infinitesimal motion of (G,q) is trivial (Lemma 23.117, with $W = V$).

Conversely, suppose that (G,q) is infinitesimally rigid at every generic q. As K is infinite, some normal assignment q is generic (Lemma 28.8). Then (G,q) is infinitesimally rigid, so its motions are constant on V, and $\dim Z(G,q) = D$ (Lemma 23.124, with $W = V$). As the screw assignments form a space of dimension $D|V|$, $\operatorname{rank} R(G,q) = D(|V|-1)$, and Theorem 28.12 gives $\operatorname{def}(\tilde{G}) = 0$. □

Formalization note 28.14. The formalization writes $d = k+1$ here, as in the note after Definition 23.1; the pinned theorems of Theorem 28.12 and Corollary 28.13 take d and the frameworks with $k = d-1$. They range over the panel-hinge frameworks on G with generic normals, whatever their selectors (Formalization note 28.6). They assume that every body is a vertex of G, and nothing of the edge labels (Remark 23.40): they apply to every simple graph on the bodies, a single edge or a tree among them.

Jackson and Jordán's Theorem 7.2 states that if $(d-1)\cdot G$ contains d edge-disjoint spanning trees, then every generic panel-and-hinge realization of the graph G is infinitesimally rigid [JJ10, §7]. Their condition implies $\operatorname{def}(\tilde{G}) = 0$. Indeed, $\tilde{G} = (D-1)\cdot G$ has $D - d = \binom{d}{2}$ further copies of each edge, and these hold $\binom{d}{2}$ edge-disjoint copies of a spanning tree of G. So $\tilde{G}$ contains D edge-disjoint spanning trees, as in the proof of their Theorem 6.5, and $\operatorname{def}(\tilde{G}) = 0$ by KT Proposition 2.2 (Lemma 20.14). Hence Corollary 28.13 contains their theorem for simple graphs, under the weaker hypothesis $\operatorname{def}(\tilde{G}) = 0$, which it shows to be necessary.

28.4. **Every generic body-and-bar realization.** As in Section 16, d is the dimension, $D = \binom{d+1}{2}$, and the vertices and edges of a multigraph are its bodies and bars. Jackson and Jordán realize each bar e as a line segment of $\mathbb{R}^d$, with distinct endpoints p_e and p'_e [JJ10, §§2.1 and 3]. A bar e that joins the bodies u and v imposes one linear condition on their screws $S(u), S(v) \in \mathbb{R}^D$: the difference $S(u) - S(v)$ is orthogonal to the vector of 2×2 minors of the matrix with rows $(p_e, 1)$ and $(p'_e, 1)$. So their realization is a body-bar framework (Definition 16.20) with this vector on

each bar, while Tay's frameworks allow any vector of $\mathbb{R}^D$. In this section the realization parameters are the $2d$ coordinates of the endpoints of each bar. So "almost all realizations" refers to the choice of the endpoints, as in Jackson and Jordán's paper.

The coordinates of this vector are polynomials in the coordinates of the endpoints. So, as for placements (Section 28.1), generic choices of endpoints exist, and the choices that are not generic lie in the zero set of a nonzero polynomial. The generic form of Tay's theorem then needs a witness. For a finite multigraph G and a set E' of bars such that $G \restriction E'$ is (D, D)-sparse, we need endpoints at which the rows of the bars in E' are linearly independent. This is Jackson and Jordán's Lemma 5.1. We derive it from the framework of Proposition 16.25 by a change of coordinates of $\mathbb{R}^D$, following Whiteley. Genericity then gives Tay's theorem at every generic choice of endpoints (Corollary 28.26).

We put the coordinate 1 of the rows first, and call the vector of minors the two-extensor of the segment.

Definition 28.15 (The two-extensor of a segment)**.** For $p \in \mathbb{R}^d$ let $h(p) = (1, p) \in \mathbb{R}^{d+1}$, with coordinates indexed by $0, 1, \dots, d$. Index the D coordinates of $\mathbb{R}^D$ by the pairs (i, j) with $0 \le i < j \le d$, through a fixed bijection. The *two-extensor* of an ordered pair of points $p, p' \in \mathbb{R}^d$ is the vector $T(p, p') \in \mathbb{R}^D$ with

$$T(p, p')_{(i,j)} \;=\; h(p)_i\, h(p')_j - h(p)_j\, h(p')_i, \qquad 0 \le i < j \le d.$$

We call the coordinates $(0, j)$ of $T(p, p')$ its *direction coordinates* and the others its *moment coordinates*. As $h(p)_0 = 1$, they are

$$T(p, p')_{(0,j)} \;=\; p'_j - p_j \qquad \text{and} \qquad T(p, p')_{(i,j)} \;=\; p_i\, p'_j - p_j\, p'_i \quad (i \ge 1),$$

polynomials of degree one and two in the $2d$ coordinates of p and p'. The vector $T(p, p')$ is the coordinate vector of the join $h(p) \vee h(p')$ (Definition 18.4), whence its name. Its coordinates are taken in the basis of $\bigwedge^2 \mathbb{R}^{d+1}$ formed by the products of two standard basis vectors in increasing order, as in Lemma 18.6. Jackson and Jordán's vector of minors puts the coordinate 1 last, as Definition 18.1 does. It has the same coordinates as $T(p, p')$ up to their order, with the direction coordinates negated. Such a change of the coordinates of $\mathbb{R}^D$ does not change which sets of rows are linearly independent (Lemma 28.23 below).

Not every vector of $\mathbb{R}^D$ is the two-extensor of a segment. If the direction coordinates of $T(p, p')$ vanish, then $p = p'$, and the moment coordinates vanish too. So the standard basis vector of $\mathbb{R}^D$ at a moment coordinate (i, j) is the two-extensor of no pair of points. It is the coordinate vector of the product of the i-th and j-th standard basis vectors of $\mathbb{R}^{d+1}$, the line at infinity through the points at infinity of the i-th and j-th coordinate axes. Hence the framework of Proposition 16.25, which puts standard basis vectors on the bars, does not in general come from a choice of endpoints.

Formalization note 28.16. The formalization calls the dimension n, as in the note after Definition 16.20. For the bijection of the pairs (i, j) with the coordinates of $\mathbb{R}^D$, it takes the one that the equality of their numbers provides (`pairIdxEquiv`).

Definition 28.17 (Endpoint realization; generic endpoint assignment)**.** Let G be a multigraph, and fix an orientation of its bars (Definition 16.22). An *endpoint assignment* q of G gives each bar e an ordered pair (p_e, p'_e) of points of $\mathbb{R}^d$. Its *endpoint realization* is the body-bar framework (G, b) in $\mathbb{R}^d$ with $b_e = T(p_e, p'_e)$ for every bar e. The assignment q is *generic for row independence* if every set of bars whose rows are linearly independent at the endpoint realization of some endpoint assignment has linearly independent rows at the endpoint realization of q.

Reversing a bar changes only the sign of its row (Section 16.4), so genericity does not depend on the orientation.

Formalization note 28.18. The formalization gives a pair of endpoints to every edge label, not only to the bars of G: an endpoint assignment is a real function of a label, one of its two ends and a coordinate. The pairs at labels that are not bars of G do not enter the rows. The bodies and the labels are drawn from two fixed sets, which the pinned lemmas below assume finite. So the polynomial of Lemma 28.20 has $2d$ variables for each label, not only for each bar. The first pin is the endpoint realization, and the second is genericity.

Lemma 28.19 (The two-extensors are polynomial in the endpoints)**.** *Let G be a multigraph, let e be a bar of G, and let $0 \le i < j \le d$. Then there is a polynomial P in the coordinates of the endpoints of the bars of G such that $P(q) = T(p_e, p'_e)_{(i,j)}$ for every endpoint assignment q of G, where (p_e, p'_e) is the pair of points that q gives e.*

Proof. By Definition 28.15, $T(p_e, p'_e)_{(i,j)} = h(p_e)_i \, h(p'_e)_j - h(p_e)_j \, h(p'_e)_i$. Each coordinate of $h(p_e) = (1, p_e)$ and of $h(p'_e)$ is 1 or a coordinate of q. So this is a polynomial of degree at most two in the coordinates of q. □

Lemma 28.20 (Abundance of generic endpoint assignments)**.** *Let G be a finite multigraph. Then there is a nonzero polynomial P in the $2d\,|E(G)|$ endpoint coordinates such that every endpoint assignment q of G with $P(q) \neq 0$ is generic for row independence. In particular, the endpoint assignments that are not generic lie in the zero set of a nonzero polynomial.*

Proof. The row of a bar e has the coefficient $-(b_e)_c \, (a_e)_v$ at a body v and a coordinate c of $\mathbb{R}^D$, where a_e is the signed incidence vector of e (the remarks after Definition 16.22). So each coefficient is 0 or $\pm(b_e)_c$. At the endpoint realization of q, it is 0 or $\pm$ a coordinate of $T(p_e, p'_e)$, a polynomial in the coordinates of q (Lemma 28.19).

The argument of Lemma 28.1 now applies. There are finitely many sets of bars, as $E(G)$ is finite. Let F be a set of bars whose rows are linearly independent at some endpoint assignment q^F. Some $|F| \times |F|$ minor of the coefficient matrix of these rows is nonzero at q^F. As a polynomial P_F in the coordinates of q, it is nonzero, and the rows of F are linearly independent at every q with $P_F(q) \neq 0$. The product P of the polynomials P_F is nonzero, and every q with $P(q) \neq 0$ is generic. □

Lemma 28.21 (Existence of generic endpoint assignments)**.** *Let G be a finite multigraph. Then some endpoint assignment of G is generic for row independence.*

Proof. The polynomial P of Lemma 28.20 is nonzero. As $\mathbb{R}$ is infinite, P has a non-root, and every non-root of P is generic. □

Each bar has one row. So if q is generic for row independence, then for every set F of bars the rows of F have at q the largest rank over all endpoint assignments. Indeed, a subset of F whose rows are linearly independent at some assignment has linearly independent rows at q. Conversely, an assignment with this property is generic. Jackson and Jordán call a body-and-bar realization generic if its rigidity matrix and the submatrices formed by its rows at sets of bars have the largest rank over all realizations of the multigraph [JJ10, §5]. Their bars have distinct endpoints. Let $d \geq 1$, and let G be finite and without loops. Then the row of each bar is nonzero at some endpoint assignment, so a generic assignment gives every bar distinct endpoints. As a generic assignment exists (Lemma 28.21), the largest ranks over their realizations are those over all endpoint assignments. So the two notions of genericity agree.

Let $c_0 = 0$, and let $c_1, \dots, c_d$ be the standard basis vectors of $\mathbb{R}^d$, the vertices of a simplex. Jackson and Jordán put the bars of each of D disjoint forests F_{ij}, $0 \leq i < j \leq d$, on the edge from c_i to c_j. They prove that the rows of these bars are linearly independent by an elimination on the coordinates of the two-extensors of the edges. Whiteley pointed out that their lemma follows from the framework in his proof of Theorem 8 [Whi88], which is that of Proposition 16.25 [JJ10, Remark after Lemma 5.1]. This framework places the bars on the edges of a projective simplex with one vertex at the origin and the others at infinity on the coordinate axes. A projective transformation, which preserves independence, takes this simplex to the one with vertices $c_0, \dots, c_d$. We use a linear form of the argument. The two-extensors of the edges of the simplex form a basis of $\mathbb{R}^D$ (Lemma 28.22). An invertible linear map of $\mathbb{R}^D$, applied to the vectors on all the bars, preserves linear independence of rows (Lemma 28.23).

Lemma 28.22 (The coordinate-segment two-extensors form a basis)**.** *Let $c_0 = 0 \in \mathbb{R}^d$, and for $1 \leq i \leq d$ let c_i be the i-th standard basis vector of $\mathbb{R}^d$. Then the D two-extensors $T(c_i, c_j)$, $0 \leq i < j \leq d$, are linearly independent, and so form a basis of $\mathbb{R}^D$.*

Proof. Let u_{ij} be the standard basis vector of $\mathbb{R}^D$ at (i, j). By Definition 28.15, $T(c_0, c_k) = u_{0k}$ for $1 \leq k \leq d$, and

$$T(c_h, c_k) \;=\; u_{hk} + u_{0k} - u_{0h}, \qquad 1 \leq h < k \leq d.$$

Jackson and Jordán list these entries in the proof of their Lemma 5.1, with the signs of the direction coordinates reversed. So every u_{0k} is one of the two-extensors, and $u_{hk} = T(c_h, c_k) - u_{0k} + u_{0h}$ is a linear combination of them. Hence the D two-extensors span $\mathbb{R}^D$, and D vectors that span $\mathbb{R}^D$ form a basis. □

Lemma 28.23 (Row independence under an invertible map of $\mathbb{R}^D$)**.** *Let (G, b) be a body-bar framework in $\mathbb{R}^d$, and let M be an invertible linear map of $\mathbb{R}^D$. Let (G, Mb) be the framework with the vector Mb_e on each bar e. If the rows of a set of bars are linearly independent in (G, b), then they are linearly independent in (G, Mb).*

Proof. Let $M^\top$ be the transpose of M. In (G, Mb) the row of a bar e takes a motion m to

$$\langle Mb_e,\, m(u_e) - m(v_e)\rangle \;=\; \langle b_e,\, M^\top m(u_e) - M^\top m(v_e)\rangle.$$

So it is the row of e in (G, b), composed with the map Φ of motions that applies $M^\top$ to the vector of every body. As $M^\top$ is invertible, so is Φ. Hence composition with Φ is an injective linear map of the space of linear functionals on motions, and it preserves linear independence. □

Lemma 28.24 (Independent rows at the coordinate segments; Jackson–Jordán Lemma 5.1)**.** *Let G be a finite multigraph, and let $E' \subseteq E(G)$ be a set of bars such that $G \restriction E'$ is (D, D)-sparse. Then there is an endpoint assignment of G at whose endpoint realization the rows of the bars in E' are linearly independent.*

Proof. If E' is empty there is nothing to prove, so let E' be non-empty. By Lemma 16.11, E' is the union of D pairwise disjoint forests. Index them by the pairs, as F_{ij} for $0 \le i < j \le d$. Give each bar e a pair $\iota(e)$: the pair (i, j) with $e \in F_{ij}$ if $e \in E'$, and any pair otherwise.

Let b_e be the standard basis vector of $\mathbb{R}^D$ at $\iota(e)$, as in Proposition 16.25. In (G, b) the rows of the bars in E' are linearly independent. Indeed, up to sign the row of e is $a_e^{(\iota(e))}$ (Section 16.4). The vectors $a_e^{(ij)}$ with $e \in F_{ij}$ are linearly independent over $\mathbb{R}$, as the sets F_{ij} are forests, by the argument of Lemma 16.9.

Let M be the linear map of $\mathbb{R}^D$ that sends the standard basis vector at (i, j) to $T(c_i, c_j)$. It is invertible, by Lemma 28.22. Let q give each bar e with $\iota(e) = (i, j)$ the endpoints c_i and c_j. Then $Mb_e = T(c_i, c_j)$ for every bar e, so (G, Mb) is the endpoint realization of q. Its rows at the bars of E' are linearly independent, by Lemma 28.23. □

Theorem 28.25 (Independence at a generic endpoint assignment)**.** *Let G be a finite multigraph, and let q be an endpoint assignment of G that is generic for row independence. Let $E' \subseteq E(G)$. Then the rows of the bars in E' at the endpoint realization of q are linearly independent if and only if $G \restriction E'$ is (D, D)-sparse.*

Proof. If $G \restriction E'$ is (D, D)-sparse, then the rows of E' are linearly independent at some endpoint assignment (Lemma 28.24), and so at q, as q is generic.

By the argument of Lemma 16.26, the converse holds at every endpoint assignment. Suppose that the rows of E' are linearly independent, and let E'' be a non-empty subset of E'. Let $A_{E''}$ be the span of the vectors a_e with $e \in E''$. Up to sign, the row of a bar $e \in E''$ is $\sum_c (b_e)_c\, a_e^{(c)}$, which lies in $A^D_{E''}$. This space has dimension $D\, r(E'')$ (Lemma 16.6 and Lemma 16.7), and it contains the $|E''|$ linearly independent rows of E''. As $r(E'') \le |V(E'')| - 1$, we get $|E''| \le D|V(E'')| - D$. So $G \restriction E'$ is (D, D)-sparse. □

Corollary 28.26 (Tay's theorem at every generic endpoint assignment; Tay, Jackson–Jordán)**.** *Let G be a finite multigraph, and let q be an endpoint assignment of G that is generic for row independence. Then the endpoint realization of q is independent if and only if G is (D, D)-sparse. It is isostatic, that is, independent and infinitesimally rigid, if and only if G is (D, D)-tight.*

Proof. Take $E' = E(G)$ in Theorem 28.25. As $G \restriction E(G) = G$, the rows of all the bars are linearly independent, that is, the realization is independent (Definition 16.24), if and only if G is (D, D)-sparse. If the realization is independent, the rank of its rigidity map is $|E(G)|$. Then it is infinitesimally rigid if and only if $|E(G)| = D|V(G)| - D$ (Definition 16.23). So it is isostatic if and only if G is (D, D)-sparse with $|E(G)| = D|V(G)| - D$, that is, (D, D)-tight (Definition 16.1). □

So Tay's theorem (Theorem 16.27) holds at every generic endpoint assignment. So does rigidity from spanning trees. Let G be a finite multigraph that has at least one body and contains D edge-disjoint spanning trees, and let F be their union. Then F has $D(|V(G)| - 1)$ bars. As F is a union of D forests, $G \restriction F$ is (D, D)-sparse (Lemma 15.8 and Theorem 16.2). By Theorem 28.25, the rows of F are linearly independent at every generic endpoint assignment. So the rank of the rigidity map there is at least $D|V(G)| - D$. It is at most $D|V(G)| - D$ (the remarks after Definition 16.22), so every generic endpoint realization of G is infinitesimally rigid.

By Theorem 28.25 and Theorem 16.2, a set of bars has linearly independent rows at a generic endpoint assignment if and only if it is independent in the union of D copies of the cycle matroid of G. Tay proved this for his generic frameworks, in which the vectors on the bars are independent indeterminates [Tay84, Corollary 5.4]. So the rank of the rigidity map at a generic endpoint assignment is the rank of this union. For bars that are segments, Jackson and Jordán's Theorem 5.2, which they credit to Tay, gives this rank as

$$D\,(|V(G)| - 1) - \max_P (D\,(|P| - 1) - e_G(P)),$$

where P ranges over the partitions of $V(G)$, and $e_G(P)$ is the number of bars that join different parts of P [JJ10, §5]. That the rank of the union has this value is their Theorem 4.1(a), which they credit to Tutte and Nash-Williams.

28.5. **Every generic body-and-hinge realization.** As in Section 19, d is the dimension, $D = \binom{d+1}{2}$, and K is a field. Jackson and Jordán realize a multigraph $G = (V, E)$ as a body-and-hinge framework by giving each edge a hinge, a $(d-2)$-dimensional affine subspace of $\mathbb{R}^d$ [JJ10, §3.3]. This is a body-hinge framework in the sense of Definition 19.3. We give each hinge by $d - 1$ points of K^d that span it (Definition 28.27). So the realization parameters are the $d(d-1)\,|E|$ coordinates of these points, and "almost all realizations" refers to the choice of the points. The coordinates of the supporting extensor of a hinge are polynomials in the coordinates of its points (Lemma 28.29). So, as for normals and endpoints, the choices of hinge points that are not generic lie in the zero set of a nonzero polynomial (Lemma 28.31), and over an infinite field generic choices exist.

Our main result is Jackson and Jordán's rank formula, KT Proposition 2.3 (Proposition 28.42), for every multigraph, with parallel edges and loops allowed. We prove it first at every hinge-point assignment that is generic for row independence (Theorem 28.40), and then at every realization that is generic in the sense of Jackson and Jordán and of Katoh and Tanigawa, with the largest ranks over all realizations (Definition 19.16, Corollary 28.41). An assignment that is generic for row independence

is generic in their sense (Lemma 28.34). Jackson and Jordán prove the formula for every multigraph (their Theorem 6.1). Our proof starts from the realization that KT Theorem 5.6 gives for every multigraph (Theorem 23.66). At deficiency zero the formula gives the theorem of Tay and Whiteley at every generic hinge-point assignment, in either sense (Corollary 28.44).

For one hinge-point assignment at the target rank $D(|V|-1)-\mathrm{def}(\tilde{G})$ we start, as for normals, from KT Theorem 5.6 for multigraphs (Theorem 23.66). It gives a hinge-coplanar panel realization at the target rank, whose hinges are nonzero. The hinge of an edge with ends u and v lies in the panels of u and v, so it is an extensor of $d-1$ vectors of the hyperplane $n_u^\perp$ of K^{d+1}, where n_u is the normal of u (Lemma 28.38). If these vectors lie in the hyperplane of the vectors with last coordinate 0, the hinge lies at infinity. Then no points of K^d span it, as their homogeneous coordinate vectors have last coordinate 1 (Definition 18.1). Such a hinge is the body-and-hinge counterpart of the line at infinity of Section 28.4. The construction of the realization does not keep its hinges off the hyperplane at infinity, so we change coordinates. One invertible linear map of K^{d+1} moves every hinge off the hyperplane at infinity (Lemma 28.37). Jackson and Jordán make a similar choice of coordinates in the proof of their Lemma 7.1, where they choose the origin off finitely many hyperplanes. The map induces an invertible linear map of the screw space, which does not change the rank of the rigidity matrix (Lemma 28.35). Each moved hinge is spanned by points of K^d (Lemma 28.36), and these points form a hinge-point assignment at the target rank (Lemma 28.39). Jackson and Jordán construct a realization at the largest rank directly, with each hinge spanned by $d-1$ of the vertices $0, e_1, \dots, e_d$ of a simplex (their Theorem 6.4). We use KT Theorem 5.6 instead, which the earlier chapters prove.

Definition 28.27 (Generic hinge-point assignment)**.** Let $G=(V,E)$ be a multigraph. A *hinge-point assignment* q of G gives each edge e a family q_e of $d-1$ points $q_{e,1}, \dots, q_{e,d-1}$ of K^d. Its framework (G,q) is the body-hinge framework whose hinge at e has the supporting extensor $C(q_e)=C(q_{e,1}, \dots, q_{e,d-1})$ (Definition 18.8). The *annihilator-row family* of q is the family of linear forms

$$S \;\mapsto\; C_t\, x_{t'} - C_{t'}\, x_t, \qquad \text{where } x = S(u)-S(v) \text{ and } C = C(q_e),$$

on screw assignments S, indexed by an edge e of G, with ends u and v taken in a fixed order, and a pair (t,t') of indices of the standard basis of $\bigwedge^{d-1} K^{d+1}$. Here y_t is the coordinate of y at the t-th basis vector. The assignment q is *generic for row independence* if every subfamily of the annihilator-row family that is linearly independent at some hinge-point assignment is linearly independent at q.

The hinge of e in (G,q) is the affine span of the points of q_e. It is nondegenerate if $C(q_e)\neq 0$ (the remarks after Definition 23.5). This holds exactly when the points of q_e are affinely independent (the remark after Definition 18.8). The forms are those of Definition 28.5, with $C(q_e)$ in place of the panel support extensor. Exchanging the ends of e changes the signs of the forms at e, so genericity does not depend on the order of the ends. As a form at e reads only the points of q_e, genericity depends only on the points that q gives the edges of G. When the hinge of e is nondegenerate,

the forms at e span the same space as the rows of $R(G,q)$ at e, by the argument in the proof of Lemma 23.91. Genericity transfers linear independence only from hinge-point assignments. So a witness with a hinge at infinity has to be moved first, as the opening of this subsection explains.

Formalization note 28.28. The formalization writes $d = k+1$, as in the note after Definition 19.3. As in Formalization note 28.6, it indexes the family by all the edge labels, and reads the ends of each label from an endpoint selector, not from G. A hinge-point assignment gives points to every label, not only to the edges of G. A body-hinge framework has no selector of its own, so genericity over G asks about the subfamilies indexed by edges of G at every selector that gives each edge of G its two ends. The first pin is the family, and the third is genericity. The second identifies the family with the annihilator rows of the framework (G,q), read with the same selector, for every multigraph G; it holds by definition. The fourth shows that genericity is the condition at any one such selector, as exchanging the ends of an edge changes signs only, and the fifth that it reads only the points at the edges of G. The pinned lemmas on abundance and existence take G and finite sets of bodies and labels. Their polynomial has $d(d-1)$ variables for each label, and is a product over all the selectors, finitely many. The pinned lemma on nondegeneracy concludes at the edges of G with two distinct ends, not at every label.

Lemma 28.29 (The hinge extensors are polynomial in the hinge points)**.** *Let $G = (V,E)$ be a multigraph with E finite, let e be an edge of G, and let t be an index of the standard basis of $\bigwedge^{d-1} K^{d+1}$. Then there is a polynomial in the $d(d-1)|E|$ coordinates of a hinge-point assignment of G whose value at every hinge-point assignment q is the coordinate $C(q_e)_t$ of $C(q_e)$ at the t-th basis vector.*

Proof. Let $A(q_e)$ be the $(d-1)\times(d+1)$ matrix whose i-th row is the homogeneous coordinate vector $\bar{q}_{e,i} = (q_{e,i}, 1)$ (Definition 18.1). As $C(q_e)$ is the join $\bar{q}_{e,1} \vee \cdots \vee \bar{q}_{e,d-1}$, its coordinate at t is the minor of $A(q_e)$ on the columns of t (Lemma 18.6). Each entry of $A(q_e)$ is a coordinate of q or the constant 1. So the same minor of the matrix of indeterminates is a polynomial, and its value at q is the minor of $A(q_e)$. □

Formalization note 28.30. The pins do not use the Plücker coordinates or Lemma 18.6. The pin computes the coordinate at t directly, as the pairing of the join with the dual basis vector at t, which a library lemma evaluates as the minor. Its polynomial is the minor of the matrix of indeterminates (`hingeExtensorPoly`), with $d(d-1)$ variables for each edge label.

Lemma 28.31 (Abundance of generic hinge-point assignments)**.** *Let $G = (V,E)$ be a multigraph with V and E finite. Then there is a nonzero polynomial P in the $d(d-1)|E|$ coordinates of a hinge-point assignment such that every hinge-point assignment q with $P(q) \neq 0$ is generic for row independence and gives each edge $d-1$ affinely independent points. In particular, the hinge-point assignments that are not generic, or that give some edge affinely dependent points, lie in the zero set of a nonzero polynomial.*

Proof. Fix the standard basis of the space of screw assignments. In its dual basis, each coordinate of a member of the annihilator-row family at an edge e is 0 or $\pm$ a coordinate of $C(q_e)$. So it is a polynomial in the coordinates of q (Lemma 28.29).

The argument of Lemma 28.1 now applies. The family has finitely many indices, as E is finite. Let s be a set of indices whose members are linearly independent at some hinge-point assignment q^s. Some $|s| \times |s|$ minor of the coordinate matrix of these members is nonzero at q^s. As a polynomial P_s in the coordinates of q, it is nonzero, and the members indexed by s are linearly independent at every q with $P_s(q) \neq 0$.

For the second property, let q^0 give every edge the first $d-1$ standard basis vectors of K^d. They are linearly independent, hence affinely independent. So $C(q_e^0) \neq 0$ for every edge e (the remark after Definition 18.8). Let t be an index with $C(q_e^0)_t \neq 0$, the same for every e. Let R_e be the polynomial of Lemma 28.29 whose value at q is $C(q_e)_t$. It is nonzero, as $R_e(q^0) \neq 0$. If $R_e(q) \neq 0$, then $C(q_e) \neq 0$, so the points of q_e are affinely independent.

The product P of the polynomials P_s and R_e is nonzero, as a finite product of nonzero polynomials. Every q with $P(q) \neq 0$ has both properties. □

Lemma 28.32 (Existence of generic hinge-point assignments)**.** *Let K be infinite, and let $G = (V, E)$ be a multigraph with V and E finite. Then some hinge-point assignment of G is generic for row independence and gives each edge $d-1$ affinely independent points.*

Proof. The polynomial P of Lemma 28.31 is nonzero. Over an infinite field a nonzero polynomial has a non-root, and every non-root of P is such an assignment. □

Lemma 28.33 (Generic hinge-point assignments have nondegenerate hinges)**.** *Let $d \geq 2$, and let $G = (V, E)$ be a multigraph, with parallel edges and loops allowed. Let q be a hinge-point assignment of G that is generic for row independence. Then the hinge of (G, q) at every edge with two distinct ends is nondegenerate.*

Proof. Let e be an edge of G, with ends $u \neq v$. Let q^0 give every edge the first $d-1$ standard basis vectors of K^d. They are linearly independent, hence affinely independent, so $C^0 = C(q_e^0)$ is not zero (the remark after Definition 18.8). Let t be an index with $C_t^0 \neq 0$. As $d \geq 2$, the screw space has dimension $D \geq 3$, so there is an index $t' \neq t$. As $u \neq v$, there is a screw assignment S for which $S(u) - S(v)$ is the t'-th basis vector. At S the member of the annihilator-row family of q^0 at (e, t, t') takes the value $C_t^0 \neq 0$.

So this member alone is a linearly independent family at q^0. As q is generic, the member of the family of q at (e, t, t') is not zero. If $C(q_e)$ were zero, every member of the family of q at e would be zero. So the hinge of e in (G, q) is nondegenerate. □

Jackson and Jordán call a realization of G generic if its rigidity matrix, and the submatrix of the rows of every set of edges, have the largest rank over all realizations of G [JJ10, §6]. This is the genericity of Definition 19.16. Each hinge of a realization is a $(d-2)$-dimensional affine subspace, the affine span of $d-1$ affinely independent points. A different choice of the points does not change the hinge constraint (the remark after Definition 19.3). So the realizations of G are, up to this choice, the

frameworks (G, q) whose hinges are nondegenerate, and Definition 19.16 asks this of the hinges at the links between distinct bodies.

Lemma 28.34 (Generic for row independence implies generic)**.** *Let $d \geq 2$, let G be a multigraph with V and E finite, loops allowed, and let q be a hinge-point assignment of G that is generic for row independence. Then (G, q) is generic in the sense of Definition 19.16. If K is infinite, some hinge-point assignment of G is generic in this sense.*

Proof. By Lemma 28.33, every hinge of (G, q) at a link between distinct bodies is nondegenerate. Let F be a set of edges, and let (G, q') be a realization. Replace the points of each degenerate hinge of q', which lies at a loop, by a fixed affinely independent family. This changes no row, as the rows of a loop are zero, and gives every hinge a nondegenerate one. The members of the annihilator-row family of the new assignment at F span the rows of its rigidity matrix at F. So some of these members, as many as the rank of these rows, are linearly independent. As q is generic, the members of the family of q with the same indices are linearly independent too. They lie in the span of the rows of $R(G, q)$ at F (the remark after Definition 19.5), whose dimension is therefore at least that rank. Over all hinge-point assignments with a degenerate hinge allowed, the rows at F can have a larger rank, as a degenerate hinge welds its two bodies together (the remarks after Definition 23.5); this is why the realizations are required to be nondegenerate. Over an infinite field, a hinge-point assignment generic for row independence exists (Lemma 28.32), and it is generic in this sense. □

We now construct a hinge-point assignment at the target rank. For $d = 3$, part (1) of Theorem 25.7 shows that a linear automorphism of the screw space, applied to every hinge, does not change the rank of the rigidity matrix. The next lemma shows this in every dimension, by comparing the rows.

Lemma 28.35 (Row rank under a change of screw coordinates)**.** *Let (G, p) be a body-hinge framework in K^d, and let Λ be a linear automorphism of the screw space $\bigwedge^{d-1} K^{d+1}$. Let $(G, \Lambda p)$ be the body-hinge framework on G whose supporting extensor at each edge e is $\Lambda\, C(p(e))$. Then* $\operatorname{rank} R(G, \Lambda p) = \operatorname{rank} R(G, p)$.

Proof. Let e be an edge with ends u and v. The rows of $R(G, p)$ at e span the linear forms $S \mapsto r(S(u) - S(v))$ on screw assignments, where r ranges over the linear forms on the screw space that vanish on $C(p(e))$ (the remark after Definition 19.5). A form r vanishes on $C(p(e))$ if and only if $r \circ \Lambda^{-1}$ vanishes on $\Lambda\, C(p(e))$. So the rows of $R(G, \Lambda p)$ at e span the forms $S \mapsto r(\Lambda^{-1}(S(u) - S(v)))$. These are the forms at e for (G, p), composed with the map $\Phi \colon S \mapsto \Lambda^{-1} \circ S$ of screw assignments.

The map Φ is linear and invertible. So composition with Φ is an invertible linear map of the space of linear forms on screw assignments. It maps the row space of $R(G, p)$ onto that of $R(G, \Lambda p)$, and the two have the same dimension. In Lemma 28.23 the corresponding map sends the row of each bar to the row of the same bar. Here it sends the rows at each edge onto the rows at that edge, but not each form $C_t\, x_{t'} - C_{t'}\, x_t$ of the annihilator-row family (Definition 28.27) to such a

form. These forms are indexed by pairs of coordinates of the screw space, and Λ does not preserve the coordinates. □

Lemma 28.36 (Points that span an extensor off the hyperplane at infinity)**.** *Let $u_1, \dots, u_{d-1} \in K^{d+1}$, and suppose that some u_i has nonzero last coordinate. Then there are points $p_1, \dots, p_{d-1}$ of K^d and a nonzero scalar c with*

$$C(p_1, \dots, p_{d-1}) \;=\; c\,(u_1 \vee \cdots \vee u_{d-1}).$$

Proof. Let $\lambda \neq 0$ be the last coordinate of u_i, and replace u_i by $\lambda^{-1}u_i$. As the join is linear in each vector, this multiplies it by λ^{-1} (the remarks after Definition 18.3). Now add to each other u_j the vector $(1 - \mu_j)\,u_i$, where μ_j is the last coordinate of u_j. Adding a multiple of one vector to another does not change the join, and u_j gets the last coordinate 1. So every vector now has last coordinate 1, and is the homogeneous coordinate vector $\bar{p}_j$ of a point p_j of K^d (Definition 18.1). Hence $C(p_1, \dots, p_{d-1}) = \bar{p}_1 \vee \cdots \vee \bar{p}_{d-1}$ is $\lambda^{-1}(u_1 \vee \cdots \vee u_{d-1})$ (Definition 18.8). □

Lemma 28.37 (Moving finitely many vectors off the hyperplane at infinity)**.** *Let K be infinite, and let $w_1, \dots, w_m$ be nonzero vectors of K^{d+1}. Then there is an invertible linear map g of K^{d+1} such that every $g(w_i)$ has nonzero last coordinate.*

Proof. For each i the linear form $x \mapsto w_i \cdot x$ is nonzero, as $w_i \neq 0$. So the product of these forms is a nonzero polynomial in the coordinates of x. As K is infinite, it has a non-root n, and $w_i \cdot n \neq 0$ for every i. Let φ be the linear form $x \mapsto n \cdot x$, and let ψ be the last coordinate. Both are nonzero, so their kernels have dimension d. Extending a linear isomorphism of $\ker\varphi$ onto $\ker\psi$, we obtain an invertible linear map g of K^{d+1} with $g(\ker\varphi) = \ker\psi$. If $g(w_i)$ had last coordinate 0, then $g(w_i) \in \ker\psi$, so $w_i \in \ker\varphi$, which is false. □

Lemma 28.38 (A nonzero extensor in a panel is decomposable)**.** *Let $d \geq 2$ and let $C \neq 0$ be an extensor that lies in the panel with normal n (Definition 23.24). Then $C = x_1 \vee \cdots \vee x_{d-1}$ for some vectors $x_1, \dots, x_{d-1}$ of K^{d+1}, with $x_1 \neq 0$.*

Proof. By definition, $C = x_1 \vee \cdots \vee x_{d-1}$ for some vectors of $n^\perp$. As $C \neq 0$, the vectors are linearly independent (the remark after Definition 18.3), so $x_1 \neq 0$. □

Lemma 28.39 (Independent annihilator rows at some hinge-point assignment)**.** *Let $d \geq 2$, let K be an infinite field, and let $G = (V, E)$ be a multigraph, with parallel edges and loops allowed, with V finite and nonempty. Then there is a hinge-point assignment q of G at which some $D(|V|-1) - \mathrm{def}(\tilde{G})$ members of the annihilator-row family of q are linearly independent.*

Proof. Let $N = D(|V| - 1) - \mathrm{def}(\tilde{G})$. By KT Theorem 5.6 for multigraphs (Theorem 23.66), G has a hinge-coplanar panel realization (G, p) at the target rank N. Its hinge C_e at every edge e is nonzero and lies in a panel, so $C_e = w_{e,1} \vee \cdots \vee w_{e,d-1}$ for some vectors $w_{e,i}$ of K^{d+1} with $w_{e,1} \neq 0$ (Lemma 28.38).

By Lemma 28.37, there is an invertible linear map g of K^{d+1} such that $g(w_{e,1})$ has nonzero last coordinate for every edge e. Let $\Lambda = \Lambda_g$ be the screw map of g (Definition 25.5), a linear automorphism of the screw space. The hinge of $(G, \Lambda p)$ at e is $gw_{e,1} \vee \cdots \vee gw_{e,d-1}$. The rigidity matrix of $(G, \Lambda p)$ has rank N (Lemma 28.35),

and its hinges are nondegenerate, as Λ is injective. By Lemma 28.36, there are points $q_{e,1}, \dots, q_{e,d-1}$ of K^d and a scalar $c_e \neq 0$ with $C(q_e) = c_e \Lambda C_e$. Let q be the hinge-point assignment that gives each edge e the points q_e.

At each edge e of $(G, \Lambda p)$, consider the forms $S \mapsto C_t x_{t'} - C_{t'} x_t$ of Definition 28.27, with $C = \Lambda C_e$. As the hinge is nondegenerate, they span the same space as the rows of $R(G, \Lambda p)$ at e (the argument in the proof of Lemma 23.91). So these forms, at all the edges, span the row space of $R(G, \Lambda p)$, of dimension N, and some N of them are linearly independent (Lemma 23.110). The member of the annihilator-row family of q at an index (e, t, t') is c_e times the form at (e, t, t'), as $C(q_e) = c_e \Lambda C_e$. Multiplying the members of a family by nonzero scalars does not affect their linear independence. So the members of the family of q at these N indices are linearly independent. □

Theorem 28.40 (The rank formula at every generic hinge-point assignment)**.** *Let $d \geq 2$, let K be an infinite field, and let $G = (V, E)$ be a multigraph, with parallel edges and loops allowed, with V finite and nonempty. Let q be a hinge-point assignment of G that is generic for row independence. Then*

$$\operatorname{rank} R(G, q) \;=\; D(|V| - 1) - \operatorname{def}(\tilde{G}).$$

Proof. The hinges of (G, q) at the edges with distinct ends are nondegenerate (Lemma 28.33), and Lemma 20.16 asks nothing of a loop. So $\operatorname{rank} R(G, q) \leq D(|V| - 1) - \operatorname{def}(\tilde{G})$ (Lemma 20.16).

For the reverse inequality, let q^0 be a hinge-point assignment at which some $D(|V| - 1) - \operatorname{def}(\tilde{G})$ members of the annihilator-row family are linearly independent (Lemma 28.39). As q is generic, the members of the family of q with the same indices are linearly independent. Each of them lies in the row space of $R(G, q)$. Indeed, a member at an edge e with ends u and v is a linear form in $S(u) - S(v)$ that vanishes at $C(q_e)$, and so a linear combination of the rows of $R(G, q)$ at e (the remark after Definition 19.5). Hence $\operatorname{rank} R(G, q) \geq D(|V| - 1) - \operatorname{def}(\tilde{G})$. □

Corollary 28.41 (A generic realization has $\operatorname{def}(\tilde{G})$ degrees of freedom)**.** *Let $d \geq 2$, let K be an infinite field, and let $G = (V, E)$ be a multigraph, with parallel edges and loops allowed, with V finite and nonempty. Let q be a hinge-point assignment of G that is generic in the sense of Definition 19.16. Then* $\operatorname{rank} R(G, q) = D(|V| - 1) - \operatorname{def}(\tilde{G})$*. So (G, q) has $\operatorname{def}(\tilde{G})$ degrees of freedom.*

Proof. Some q_0 is generic for row independence (Lemma 28.32), so by Theorem 28.40, $\operatorname{rank} R(G, q_0) = D(|V| - 1) - \operatorname{def}(\tilde{G})$. The hinges of q_0 at the links between distinct bodies are nondegenerate, so q_0 is a realization (Lemma 28.33). As q is generic in the sense of Definition 19.16, the rank of $R(G, q)$ is at least that of $R(G, q_0)$. It is at most $D(|V| - 1) - \operatorname{def}(\tilde{G})$ by Lemma 20.16, as the hinges of q at the links between distinct bodies are nondegenerate. □

Proposition 28.42 (KT Proposition 2.3)**.** *Let $d \geq 2$, let K be an infinite field, and let $G = (V, E)$ be a multigraph, with parallel edges and loops allowed, with V finite and nonempty. Let q be a hinge-point assignment of G that is generic in the sense*

of Definition 19.16, and let c be an integer. Then the following are equivalent, as Katoh and Tanigawa state [KT11, p. 655]*, crediting Jackson and Jordán* [JJ10]*:*

(i) $\operatorname{rank} R(G,q) = D(|V|-1) - c$*;*
(ii) $\operatorname{def}(\tilde{G}) = c$*;*
(iii) $\operatorname{rank} M(\tilde{G}) = D(|V|-1) - c$*.*

The same holds at every q that is generic for row independence (Definition 28.27), the form the rest of the development uses.

Proof. By Corollary 28.41, $\operatorname{rank} R(G,q) = D(|V|-1) - \operatorname{def}(\tilde{G})$, which is (i) $\Leftrightarrow$ (ii). By Theorem 20.12, $\operatorname{rank} M(\tilde{G}) = D(|V|-1) - \operatorname{def}(\tilde{G})$, which is (ii) $\Leftrightarrow$ (iii). A q that is generic for row independence is generic in this sense (Lemma 28.34), and Theorem 28.40 gives the formula there directly. □

Corollary 28.43 (Every generic body-and-hinge realization of a rigid graph; Jackson–Jordán)**.** *Let $d \geq 2$, let K be an infinite field, and let $G = (V, E)$ be a multigraph, with parallel edges and loops allowed, with V finite and nonempty. Then (G, q) is infinitesimally rigid at every hinge-point assignment q of G that is generic in the sense of Definition 19.16 if and only if* $\operatorname{def}(\tilde{G}) = 0$*. The same holds with the assignments that are generic for row independence (Definition 28.27) in place of these.*

Proof. Suppose first that $\operatorname{def}(\tilde{G}) = 0$, and let q be generic in the sense of Definition 19.16. By Corollary 28.41, $\operatorname{rank} R(G,q) = D(|V|-1)$, so $\dim Z(G,q) = D|V| - D(|V|-1) = D$. Hence every infinitesimal motion of (G,q) is trivial (Lemma 23.117, with $W = V$).

Conversely, suppose that (G,q) is infinitesimally rigid at every such q. As K is infinite, some hinge-point assignment q is generic in this sense (Lemma 28.34). Then (G,q) is infinitesimally rigid, so its motions are constant on V, and $\dim Z(G,q) = D$ (Lemma 23.124, with $W = V$). As the screw assignments form a space of dimension $D|V|$, $\operatorname{rank} R(G,q) = D(|V|-1)$, and Corollary 28.41 gives $\operatorname{def}(\tilde{G}) = 0$.

At the assignments that are generic for row independence the argument is the same, with Theorem 28.40 in place of Corollary 28.41, and with an assignment that is generic for row independence, which exists as K is infinite (Lemma 28.32). □

Corollary 28.44 (Generic body-and-hinge rigidity from spanning trees; Tay, Whiteley, Jackson–Jordán)**.** *Let $d \geq 2$, let K be an infinite field, and let $G = (V, E)$ be a multigraph, with parallel edges and loops allowed, with V finite and nonempty. Then (G, q) is infinitesimally rigid at every hinge-point assignment q of G that is generic in the sense of Definition 19.16 if and only if $\tilde{G} = (D-1) \cdot G$ contains D edge-disjoint spanning trees. The same holds with the assignments that are generic for row independence (Definition 28.27) in place of these.*

Proof. By Corollary 28.43, the condition on the generic hinge-point assignments, in either sense, holds if and only if $\operatorname{def}(\tilde{G}) = 0$. As $D \geq 3$, KT Proposition 2.2 (Lemma 20.14) shows that this holds if and only if $\tilde{G}$ contains D edge-disjoint spanning trees. □

Formalization note 28.45. The formalization states Proposition 28.42, Corollary 28.43 and Corollary 28.44 twice: first at the assignments that are generic in the sense of Definition 19.16, Katoh and Tanigawa's, then at those that are generic for row independence. The proposition's second form follows from its first (Lemma 28.34). It writes $d = k + 1$, as in the note after Definition 19.3. The pinned results of this subsection from Lemma 28.39 on take the dimension in $\operatorname{def}(\tilde{G})$ as a separate parameter, as in the note after Definition 23.1. They assume that every body is a vertex of G, and state rigidity on $V(G)$, which is then the set of all bodies. They assume nothing of the edge labels (Remark 23.40): they apply to every multigraph on the bodies, a single edge, a tree or a double edge among them. The pinned witness lemma takes a selector that gives each edge of G its two ends. The rank formula and its corollaries take none, as genericity over G asks about every such selector (the note after Definition 28.27).

Tay [Tay89, §7] and Whiteley [Whi88, Theorem 10] characterized, independently, the multigraphs G that have an infinitesimally rigid body-and-hinge realization in $\mathbb{R}^d$: those for which $(D-1)\cdot G$ contains D edge-disjoint spanning trees. Jackson and Jordán state this as their Corollary 6.3, a consequence of their Theorem 6.1 [JJ10, §6]. The body-hinge chapter proves the characterization in its existence form, for frameworks given by bundles of bars (Theorem 17.5). Corollary 28.44 is its generic form for frameworks whose hinges are spanned by points, for multigraphs.

29. Pencil realizations: hinges concurrent within a common panel

In dimension three, this chapter and Section 30 strengthen KT Theorem 5.6 in the hinge-coplanar model of Section 23.5. A pencil realization gives each body, besides its panel, a point of that panel, through which every hinge at the body passes (Definition 29.2). The hinges at a body then form a pencil of lines through a point in a plane. We show that, over every infinite field, every spanning multigraph G has a pencil realization of rank $6(|V(G)|-1)-\operatorname{def}(\tilde{G})$, the deficiency rank of Theorem 23.66. If G is simple, the realization can be chosen with the points of adjacent bodies projectively distinct. If moreover G has a nondegenerate pencil realization (Definition 29.23), it has one of that rank. These claims make up Theorem 29.34. Besides Section 23.5, the reader needs the deficiency and rigid subgraphs of Section 20.

Section 29.1 and Section 29.2 give the definitions. Section 29.4 places the two independent hinges of Katoh and Tanigawa's base case (KT Lemma 5.3) through one point of their common panel. Section 29.6 gives an added edge a hinge through the points of both its ends. Such a hinge exists only when the point of each end lies in the panel of the other (Lemma 29.12). These conditions can fail at the edges that Katoh and Tanigawa add to a realization of a minimal spanning subgraph. So we induct over every multigraph, not only over the minimal ones (Theorem 29.15). Section 29.7 states this induction and proves its cases at a loop, on at most two bodies and at a cut edge. Section 29.8 adds the distinct points and the nondegenerate realizations to the induction statement. Katoh and Tanigawa's induction statement likewise asks for nonparallel realizations (KT Theorem 5.5). The cases at a loop, on at most

two bodies and at a cut edge hold for the strengthened statement too, so it follows from the two remaining cases (Theorem 29.30). Section 29.9 proves these two cases, where G has a proper rigid subgraph and where it is two-edge-connected without one. At a simple two-edge-connected multigraph both follow from two statements (Definition 29.31), which Section 30 proves by lifting pictures of G in the plane.

Two parts of the chapter are not used in the proof of Theorem 29.34. Section 29.3 shows that the polarity of Lemma 25.9 takes a pencil realization to one with the panel normals and the points exchanged. This chapter uses the polarity nowhere else. Section 29.5 realizes the cycles on three and four bodies as pencil realizations.

29.1. **Extensors through a point.** Hinge-coplanarity is a containment condition: a screw element lies in the panel with normal n when it is the extensor of points of the hyperplane $n^{\perp}$ (Definition 23.24). We dualize this to concurrency. A screw element passing through a point replaces containment in the hyperplane by containment of the point in the element's span: at $k = 2$ this says a line — the extensor of two points — contains the point.

Definition 29.1 (Extensor through a point)**.** A screw element $C \in \bigwedge^{k} K^{k+2}$ *passes through* the homogeneous point $q \in K^{k+2}$ when C is the extensor of k points $p_1, \dots, p_k \in K^{k+2}$ whose span contains q.

For $k \geq 2$ the zero element passes through every point (take every $p_i = q$), so we treat the nonzero-ness of C as a separate hypothesis wherever the notion is used below, where $k = 2$. For $k = 1$ the extensor of q is q, and the zero element passes through no nonzero point.

29.2. **Pencil realizations.** A pencil realization asks that, at every body, the incident hinges lie in the body's panel and pass through one point of that panel. This is the classical picture of a pencil of lines through a point in a plane. In the molecular reading (Section 25.1) the panel is the plane of an atom's bonds and the point is the atom's centre.

Definition 29.2 (Pencil realization)**.** A hinge-coplanar panel realization of a multigraph G (Definition 23.63), with normal assignment $v \mapsto n_v$, has a *pencil realization* when it is given in addition a per-body nonzero homogeneous point $v \mapsto q_v \in K^{k+2}$, incident to its own panel ($q_v \cdot n_v = 0$). Every link's supporting extensor must then pass through the points of both its endpoints (Definition 29.1). We call this requirement the *pencil condition*.

The incidence $q_v \cdot n_v = 0$ is forced at any body with a nonzero hinge: the hinge's spanning points lie in $n_v^{\perp}$ and contain q_v in their span, so $q_v \in n_v^{\perp}$. We require it at every body, isolated ones included, so that the normal and the point play symmetric roles under the polarity below (Section 29.3). Two hinges in one panel always pass through a common point of it (Lemma 29.8). So at a body of degree at most two, asking the hinges to pass through one point adds nothing to hinge-coplanarity. The requirement first has content at a body with three or more hinges.

29.3. **The polarity and self-duality.** Work at grade $k = 2$ ($d = 3$) for the rest of this subsection, the setting of Lemma 25.9. There the polarity sends the extensor of two homogeneous points $v_0, v_1 \in K^4$ to the panel-support extensor $C(v_0, v_1)$ (Definition 23.1), over every field K: the join-of-two-points $\leftrightarrow$ meet-of-two-panels identity. The two transport lemmas below also hold over every field K. They show that the polarity turns passing through the point q into lying in the panel with normal q, and lying in that panel into passing through q.

Lemma 29.3 (Passing through a point transports to lying in a panel)**.** *If a screw element $C \in \bigwedge^2 K^4$ passes through the homogeneous point $q \in K^4$ (Definition 29.1), then its polarity image lies in the panel with normal q (Definition 23.24).*

Proof. If $C = 0$ the image is 0, which lies in every panel. Otherwise C is the extensor of an independent pair v_0, v_1 with q in their span; the polarity (Lemma 25.9) sends C to the panel-support extensor $C(v_0, v_1)$ (Definition 23.1), which by the point-join $\leftrightarrow$ panel-meet duality is the extensor of an independent pair p_0', p_1' orthogonal to both v_0 and v_1 (Lemma 23.3 and the remark after it). Since q lies in the span of v_0, v_1 and the dot product is linear, p_0' and p_1' are also orthogonal to q, so the image lies in the panel with normal q. □

Lemma 29.4 (Lying in a panel transports to passing through a point)**.** *If a screw element $C \in \bigwedge^2 K^4$ lies in the panel with normal $q \in K^4$ (Definition 23.24), then its polarity image passes through q (Definition 29.1).*

Proof. If $C = 0$ the image is 0, which passes through every point (take the constant family at q). Otherwise C is the extensor of an independent pair v_0, v_1 both orthogonal to q; the polarity sends C to the panel-support extensor $C(v_0, v_1)$ (Definition 23.1), which is the extensor of an independent pair p_0', p_1' orthogonal to both v_0 and v_1 (Lemma 23.3 and the remark after it). The orthogonal complement of $\{v_0, v_1\}$ is 2-dimensional, hence equal to the span of the independent pair p_0', p_1'; as q also lies in this complement, $q \in \operatorname{span}\{p_0', p_1'\}$, i.e. the image passes through q. □

Lemma 29.5 (Self-duality of pencil realizations)**.** *Over any field K, transporting a pencil realization (F, n, q) of a multigraph G along the polarity Λ of Lemma 25.9 (field-general as a screw-space automorphism, Theorem 25.7) produces a pencil realization $(\Lambda F, q, n)$ of G, with the roles of the normal and the point exchanged.*

Proof. The transported framework has the same multigraph and nonzero supporting extensors, since the polarity is a linear automorphism (Theorem 25.7). Its new normal assignment is q, each nonzero by hypothesis, and the incidence $n_v \cdot q_v = 0$ read the other way gives $q_v \cdot n_v = 0$, so it is preserved. Let C be the supporting extensor of a link uw. As C passes through q_u and q_w, its image lies in the panels with normals q_u and q_w (Lemma 29.3). As C lies in the panels with normals n_u and n_w, its image passes through n_u and n_w (Lemma 29.4). □

We record this symmetry for its own sake: the proof of Theorem 29.34 does not use it. It is one reason why Definition 29.2 allows the points of a link's two endpoints to coincide: requiring them to be projectively distinct would break the symmetry. Section 29.8 gives a second reason, and imposes distinctness as a separate condition.

29.4. **The coincident-panel pencil pair.** This subsection supplies the pencil condition at the base case of Theorem 29.15: every loopless multigraph on at most two vertices (Lemma 29.21). Katoh–Tanigawa's two-vertex base realization of a parallel edge pair (Lemma 23.34, KT Lemma 5.3, p. 669) places both hinges in a single shared panel $n^\perp$ with two linearly independent supporting extensors (Lemma 23.31), which fill the whole screw space K^6 and make the two bodies rigid. On the pencil condition the same construction supplies a common concurrency point at no extra cost: two lines of a flat pencil already share the point of the pencil.

Lemma 29.6 (Two independent extensors through a common point of a panel)**.** *For any normal $n \in K^4$ there is a nonzero point $q \in n^\perp$ and two screw elements $C, C' \in \bigwedge^2 K^4$, each lying in the panel with normal n (Definition 23.24) and passing through q (Definition 29.1), whose* 2*-extensors are linearly independent.*

Proof. The panel $n^\perp$ has dimension at least 3 in K^4, so it contains three linearly independent vectors q, a, b. Take the two pencil lines $q \vee a$ and $q \vee b$ through q: both lie in $n^\perp$ and contain q in their span, so each lies in the panel and passes through q. Their extensors are linearly independent because $\{q, a, b\}$ is — the shared-vector case of the independence of extensors on two distinct point subsets behind Lemma 23.31. □

Lemma 29.7 (Two independent extensors through a prescribed point of a panel)**.** *For any normal $n \in K^4$ and any nonzero point $q \in n^\perp$, there are two screw elements $C, C' \in \bigwedge^2 K^4$, each lying in the panel with normal n (Definition 23.24) and passing through q (Definition 29.1), whose* 2*-extensors are linearly independent.*

Proof. As for Lemma 29.6, with q given in advance. The panel $n^\perp$ has dimension at least 3 in K^4, so q extends to a linearly independent triple q, y, z in $n^\perp$. The two pencil lines $q \vee y$ and $q \vee z$ through q both lie in $n^\perp$ and contain q in their span, so each lies in the panel and passes through q; their extensors are linearly independent because $\{q, y, z\}$ is — the shared-vector case behind Lemma 23.31. □

Katoh–Tanigawa realize the base of their induction directly, at one and two bodies (Lemma 23.36). Asking the hinges at a body to pass through one point has content only at degree three or more, so on two bodies it has content only when three or more edges join them. Lemma 29.21 handles such a class by placing both bodies on a single panel at a fixed normal, with two independent hinges through a common point of the panel (Lemma 29.6) making the two bodies rigid (Lemma 23.30); each further edge reuses one of the two hinges. It assembles this with the edgeless and single-edge cases into the base case of Theorem 29.15.

29.5. **Pencil cycles: degree-two concurrency is automatic.** Katoh–Tanigawa's cycle realization (Lemma 23.80, KT Lemma 5.4) places a cycle of bodies with each hinge the meet of two adjacent panels. Every body has degree two, and at a body of degree at most two the concurrency requirement costs nothing: two lines lying in a common plane always meet in a point of that plane. The following records this projective fact in the containment model, so that the cycle realization — indeed any hinge-coplanar realization (Definition 23.63) restricted to its degree-two bodies —

already admits the per-body concurrency point a pencil realization asks for. The same fact accounts for two features of the nondegeneracy conditions in Section 29.8. A pencil hub there is a body of degree at least three, since only there does passing through one point constrain the hinges. At a body of degree two, whose two hinges pass through a common point for free, a further condition keeps them from coinciding. Theorem 29.15 does not call on the two pencil realizations of a cycle built below; they give a direct example of the same fact, on the cycle realization Katoh–Tanigawa already use.

Lemma 29.8 (Two coplanar hinges share a concurrency point)**.** *Let $n \in K^4$ be nonzero and let $C_1, C_2 \in \bigwedge^2 K^4$ be nonzero screw elements each lying in the panel with normal n (Definition 23.24). Then there is a nonzero point $q \in n^\perp$ through which both pass (Definition 29.1).*

Proof. Each hinge is the extensor of an independent pair, spanning a 2-dimensional subspace of the panel $n^\perp$, which has dimension 3 in K^4. Two 2-dimensional subspaces of a 3-dimensional space meet in dimension at least $2+2-3=1$ (the modular law), so their intersection contains a nonzero point q. Lying in each hinge's span, q passes through both hinges; and lying in $n^\perp$ it is orthogonal to n. □

The range $3 \le m \le 4$ is exactly where the shared-normal family exists at $d = 3$ (Lemma 23.62), so the triangle and the quadrilateral are both realized. Since every body of a cycle has degree two, the concurrency fact above (Lemma 29.8) supplies each body's point directly, alongside the shared-normal family's panel data, giving a pencil realization outright.

Lemma 29.9 (Pencil realization of a cycle)**.** *Let G be presented as a cycle of m bodies (Definition 21.64) with $3 \le m \le 4$. Then G has a pencil realization (Definition 29.2) that is infinitesimally rigid on all of its bodies $V(G)$ (Definition 19.15).*

Proof. At grade $k = 2$ the cyclic shared-normal family (Lemma 23.62) supplies m panel normals $n_0, \dots, n_{m-1}$ — the range $3 \le m \le 4 = k+2$ is exactly its hypothesis — with each consecutive pair n_i, n_{i+1} linearly independent. The i-th cycle edge is assigned the panel meet $C(n_i, n_{i+1})$ (Definition 23.1), nonzero and lying in both endpoint panels $n_i^\perp$, $n_{i+1}^\perp$ (Definition 23.24); every edge label off the cycle is given a fixed nonzero fallback, so all supporting extensors are nonzero. Body i has exactly two incident hinges, $C(n_{i-1}, n_i)$ and $C(n_i, n_{i+1})$, both lying in the panel $n_i^\perp$; by Lemma 29.8 they share a nonzero point $q_i \in n_i^\perp$, through which both pass (Definition 29.1). Assign body i the point q_i. Then each point is nonzero and incident to its own panel, and the edge $C(n_i, n_{i+1})$ passes through the points of both its endpoints: through q_i as body i's second hinge, and through q_{i+1} as body $i+1$'s first hinge (its first incident hinge is $C(n_i, n_{i+1})$ since $(i+1)-1 = i$). The m cyclic meets are linearly independent (Lemma 23.62), so the framework is infinitesimally rigid on the cycle's bodies, which are all of $V(G)$ (Lemma 23.77). □

The coplanar realization below is its corollary, discarding the per-body concurrency point: a hinge-coplanar panel realization needs only the panel data, which the pencil realization above already supplies, built directly from the cyclic family

of shared panel normals rather than through the meet model of the original cycle realization.

Lemma 29.10 (Coplanar realization of a cycle)**.** *Let G be presented as a cycle of m bodies (Definition 21.64) with $3 \le m \le 4$. Then G has a hinge-coplanar panel realization (Definition 23.63) that is infinitesimally rigid on all of its bodies $V(G)$ (Definition 19.15).*

Proof. Immediate corollary of the pencil realization of the cycle (Lemma 29.9), discarding the per-body concurrency point: a pencil realization is in particular hinge-coplanar, and rigidity is unaffected by dropping a conjunct. □

29.6. **The two-pencil extension step.** Katoh–Tanigawa's induction extends a hinge-coplanar realization across a re-added edge by supplying a supporting extensor in the meet of the two endpoint panels (p. 670): for any two normals the common perp $n_u^\perp \cap n_v^\perp \subseteq K^4$ has dimension at least $4 - 2 = 2$, so it contains a nonzero hinge lying in both panels, with no transversality of the normals needed. On the pencil condition the re-added hinge must additionally pass through the concurrency point of each endpoint. Where in the coplanar model a common hinge always exists, the pencil model asks for one through two prescribed points as well, and this constrains the two bodies: a hinge lying in both panels and passing through both points confines each point to the common perp, so each concurrency point must lie not only in its own panel but in the other body's panel too. The following existence step assumes exactly these two *cross-incidences*.

Lemma 29.11 (Two-pencil extension hinge)**.** *Let $n_u, n_v \in K^4$ be panel normals and $q_u, q_v \in K^4$ concurrency points with q_u incident to its own panel ($q_u \cdot n_u = 0$) and q_v to its own ($q_v \cdot n_v = 0$), and suppose $q_u \neq 0$. If the two cross-incidences $q_u \cdot n_v = 0$ and $q_v \cdot n_u = 0$ hold — each concurrency point lies in the other body's panel — then there is a nonzero $C \in \bigwedge^2 K^4$ lying in both panels (Definition 23.24) and passing through both points (Definition 29.1).*

Proof. Under the four incidences both points lie in the common perp $W = n_u^\perp \cap n_v^\perp$, which has dimension at least 2 in K^4 regardless of whether the normals are independent. It suffices to exhibit a 2-dimensional subspace of W containing both points; its extensor is then a nonzero hinge in both panels through both points. If q_u and q_v are linearly independent their span is such a subspace. Otherwise q_v lies on the line through q_u, and since W has dimension at least 2 while that line has dimension 1, W contains a vector w independent from q_u; the span of q_u and w is a 2-dimensional subspace of W containing q_u, hence also q_v. □

The two cross-incidences are not only sufficient but necessary, so the extension hinge exists precisely when the two bodies' concurrency points are mutually panel-incident.

Lemma 29.12 (Two-pencil extension biconditional)**.** *With the hypotheses of Lemma 29.11 — the points q_u, q_v incident to their own panels and $q_u \neq 0$ — the two cross-incidences $q_u \cdot n_v = 0$ and $q_v \cdot n_u = 0$ hold if and only if there is a nonzero screw element lying in both panels (Definition 23.24) and passing through both points (Definition 29.1).*

Proof. The forward implication is Lemma 29.11. For the converse, suppose such a nonzero hinge C exists. Two point pairs with the same nonzero 2-extensor span the same plane (Lemma 22.18), the converse of the point-pair proportionality of Lemma 22.16. So the points through which C passes and the points spanning it inside a panel span one and the same plane; since C passes through q_u and lies in the panel of n_v, the point q_u lies in a span of vectors orthogonal to n_v, giving $q_u \cdot n_v = 0$, and symmetrically $q_v \cdot n_u = 0$. □

These cross-incidences are the chapter's first condition that the coplanar model's extension step does not need, and three later arguments turn on them. Katoh–Tanigawa reach a graph that is not minimal by adding edges to a realization of a minimal spanning subgraph. For pencil realizations the cross-incidences of the added edges cannot always be met, so we run the induction on every multigraph (Section 29.7). The cut-edge case meets the cross-incidences of its crossing edge by first repositioning one side along a projective automorphism (Lemma 29.18). At K_4 the necessity direction of Lemma 29.12 forces every vertex's panel normal into a single hyperplane, so no simple realization of K_4 can give all four vertices independent panel normals (Section 29.8).

29.7. **Reduction on all spanning multigraphs.** This subsection states the induction behind Theorem 29.34 and proves three of its five cases. Katoh–Tanigawa's induction runs over minimal graphs, those from which no edge can be deleted without changing the deficiency (Theorem 21.56). They reach any other graph by realizing a minimal spanning subgraph with the same deficiency and adding the remaining edges, each with a hinge in the meet of its two panels (Theorem 23.66). The added edges need not raise the rank, since the subgraph already attains the deficiency rank.

For pencil realizations this step fails, and the failure lies in the smaller graph. An added edge needs its two cross-incidences (Lemma 29.12), and imposing them can lower the rank of the smaller graph. Take K_4 and a 4-cycle in it, both of deficiency 0. A pencil realization of the 4-cycle has rank 14 plus the dimension of the span of its four hinges, so it attains its deficiency rank 18 only when the hinges are linearly independent. The cross-incidences of the two diagonals, with those the cycle edges satisfy already, put all four points in every panel, so the points span at most a plane. If they span a plane, every panel is that plane, and the hinges lie in the three-dimensional space of its lines. If they span a line, at least two edges of the cycle join bodies with different points, and both of their hinges are that line. If they coincide, all four hinges pass through one point. In each case the hinges span at most three dimensions, so no pencil realization of the 4-cycle at its deficiency rank admits the diagonals.

We therefore run the induction on every multigraph. Its case analysis still needs a vertex of degree two when there is no proper rigid subgraph. Katoh–Tanigawa find one by an edge count that uses minimality (KT Lemmas 4.5 and 4.6, pp. 663–664). Here a counting fact takes its place: a loopless multigraph whose vertices all have degree at least three has a proper rigid subgraph (Lemma 29.14). So a graph without one has a vertex of degree at most two, and two-edge-connectivity forces that degree to be exactly two. The statement the induction proves is the following.

Definition 29.13 (Pencil realization at the deficiency rank)**.** A *pencil realization at the deficiency rank* of a multigraph G is a pencil realization (F, n, q) of G (Definition 29.2) whose rigidity rows span dimension

$$\dim \operatorname{span}(\text{rigidity rows of } F) = D(|V(G)| - 1) - \operatorname{def}(\tilde{G}),$$

the target of Theorem 23.66 restricted to the pencil condition.

Lemma 29.14 (Proper rigid subgraph from a minimum degree of three)**.** *Let G be a loopless multigraph with $|V(G)| \geq 3$ and $D = \binom{d+1}{2} \geq 4$. If every vertex of G has degree at least three, G has a proper rigid subgraph.*

Proof. Let v be a vertex of minimum degree $\delta \geq 3$. The handshake identity gives $\delta\,|V(G)| \leq 2\,|E(G)|$, so the edges avoiding v number $|E(G)| - \delta \geq \delta(|V(G)| - 2)/2$, and the $(D-1)$-fold fiber of these edges in $M(\tilde{G})$ has size $(D-1)(|E(G)| - \delta) > D(|V(G)| - 2)$ once $D \geq 4$ — strictly more than the (D, D)-sparsity bound on any vertex set omitting v. The fiber is therefore dependent in $M(\tilde{G})$ and contains a circuit, whose induced subgraph is rigid (Lemma 21.2), spans at least two vertices, and omits v, hence is proper. □

Theorem 29.15 (Reduction on all spanning multigraphs; cf. KT Theorem 4.9)**.** *A property P of multigraphs that is closed under five cases holds of every nonempty multigraph. The five cases are:* (i) *G has a loop (delete it and reinsert it);* (ii) *G is loopless with $|V(G)| \leq 2$ (base case);* (iii) *G is loopless, $|V(G)| \geq 3$, and not two-edge-connected (cut-edge decomposition, Definition 20.20);* (iv) *G is loopless, $|V(G)| \geq 3$, and has a proper rigid subgraph (Definition 20.17);* (v) *G is loopless, two-edge-connected, $|V(G)| \geq 3$, has no proper rigid subgraph, and (by Lemma 29.14) some vertex has degree exactly two. Case (i) is handed the property at every nonempty multigraph smaller than G in the lexicographic order on $(|V(G)|, |E(G)|)$. Cases (iii), (iv) and (v) are handed it at every nonempty multigraph on fewer vertices than G.*

Proof. Strong induction on $(|V(G)|, |E(G)|)$, lexicographically. Its hypothesis at G gives the property at every smaller nonempty multigraph, in particular at every one on fewer vertices. Deleting a loop keeps $|V(G)|$ and lowers $|E(G)|$, so it gives a smaller multigraph. Case (v)'s degree-two vertex follows from Lemma 29.14 (no proper rigid subgraph forces a vertex of degree at most two) and two-edge-connectivity (degree at least two, Lemma 20.21). □

We take the five cases in the order of the lemmas below. The loop case (i) is Lemma 29.16. The cut-edge case (iii) is Lemma 29.19, after two lemmas on projective automorphisms. The base case (ii) is Lemma 29.21. The contraction and split cases, (iv) and (v), are left for later subsections. Section 29.8 strengthens the statement that the induction proves. Section 29.9 then proves both cases by one argument, which rests on Section 30. That argument and Section 30 contract proper rigid subgraphs, using Lemma 29.22: contraction leaves the deficiency unchanged.

Lemma 29.16 (The loop case)**.** *Let e be a loop of G at a vertex v. If $G - e$ has a pencil realization at the deficiency rank (Definition 29.13), then so does G.*

Proof. Keep the realization of $G - e$, and give e a nonzero hinge that lies in the panel of v and passes through the point of v. It exists by Lemma 29.11 with $u = v$, where both cross-incidences become the incidence of the point of v with its own panel. The loop adds only zero rows, since both of its ends are v. A loop crosses no partition of $V(G)$, so deleting it leaves the deficiency unchanged, and the rank is still the deficiency rank. □

The cut-edge case moves one side of a realization by a projective automorphism. The next lemma shows that such a move keeps a pencil realization at the deficiency rank, and the one after it chooses the automorphism.

Lemma 29.17 (Projective transport of a pencil realization)**.** *Let g be a linear automorphism of K^4 and h a contragredient of it, meaning $\langle gx, hy\rangle = \langle x, y\rangle$ for the standard pairing. Transporting a pencil panel realization (F, n, q) by acting on hinges through the induced automorphism of the screw space, on the concurrency points q by g, and on the panel normals n by h, again gives a pencil panel realization on the same multigraph. Its rigidity-row span has the same dimension, so a pencil realization at the deficiency rank transports to one.*

Proof. Panel containment and through-point incidence transport because the induced screw automorphism sends the extensor of spanning points $p_0, \dots$ to the extensor of $gp_0, \dots$; the contragredient identity turns each incidence $\langle p_i, n\rangle = 0$ into $\langle gp_i, hn\rangle = 0$, and g sends the span containing q to the one containing gq. The panel–point incidence $\langle q, n\rangle = 0$ is likewise preserved, and the automorphisms send nonzero data to nonzero data. Rank invariance is the change-of-screw-coordinates identity (Lemma 28.35): this lemma is the collineation companion of Lemma 29.5, whose polarity is instead a correlation, exchanging the normal and point roles rather than repositioning within them; both rest on the same field-general screw-space automorphism mechanism (Theorem 25.7). □

Lemma 29.18 (Existence of a repositioning automorphism)**.** *Fix the panel/point data of the two sides at a crossing edge uv: normal $n_1(u)$ and point $pt_1(u) \neq 0$ on the V_1 side, normal $n_2(v)$ and point $pt_2(v) \neq 0$ on the V_2 side. Then there is a linear automorphism g of K^4 with a contragredient h ($\langle gx, hy\rangle = \langle x, y\rangle$ for the standard pairing) meeting the two cross-incidences of Lemma 29.12 once the V_2 side is transported by (g, h): $\langle pt_1(u), h\, n_2(v)\rangle = 0$ and $\langle g\, pt_2(v), n_1(u)\rangle = 0$; moreover the transported point $g\, pt_2(v)$ is projectively distinct from $pt_1(u)$, which the construction supplies for free.*

Proof. Explicit, over any field. Every automorphism of K^4 has a contragredient, the inverse-transpose. Pick $a \in n_1(u)^\perp$ with $\{a, pt_1(u)\}$ independent and $b \in n_2(v)^\perp$ with $\{b, pt_2(v)\}$ independent, possible because each hyperplane $\perp$ is at least 3-dimensional and so is not contained in a line. Let g be the frame automorphism sending $pt_2(v) \mapsto a$ and $b \mapsto pt_1(u)$. Then $g\, pt_2(v) = a \in n_1(u)^\perp$ is the second incidence, and $g\, b = pt_1(u)$ with the contragredient identity turns the first into $b \in n_2(v)^\perp$. The distinctness clause is the independence of $\{a, pt_1(u)\}$ chosen at the start, read through $g\, pt_2(v) = a$. □

Lemma 29.19 (The cut-edge case)**.** *Let G be a multigraph that is not two-edge-connected, so that at most one edge crosses some nonempty proper subset $V_1 \subsetneq V(G)$, and let $V_2 = V(G) \setminus V_1$. If $G[V_1]$ and $G[V_2]$ each have a pencil realization at the deficiency rank, then so does G. The same holds for realizations at the deficiency rank in which the two points of every link are projectively distinct.*

Proof. Realize the two sides separately. If no edge crosses, together they realize G. Suppose one edge uv crosses, with $u \in V_1$. The two realizations were chosen independently, so the cross-incidences of uv need not hold, and without them uv has no hinge (Lemma 29.12). So we first move the side V_2 by the projective automorphism of Lemma 29.18, after which both cross-incidences hold. The moved side is still a pencil realization at the deficiency rank (Lemma 29.17). Now Lemma 29.11 gives uv a hinge. In both cases the ranks of the parts add up to the deficiency rank of G, as in Katoh–Tanigawa's cut-edge case (Lemma 23.27). For the second form, the points along a link inside a side stay distinct under the automorphism, and at uv they are distinct by the last clause of Lemma 29.18. □

Formalization note 29.20. The two forms are one Lean theorem whose parameter selects the form; the pinned names are that theorem and its two specializations.

Lemma 29.21 (The base case, $|V(G)| \leq 2$)**.** *Every loopless multigraph with at most two vertices has a pencil realization at the deficiency rank (Definition 29.13).*

Proof. Give every body the same normal n and the same point q of its panel. With no edge the deficiency rank is 0. With one edge it is $D - 1$, which one nonzero hinge in the panel through q attains. With a class of two or more parallel edges it is D. Two edges of the class get the two independent hinges of Lemma 29.6, which make the two bodies rigid (Lemma 23.30), and every further edge reuses the second hinge. □

Lemma 29.22 (Deficiency is unchanged by contracting a proper rigid subgraph)**.** *Let H be a proper rigid subgraph of G. Contracting H to a single vertex leaves the deficiency unchanged,* $\mathrm{def}(G/\widetilde{E(H)}) = \mathrm{def}(\tilde{G})$*, without assuming G minimal.*

Proof. The matroid-contraction rank identity and the rank/deficiency bridge behind Lemma 21.6 already use no minimality; only the base/fiber-meeting half of the corresponding minimality-tracking statement did, and that half is not needed here. □

29.8. **Nondegenerate realizations and the conditioned pair.** Katoh and Tanigawa's induction statement asks for more than the rank. When the graph is simple, their realization must also be nonparallel, that is, no two of its panels are parallel (KT p. 668; KT Theorem 5.5, p. 670). This subsection defines a pencil analogue, the nondegenerate pencil realization, and strengthens the statement of the induction to include it. The strengthened statement, the conditioned pair, asks for a nondegenerate realization at the deficiency rank under two conditions on the graph. We show that neither condition can be dropped. The subsection ends with the reduction of Theorem 29.15 for the conditioned pair. Throughout, $k = 2$, so panel normals and concurrency points lie in K^4.

The pencil condition constrains the hinges at a body only when the body has three or more of them (Section 29.2). The next definition calls such a body a hub. It asks for independence among the normals and among the points around each body.

Definition 29.23 (Nondegenerate pencil realization, feasibility)**.** A vertex of a multigraph is a *pencil hub* when it has degree at least three. The *closed hub-neighbourhood* of a vertex v is the set of pencil hubs among v itself and its neighbours, and the *closed neighbourhood* of v is v together with all of its neighbours, hub or not. A pencil realization of G (Definition 29.2) is *nondegenerate* when the two concurrency points of every link are projectively distinct; at every vertex, the panel normals of its closed hub-neighbourhood are linearly independent; and at every non-hub vertex, the concurrency points of its closed neighbourhood are linearly independent. A multigraph is *nondegeneracy-feasible* when it has a nondegenerate pencil realization.

Formalization note 29.24. The five pins are the five terms defined, in order: pencil hubs, the closed hub-neighbourhood, the closed neighbourhood, nondegenerate pencil realizations and nondegeneracy-feasibility.

The first condition makes the hinge of every link the line through the points of its two ends (Lemma 22.18). The other two are limited by the incidences of a pencil realization. The hinge of an edge vw is nonzero, lies in the panel of w and passes through q_v, so $q_v \cdot n_w = 0$ (Lemma 29.12). Thus the normals of v and its neighbours lie in the three-dimensional space $q_v^\perp$, and their points lie in the panel $n_v^\perp$, also of dimension three. Hence the normals of a closed hub-neighbourhood are independent only if it has at most three members. At K_4 every body is a hub and every closed hub-neighbourhood has four members, so K_4 is not nondegeneracy-feasible. Likewise, the points of a closed neighbourhood are independent only if it has at most three members, which rules out every hub of a simple graph. The third condition is asked only at bodies that are not hubs. At such a body it follows from the first condition and $q_v \neq 0$, unless the body has two distinct neighbours. There it asks that the two hinges at the body be distinct lines. The first two conditions do not imply this. The path on three bodies has a pencil realization whose three points are distinct but collinear, and both of its hinges are then the same line.

In the next definition, generic names only the property stated, not an algebraic independence of coordinates.

Definition 29.25 (Generic pencil realization)**.** A multigraph G *has a generic pencil realization* when it has a nondegenerate pencil realization (Definition 29.23) whose rigidity rows attain the deficiency-rank target of Definition 29.13.

Keeping only the first condition of Definition 29.23 gives a weaker notion, which every simple multigraph with at least one body satisfies at the deficiency rank (Theorem 29.34). We do not add this condition to Definition 29.2, for two reasons. It would break the symmetry noted after Lemma 29.5. Also, some multigraphs would have no pencil realization at the deficiency rank. Two bodies joined by two edges have deficiency rank $D = 6$, which needs two independent hinges. At distinct points both hinges are the line through the two points, and the rank is 5.

Definition 29.26 (Adjacent-distinct pencil realization)**.** A multigraph G *has an adjacent-distinct pencil realization* when it has a pencil realization at the deficiency rank (Definition 29.13) in which the two concurrency points of every link are projectively distinct.

A generic pencil realization is adjacent-distinct, and an adjacent-distinct one is in particular a pencil realization at the deficiency rank.

The induction statement asks for a generic realization under two conditions. Simplicity alone, the condition of KT Theorem 5.5, is not enough, since K_4 is simple and not nondegeneracy-feasible. Nondegeneracy-feasibility alone is not enough either. Two bodies joined by two edges are nondegeneracy-feasible. Neither is a hub, so it suffices to take two distinct panels, two distinct points on their common line, and that line as both hinges. By the count above, none of their nondegenerate realizations attains the deficiency rank. So the statement asks for both conditions.

Definition 29.27 (The conditioned pair)**.** A multigraph G satisfies the *conditioned pair* at d when it has a pencil realization at the deficiency rank (Definition 29.13); when, if it is simple, it has an adjacent-distinct one (Definition 29.26); and when, if it is both simple and nondegeneracy-feasible (Definition 29.23), it has a generic one (Definition 29.25).

Formalization note 29.28. In the Lean, this definition and Definition 29.13, Definition 29.25 and Definition 29.26 take the dimension d of the deficiency (Definition 20.3, with $D = \binom{d+1}{2}$), which the formalization calls n, as an argument, while their realizations stay at $k = 2$, where the target has $D = 6$. The theorems that use them take $d = 3$, where the two agree.

We now prove the reduction for the conditioned pair, beginning with the three cases that Section 29.7 proves for the bare statement.

Lemma 29.29 (The conditioned pair at a loop, on at most two bodies and at a cut edge)**.** *Let K be an infinite field and G a multigraph.*

(1) If G has a loop, and every nonempty multigraph with fewer vertices than G, or with as many vertices and fewer edges, satisfies the conditioned pair at $d = 3$ (Definition 29.27), then so does G.
(2) If G is loopless and has one or two vertices, it satisfies the conditioned pair at $d = 3$.
(3) If G is loopless with $|V(G)| \geq 3$ and not two-edge-connected (Definition 20.20), and every nonempty multigraph on fewer vertices than G satisfies the conditioned pair at $d = 3$, then so does G.

Proof. In each case the bare conjunct is Lemma 29.16, Lemma 29.21 or Lemma 29.19. In case (1) the first of these applies to G with the loop deleted, which has as many vertices as G and fewer edges. The other two conjuncts are needed only when G is simple. A multigraph with a loop is not simple. A simple multigraph on at most two bodies has at most one edge, and a direct construction gives it a nondegenerate realization at the deficiency rank. At a cut edge, the adjacent-distinct conjunct is the second form of Lemma 29.19.

It remains to prove the generic conjunct in case (3). Let G be simple and nondegeneracy-feasible, with at most one edge crossing a nonempty proper subset V_1 of $V(G)$. A nondegenerate realization of G restricts to one of a subgraph, provided no body of the subgraph drops from degree three or more to degree two. So, except in the last case below, each smaller graph used is nondegeneracy-feasible, and the hypothesis of case (3) gives it a generic realization. If no edge crosses, the generic realizations of the two sides together form one of G. Suppose one edge uv crosses, with $u \in V_1$. If both sides have at least two bodies, we use each side together with the far end of uv, so that u keeps its degree in the first and v in the second. We move the second as in Lemma 29.19, by an automorphism that also avoids finitely many subspaces, and join the two along uv.

If one side is a single body, say v, we use $G-v$, and give v a point and a panel that avoid finitely many subspaces. The body v has degree one, so it is not a pencil hub, and it is the only neighbour that a body of $G-v$ loses. By Lemma 30.133(a), $G-v$ is then nondegeneracy-feasible, by a realization whose normals are independent on the closed hub-neighbourhood in G of every body. By part (b) of the same lemma, the generic realization of $G-v$ moves, at the same rank, to one whose normals are independent on these sets, as a realization of G requires. □

Theorem 29.30 (The conditioned pair from the contraction and split cases)**.** *Let K be an infinite field. Consider two kinds of loopless multigraphs G with $|V(G)| \geq 3$: those with a proper rigid subgraph (Definition 20.17), and those that are two-edge-connected, have no proper rigid subgraph and have a vertex of degree two. Suppose that each such G satisfies the conditioned pair at $d = 3$ (Definition 29.27) whenever every nonempty multigraph on fewer vertices than G does. Then every nonempty multigraph satisfies the conditioned pair at $d = 3$.*

Proof. Apply Theorem 29.15 with the conditioned pair at $d = 3$ as the property. Its cases at a loop, on at most two vertices and at a cut edge are Lemma 29.29. Its contraction and split cases are the two hypotheses. □

29.9. **Reduction to the main component.** This subsection proves the two hypotheses of Theorem 29.30, the contraction and split cases for the conditioned pair, and with them the pencil conjecture. At a simple two-edge-connected multigraph, both cases follow from two statements: the multigraph has an adjacent-distinct pencil realization at the deficiency rank, and a generic one when it is nondegeneracy-feasible. These are the main-component statements, proved in Section 30. A loopless two-edge-connected multigraph that is not simple has two parallel edges, and we contract them, as Katoh and Tanigawa do (KT Lemma 6.2).

The idea is to build pencil realizations by lifting planar pictures. Place each body v at a point $q_v = (x_v, y_v)$ of the plane, with adjacent bodies at distinct points and no closed neighbourhood on a line. Such a picture is admissible (Definition 30.1). Lift v to the point $p_v = (x_v, y_v, z_v, 1)$ of K^4 at a height z_v (Definition 30.6). Take p_v as the concurrency point of v, and the line through p_u and p_v as the hinge of an edge uv. Every hinge then passes through the points of its two ends. The hinges at v lie in a common panel exactly when the points of the closed neighbourhood of v lie in a plane. So the lifted configuration is a pencil realization when every closed

neighbourhood is coplanar. Over a fixed picture q this is a linear condition on the heights (Lemma 30.12), and the heights that satisfy it form a linear space $L(q)$, the lifting space (Definition 30.2).

Over the admissible pictures at which $\dim L(q)$ is least, these configurations form a vector bundle. Its closure is the main component of the configuration space (Remark 30.19). In Section 30 we show that, at every simple connected multigraph whose bodies all have degree at least two, a general configuration of the main component has the deficiency rank (Theorem 30.139). The proof is a strong induction on the number of bodies. A lifted configuration at the deficiency rank is an adjacent-distinct pencil realization (Lemma 30.11), which gives the first statement. The second does not follow in this way. When two pencil hubs have three common neighbours, as in $K_{2,3}$, no configuration over a general picture is nondegenerate (Lemma 30.127). We prove the second statement instead inside the induction of Theorem 29.15, from the conditioned pair at smaller multigraphs (Theorem 30.137).

Definition 29.31 (The two main-component statements)**.** Let K be a field. The *distinct main-component statement* asserts that every simple two-edge-connected multigraph on at least three bodies has an adjacent-distinct pencil realization at the deficiency rank at $d = 3$ (Definition 29.26). The *generic main-component statement* asserts that every such multigraph that is nondegeneracy-feasible (Definition 29.23) has a generic pencil realization at $d = 3$ (Definition 29.25).

Lemma 29.32 (A multigraph with parallel edges; cf. KT Lemma 6.2)**.** *Let K be a field and G a loopless multigraph on at least three bodies that is not simple. If every multigraph on strictly fewer bodies satisfies the conditioned pair at $d = 3$ (Definition 29.27), then G has a pencil realization at the deficiency rank (Definition 29.13).*

Proof. We follow the proof of KT Lemma 6.2 [KT11, p. 673], as in Lemma 23.102, for pencil realizations and without minimality.

Let e and f be parallel edges joining bodies a and b. Their $(D-1)$-fold copies are $2(D-1) \geq D$ parallel edges on $\{a, b\}$, hence D edge-disjoint spanning trees, so e and f span a rigid subgraph H on $\{a, b\}$, as in KT Lemma 6.2 (p. 673). It is proper, as G has at least three bodies. Contracting H removes one body and leaves the deficiency unchanged (Lemma 29.22). So by the induction hypothesis the contraction G/H has a pencil realization at the deficiency rank. Give a and b the panel and the point of the contracted body, and keep every other hinge. Then realize e and f by two linearly independent lines through that point, inside that panel. The point is fixed by the realization of the contraction, and two such lines exist through any point of a panel (Lemma 29.7). Every hinge now lies in the panels and passes through the points of both of its ends, so this is a pencil realization of G.

It remains to compute the rank. The two independent hinges make $\{a, b\}$ infinitesimally rigid (Lemma 23.30). As in Katoh and Tanigawa's proof (KT eqs. (6.3)–(6.5), pp. 673–674), the rigidity matrix is block triangular. So the rows of e and f add $D = 6$ to the rank of the contraction's realization (Lemma 19.33), and the rank is at least

$$D(|V(G/H)| - 1) - \operatorname{def}(\tilde{G}) + D \;=\; D(|V(G)| - 1) - \operatorname{def}(\tilde{G}).$$

No pencil realization of G has a larger rank (Lemma 20.16), so this is the deficiency rank. □

Theorem 29.33 (Pencil realization from the two main-component statements)**.** *Let K be an infinite field. If the distinct and the generic main-component statements (Definition 29.31) hold, then every multigraph on the whole ambient body set satisfies the conditioned pair at $d = 3$.*

Proof. By Theorem 29.30, it suffices to prove the contraction and split cases for the conditioned pair, and one argument proves both. Let G be a loopless multigraph on at least three bodies, and suppose that every nonempty multigraph on fewer bodies satisfies the conditioned pair. There are three cases.

- If G is not two-edge-connected, it satisfies the conditioned pair by the cut-edge case of Lemma 29.29.
- If G is two-edge-connected and simple, the two statements give its adjacent-distinct and generic conjuncts. An adjacent-distinct realization is in particular one at the deficiency rank, which is the bare conjunct.
- If G is two-edge-connected and not simple, its adjacent-distinct and generic conjuncts are vacuous, and its bare conjunct follows from Lemma 29.32.

□

Theorem 29.34 (The pencil realization theorem)**.** *Let K be an infinite field. Every multigraph on the whole ambient body set satisfies the conditioned pair at $d = 3$ (Definition 29.27): it has a pencil realization at the deficiency rank, with adjacent concurrency points distinct when it is simple, and a generic one when it is also nondegeneracy-feasible. More generally, so does every multigraph with at least one body.*

Proof. The two main-component statements hold over every infinite field (Theorem 30.139). So the first claim follows from Theorem 29.33. The second claim is Theorem 30.138, on which the proof of the generic statement rests. □

30. The main component of the pencil configuration space

This section proves the two main-component statements over every infinite field (Theorem 30.139). The first asks that every simple two-edge-connected multigraph on at least three bodies have an adjacent-distinct pencil realization at the deficiency rank (Definition 29.31). The second asks such a multigraph for a generic pencil realization when it is nondegeneracy-feasible. At a simple two-edge-connected multigraph, the contraction and split cases of the pencil reduction follow from the two statements. So this section completes the proof of the pencil realization theorem (Theorem 29.34). Section 29.9 derives the two cases from the statements, and sketches the idea of this section: to lift planar pictures of the graph to configurations of points in K^4. It also explains, at $K_{2,3}$, why the generic statement needs a further argument. Besides Section 29, the reader needs the deficiency and the merged deficiency of Section 20, the rank-addition lemmas of Section 19.3, and two rank lemmas of Case I of the algebraic induction (Lemma 23.86, Lemma 23.93). We also use splitting off and contraction (Definition 21.10, Definition 21.3).

We fix the following notation and terms.

- $\mathrm{def}_2(G)$ and $\mathrm{def}_3(G)$ are the deficiencies of Definition 20.3 at $d = 2$ and $d = 3$ ($D = 3$ and $D = 6$), so that $\mathrm{def}_3(G) = \mathrm{def}(\tilde{G})$. We call $\mathrm{def}_2(G)$ the *planar deficiency* of G.
- A graph satisfies the standing hypotheses (Definition 30.45) if it is finite, simple and connected, with minimum degree at least two.
- The *target* of G is its deficiency rank $\mathrm{tgt}(G) = 6(|V(G)| - 1) - \mathrm{def}(\tilde{G})$ (Definition 29.13, at $D = 6$). We say that G *has its target* at a configuration of rank $\mathrm{tgt}(G)$ (Definition 30.6).
- $L_G(q)$ is the lifting space of G at a picture q (Definition 30.2), written $L(q)$ when G is clear. A configuration (q, z) with q admissible and $z \in L_G(q)$ is *attaining* if G has its target at it, and z is then an *attaining height*.
- The configurations at which the points of every closed neighbourhood lie in a plane form the *pencil configuration space* of G. Over an admissible picture q, they are the configurations (q, z) with $z \in L(q)$ (Lemma 30.12). The *main component* is the closure of the part of this space over the main pictures (Definition 30.18). We never form it (Remark 30.19).
- *General* means off the zero set of a nonzero polynomial: in the picture coordinates, and over each picture in the heights. The general configuration of G attains (Definition 30.8) if a general configuration (q, z) with $z \in L(q)$ is attaining.
- A *flat configuration* is a configuration $(q, 0)$, with all heights zero.
- A set W of at least two bodies is *rigid* if $\mathrm{def}_3(G[W]) = 0$, that is, if $G[W]$ is a rigid subgraph (Definition 20.17). It is *planar-rigid* if $\mathrm{def}_2(G[W]) = 0$.
- A contraction step contracts a set $W \subsetneq V(G)$ of at least two bodies to one of its bodies. We call W, and the subgraph $H = G[W]$ that it induces, the *core*, and write G/H for the contraction (Definition 21.3). The core is *additive* if it is rigid and $\mathrm{def}_2(H) + \mathrm{def}_2(G/H) \leq \mathrm{def}_2(G)$. This core is not the core of Definition 27.40, the largest subgraph of minimum degree at least two: a graph that satisfies the standing hypotheses is its own core in that sense (Lemma 27.42).
- The *value* of a partition P of the bodies is its D-deficiency $\mathrm{def}_{\tilde{G}}(P)$ (Definition 20.3). The *singleton value* of a set X of bodies is $s_D(X) = D(|X| - 1) - (D - 1)\, e(X)$, the value of the partition of $G[X]$ into singletons. Here $e(X)$ is the number of edges joining two distinct bodies of X, and $s = s_3$ is the case $d = 2$.
- A graph is *smaller than* G if it satisfies the standing hypotheses and has fewer bodies than G. The *structural coverage* is the case analysis by which every graph satisfying the standing hypotheses admits a step of the induction at graphs smaller than it (Theorem 30.124).
- Let $a - x_1 - \cdots - x_k - b$ be an open ear of G on V_1 (Definition 30.63), and take a configuration of G whose hinges on the ear are nonzero. Then Λ is the span of the hinges of the ear, and ρ is the space of relative screws of a

and b in $G[V_1]$ (Definition 19.27). The inequality $(\star)$ is

$$\dim(\rho + \Lambda) \;\geq\; k + 1 + \operatorname{def}(\tilde{G}[V_1]) - \operatorname{def}(\tilde{G}).$$

Where $G[V_1]$ has its target, $(\star)$ gives G rank at least $\operatorname{tgt}(G)$ (Section 30.7). In the contraction step, ρ is instead the map of Lemma 30.57, which takes a vector (z, h) of unknowns of the rescaled lifting system to the height that agrees with z on the core and vanishes elsewhere.

- The *flat coordinates* $(\sigma(c), \tau(c)) \in K^3 \times K^3$ of a screw $c \in \bigwedge^2 K^4$ are its image under the isomorphism of Lemma 30.66.

Section 30.1 defines the configurations and proves two lemmas used throughout. If one configuration (q, z) with q a main picture and $z \in L(q)$ has rank $\operatorname{tgt}(G)$, then the general configuration of G attains (Lemma 30.20). Two nonempty Zariski-open subsets of a linear space meet (Lemma 30.26). Section 30.2 computes the rank of a flat configuration. Section 30.3 derives Jackson and Jordán's equality [JJ08b] for the dimension of the lifting space from the case $d = 2$ of KT Theorem 5.6 (Theorem 23.54). The equality then holds over every infinite field. With the flat rank, it settles the graphs whose two deficiencies agree (Corollary 30.44).

The general configuration of every graph satisfying the standing hypotheses attains. We prove this by strong induction on the number of bodies (Theorem 30.113). Each step assumes that the general configuration attains at none, one or two graphs with fewer bodies, and proves it at G. Every step ends with Lemma 30.20. Most steps also use Lemma 30.26 to find the heights of the one configuration they need. Section 30.4 describes the scheme that the steps share, and proves the steps at a cut vertex and along a chain of bridges. The five subsections after it prove the other steps:

- contraction at an additive core, which is KT Lemma 6.3 read on configurations, and at a core of planar deficiency zero as its corollary (Section 30.5);
- a cycle, and an open ear with at least five interior bodies (Section 30.6);
- an open ear with two, three or four interior bodies (Section 30.7);
- an open ear with one or two interior bodies whose ends are not adjacent (Section 30.8);
- splitting off a body of degree two whose neighbours are not adjacent (Section 30.9).

Section 30.10 proves the estimates on partitions that the structural coverage uses. Section 30.11 gives the coverage, which completes the induction (Theorem 30.124).

Section 30.12 proves the two main-component statements. The first follows from the induction, as an attaining configuration is an adjacent-distinct pencil realization (Lemma 30.125). The second is proved inside the induction of the pencil reduction (Theorem 29.15). Its step at G may assume the conditioned pair at every multigraph with fewer bodies, at least one (Theorem 30.137). This induction gives the conditioned pair at every multigraph with at least one body (Theorem 30.138), and the second statement with it.

30.1. **Planar pictures, the lifting space, and the main component.** This subsection introduces the configurations of the induction, and proves the lemma

that each step of the induction uses (Lemma 30.20). Throughout, K is an infinite field and G satisfies the standing hypotheses (Definition 30.45): it is finite, simple and connected, with minimum degree at least two. In a simple graph a body has as many distinct neighbours as its degree, so every closed neighbourhood of G has at least three members.

The idea is to lift a picture of G in the plane, as in scene analysis. The picture places each body v at a point $q_v = (x_v, y_v)$ of K^2, and a height z_v lifts it to the point $p_v = (x_v, y_v, z_v, 1)$, in homogeneous coordinates. The picture and the heights form a configuration. Take the points p_v as concurrency points and the joins $p_u \vee p_v$ as hinges. Then the configuration is a pencil realization when the points of every closed neighbourhood lie in a plane (Section 29.9). We draw the picture with adjacent bodies at distinct points and with no closed neighbourhood on a line; such a picture is admissible. The first condition makes every hinge nonzero. By the second, a plane through the points of a closed neighbourhood is not vertical, so it is the graph $\zeta = a\xi + b\eta + c$ of an affine function of (ξ, η). The closed neighbourhood is therefore coplanar exactly when the heights on it agree with an affine function of the picture (Lemma 30.12). This condition is linear in the heights. The heights that satisfy it at every body form the lifting space $L(q)$. In the language of scene analysis, $L(q)$ is the space of heights of the liftings of q, for the incidence structure whose faces are the closed neighbourhoods. The heights of one affine function at every body are the trivial liftings, which put all the faces in one plane (see Whiteley [Whi96], §8.3).

The rest of the subsection shows that one configuration can stand for the general one. A configuration over an admissible picture with $z \in L(q)$ gives a pencil realization (Lemma 30.11). The general configuration of G attains when, at a general picture and a general height in $L(q)$, this realization has the deficiency rank (Definition 30.8). The main pictures are the admissible pictures at which $\dim L(q)$ is least. If one configuration over a main picture has the deficiency rank, then the general configuration attains (Lemma 30.20). We extend its height to heights $Z(q) \in L(q)$ at general pictures q, polynomial in q, and use that the rank is lower semicontinuous. The configurations over the main pictures form a vector bundle, whose closure is the main component of the configuration space (Remark 30.19). Most steps of the induction also need one point of a linear space, such as $L(q)$, off the zero sets of two polynomials that are each nonzero somewhere on it. Such a point exists, as two nonempty Zariski-open subsets of a linear space meet (Lemma 30.26).

Definition 30.1 (Admissible planar picture)**.** A *planar picture* of G is a map $q\colon V(G) \to K^2$, written $q_v = (x_v, y_v)$; its *homogeneous picture points* are the vectors $(x_v, y_v, 1) \in K^3$. The picture is *admissible* if $q_u \neq q_v$ for every edge uv and, for every body v, three members of the closed neighbourhood of v (Definition 29.23) have linearly independent homogeneous picture points, that is, the points q_w of the closed neighbourhood do not lie on one line.

Both conditions are Zariski-open in the coordinates of the picture: $q_u \neq q_v$ fails only where $x_u - x_v$ and $y_u - y_v$ both vanish, and three picture points are collinear exactly where the determinant of their homogeneous picture points vanishes. So the admissible pictures form a Zariski-open subset of $K^{2|V(G)|}$. It is nonempty. As K is

infinite, we can place the bodies at distinct points (t, t^2) of a parabola, and no three of these points are collinear.

Definition 30.2 (The lifting space)**.** Let q be a planar picture of G. A *height* is a map $z\colon V(G) \to K$. The *lifting space* $L(q)$ is the set of heights z such that, for every body v, the restriction of z to the closed neighbourhood of v agrees with an affine function of the picture there: there are $a, b, c \in K$ with $z_w = ax_w + by_w + c$ for every w in that neighbourhood. The space $\mathrm{Aff}(q)$ of *globally affine* heights is the subspace of heights $z_w = \alpha x_w + \beta y_w + \gamma$ for a single triple $(\alpha, \beta, \gamma) \in K^3$, the same at every body.

The conditions are linear in z, so $L(q)$ is a linear subspace of $K^{V(G)}$.

Lemma 30.3 (The lifting space contains the affine heights)**.** *Let q be a picture of G. Then $\mathrm{Aff}(q) \subseteq L(q)$. If q is admissible and G has at least one body, then $\mathrm{Aff}(q)$ is three-dimensional, so $\dim L(q) \geq 3$.*

Proof. A height in $\mathrm{Aff}(q)$ agrees with one affine function of the picture at every body, and so on every closed neighbourhood; it lies in $L(q)$. The map $(\alpha, \beta, \gamma) \mapsto (\alpha x_w + \beta y_w + \gamma)_w$ is linear, with image $\mathrm{Aff}(q)$. At an admissible picture it is injective, since an affine function that vanishes at three non-collinear picture points of a closed neighbourhood is zero. So $\dim \mathrm{Aff}(q) = 3$, and $\dim L(q) \geq 3$ by the containment. □

Formalization note 30.4. In the formalization a height is a function on the whole ambient set of bodies that vanishes off $V(G)$. So $\dim L(q)$ is the dimension of a space of heights on $V(G)$, also when G does not contain every body.

The steps of the induction use one picture both for G and for smaller graphs on some of its bodies, and some steps move the picture at a body outside a smaller graph. A picture of a larger set of bodies is read as a picture of G by restriction to $V(G)$.

Lemma 30.5 (Admissibility and the lifting space read only the bodies of the graph)**.** *Let q and q' be planar pictures of a set of bodies containing $V(G)$ that agree at every body of G. Then q is admissible for G if and only if q' is, and $L_G(q) = L_G(q')$.*

Proof. Admissibility reads the picture at the two ends of each edge and on each closed neighbourhood, and the lifting space on each closed neighbourhood; all of these consist of bodies of G. □

Definition 30.6 (Configurations and their planes)**.** A planar picture q and a height z give the *configuration* $p = (q, z)$ of homogeneous points $p_v = (x_v, y_v, z_v, 1) \in K^4$. Given a choice of three bodies a_v, b_v, c_v at every body v, the *plane normal* of p at v is the vector $n_v \in K^4$ with $n_v \cdot w = \det[p_{a_v}, p_{b_v}, p_{c_v}, w]$ for every $w \in K^4$.

In the pencil realization of a configuration, the panel of a body v is the plane through the points of its closed neighbourhood. A normal of that plane is determined only up to a scalar, and we fix one by computing it from three of the points. The plane normal n_v is orthogonal to p_{a_v}, p_{b_v} and p_{c_v}, as a determinant with two equal rows vanishes. It is nonzero exactly when these three points are linearly independent, and then $n_v^\perp$ is the plane through them. The coordinates of n_v are polynomials in those of q and z.

Lemma 30.7 (The plane through three points of a closed neighbourhood contains all of it)**.** *Let q be a picture, $z \in L(q)$, and v a body of G whose three chosen bodies a_v, b_v, c_v lie in the closed neighbourhood of v. Then the plane normal n_v is orthogonal to p_w for every w in the closed neighbourhood of v.*

Proof. As $z \in L(q)$, there are $a, b, c \in K$ with $z_w = ax_w + by_w + c$ for every w in the closed neighbourhood of v. For each such w, the point p_w is therefore the image of the homogeneous picture point $(x_w, y_w, 1)$ under the linear map $(\xi, \eta, \omega) \mapsto (\xi, \eta, a\xi + b\eta + c\omega, \omega)$ from K^3 to K^4. The four points $p_{a_v}, p_{b_v}, p_{c_v}, p_w$ lie in its image, of dimension at most three. So they are linearly dependent, and $n_v \cdot p_w = \det[p_{a_v}, p_{b_v}, p_{c_v}, p_w]$ vanishes. □

Definition 30.8 (The general configuration attains)**.** The *rank* of a configuration $p = (q, z)$ of G is the rank of the rigidity matrix (Definition 19.7) of the panel-hinge framework on G whose panel normal at each body v is p_v (Definition 23.5). The *general configuration of G attains* if there is a nonzero polynomial P in the picture coordinates such that every picture q with $P(q) \neq 0$ is admissible and the heights $z \in L(q)$ at which (q, z) has rank $6(|V(G)| - 1) - \mathrm{def}(\tilde{G})$ (Definition 20.3, at $d = 3$) contain a nonempty Zariski-open subset of $L(q)$.

The panel normals in this definition are the points p_v, not the plane normals n_v. With the points, the hinge at an edge uv is the meet of the planes $p_u^\perp$ and $p_v^\perp$ (Definition 23.1). Under the polarity it corresponds to the line $p_u \vee p_v$, the hinge of the pencil realization, and it is nonzero whenever $q_u \neq q_v$ (Lemma 30.9). With the plane normals, the hinge would be zero wherever two adjacent planes coincide, and a zero hinge allows its two bodies no relative motion.

Over an infinite field a nonzero polynomial has a non-root, so the pictures q with $P(q) \neq 0$ form a nonempty Zariski-open set. Over a finite field a nonzero polynomial can vanish at every point, and the definition then holds vacuously.

Lemma 30.9 (The rank of a configuration is the rank of the joins of its points)**.** *Let $p = (q, z)$ be a configuration of G, and let F_p be the body-hinge framework on G (Definition 17.2) whose hinge at each edge uv is the line $p_u \vee p_v$ (Definition 18.3). Then the rigidity matrix of F_p has the rank of p (Definition 30.8). If q is admissible, every hinge of F_p is nonzero.*

Proof. The hinge at uv of the panel-hinge framework that defines the rank of p is the meet of the planes $p_u^\perp$ and $p_v^\perp$. It is the image of $p_u \vee p_v$ under the polarity (Lemma 25.9), an invertible linear map of the screw space over every field. So the two rigidity matrices have the same rank (Lemma 28.35). Over an admissible picture, adjacent picture points differ. A relation $\lambda\, p_u + \mu\, p_v = 0$ then has $\lambda + \mu = 0$ in the last coordinate and $\lambda\,(q_u - q_v) = 0$ in the first two, so $\lambda = \mu = 0$. Hence p_u and p_v are linearly independent, and their join is nonzero. □

Formalization note 30.10. A body-hinge framework has a hinge at every label of the ambient set of edge labels, also at labels that are not edges of G. There the formal framework F_p has one fixed nonzero line. No rigidity row reads it, and it keeps every hinge nonzero, as a pencil realization requires.

Lemma 30.11 (A configuration over an admissible picture is a pencil realization)**.** *Let q be an admissible picture of G, $z \in L(q)$ and $p = (q, z)$. Choose at every body v three members of its closed neighbourhood with non-collinear picture points, and let n_v be the resulting plane normal (Definition 30.6). Then the framework F_p of Lemma 30.9, with panel normals n_v and concurrency points p_v, is a pencil realization of G (Definition 29.2) whose adjacent points are linearly independent. If moreover p has rank $6(|V(G)|-1)-\mathrm{def}(\tilde{G})$, then G has an adjacent-distinct pencil realization (Definition 29.26, at $d = 3$).*

Proof. The three chosen configuration points at v are linearly independent, since their picture points are, so $n_v \neq 0$. And n_v is orthogonal to p_w for every w in the closed neighbourhood of v (Lemma 30.7).

The two ends of an edge uv lie in the closed neighbourhoods of both u and v. So p_u and p_v are orthogonal to n_u and to n_v: the line $p_u \vee p_v$ lies in both panels, and it passes through both points by construction. Taking $w = v$ gives the incidence $p_v \cdot n_v = 0$, and $p_v \neq 0$ since its last coordinate is 1. Adjacent points are linearly independent, and every hinge is nonzero (Lemma 30.9). By the same lemma the rank of F_p is the rank of p. □

The next lemma proves the equivalence on which the lifting space rests. It also gives the plane of each body explicitly, in the form that Lemma 30.131 uses.

Lemma 30.12 (Coplanarity is linear in the heights)**.** *Let q be admissible, z a height and $p = (q, z)$. The points of every closed neighbourhood of p lie in a plane if and only if $z \in L(q)$. In that case the plane through the points of the closed neighbourhood of each body v is unique and not vertical: if $z_w = ax_w + by_w + c$ on that neighbourhood, every nonzero vector orthogonal to its points is a multiple of $(a, b, -1, c)$.*

Proof. Suppose that $z \in L(q)$. Admissibility gives at each body v three members of its closed neighbourhood with non-collinear picture points. Their configuration points are linearly independent, as their picture points are, so their plane normal is nonzero. It is orthogonal to the points of the whole closed neighbourhood (Lemma 30.7). So these points lie in a plane.

Conversely, suppose that the points of every closed neighbourhood lie in a plane. At a body v, let $n \neq 0$ be orthogonal to them, so that $n_0x_w + n_1y_w + n_2z_w + n_3 = 0$ for every w in the closed neighbourhood of v. If n_2 were zero, (n_0, n_1, n_3) would be orthogonal to the homogeneous picture points of the three members that admissibility gives, which are linearly independent, and n would vanish. So $n_2 \neq 0$, and solving for z_w shows that z agrees with an affine function of the picture on the closed neighbourhood. Hence $z \in L(q)$.

If $z_w = ax_w + by_w + c$ on the closed neighbourhood of v, the vector $(a, b, -1, c)$ is orthogonal to its points. The points of the three members above are linearly independent, so the vectors orthogonal to them form a line. Hence every nonzero vector orthogonal to the closed neighbourhood is a multiple of $(a, b, -1, c)$, and its height coordinate is nonzero: the plane is not vertical. □

The next lemma is stated for every invertible linear map of K^4, because we need two such maps. One scales and shifts the heights (Lemma 30.14). The other is the homothety that shrinks a core along the contraction curve (Lemma 30.60).

Lemma 30.13 (A collineation of the normals preserves the rank)**.** *Let G be a graph, and let g be an invertible linear map of K^4. If panel normals p and p' in K^4 satisfy $p'_w = g(p_w)$ at every body w of G, then the rigidity matrices of the panel-hinge frameworks on G with normals p and with normals p' have the same rank.*

Proof. The hinge at an edge uv with normals p is the image of the line $p_u \vee p_v$ under the polarity (Lemma 25.9). So the rigidity matrix with normals p has the rank of the body-hinge framework whose hinge at uv is $p_u \vee p_v$, as in the proof of Lemma 30.9, and likewise for p'. Both ends of an edge of G are bodies of G. So $p'_u \vee p'_v = g(p_u) \vee g(p_v)$ is the image of $p_u \vee p_v$ under the invertible linear map of the screw space induced by g, and the two ranks agree (Lemma 28.35). □

Lemma 30.14 (Scaling and affine shifts of the heights preserve the rank)**.** *Let q be a picture of G, z a height, $t \in K$ nonzero and $a \in \mathrm{Aff}(q)$. Then the configurations (q, z) and $(q, tz + a)$ have the same rank (Definition 30.8).*

Proof. Write $a_w = \alpha x_w + \beta y_w + \gamma$. The linear map $(\xi, \eta, \zeta, \omega) \mapsto (\xi, \eta, t\zeta + \alpha\xi + \beta\eta + \gamma\omega, \omega)$ of K^4 is invertible, since $t \neq 0$. At every body w it maps the point $(x_w, y_w, z_w, 1)$ of (q, z) to the point $(x_w, y_w, tz_w + a_w, 1)$ of $(q, tz + a)$. So Lemma 30.13 applies. □

To move heights from one picture to another, we describe $L(q)$ by a linear system whose coefficients are polynomials in the picture. Besides the heights, its unknowns are the coefficients of an affine function at each body.

Definition 30.15 (The lifting system)**.** Let q be a planar picture of G. The *lifting system* $M(q)$ of G at q is the homogeneous linear system in the heights z and one coefficient vector $h_v \in K^3$ per body v that requires $z_w = h_v \cdot (x_w, y_w, 1)$ for every body v and every w in the closed neighbourhood of v. Its coefficients are polynomials in q, and the heights of its solutions form $L(q)$. If q is admissible, its solutions form a space of dimension $\dim L(q)$.

Proof. A height z lies in $L(q)$ exactly when every body v has a vector h_v with $z_w = h_v \cdot (x_w, y_w, 1)$ for every w in its closed neighbourhood (Definition 30.2). Over an admissible picture the heights of a solution determine each h_v, as three members of the closed neighbourhood of v have non-collinear picture points. So there the solutions form a space of dimension $\dim L(q)$. □

Formalization note 30.16. The three pins are the system, as a matrix of polynomials in the picture coordinates, the heights of its solutions, and the dimension of its solutions over an admissible picture. As for the lifting space (Formalization note 30.4), the formal heights and coefficient vectors are indexed by the whole ambient set of bodies, and the system requires them to vanish off $V(G)$.

Near a point, the kernel of a matrix of polynomials can only lose dimension. Where it keeps its dimension, a kernel vector at the point extends to kernel vectors

nearby that are polynomial in the parameters, and Cramer's rule writes them down. We state this in general, since the contraction step uses it for the rescaled lifting system, with the one parameter t (Definition 30.55).

Lemma 30.17 (A polynomial section of the kernel)**.** *Let $M(s)$ be a matrix whose entries are polynomials in parameters s, and let $M(s_0)\,x_0 = 0$. Then there are a polynomial D with $D(s_0) \neq 0$ and a vector $X(s)$ of polynomials with $X(s_0) = x_0$, such that* $\dim\ker M(s) \leq \dim\ker M(s_0)$ *wherever $D(s) \neq 0$, and $M(s)\,X(s) = 0$ wherever moreover* $\dim\ker M(s) \geq \dim\ker M(s_0)$.

Proof. Let π be a projection onto $\ker M(s_0)$. The stacked matrix $[M(s_0); \pi]$, with the rows of $M(s_0)$ above those of π, is injective, as π fixes every vector of $\ker M(s_0)$. Let H be a left inverse of it, and let $S(s) = H\,[M(s); \pi]$. This is a square matrix of polynomials with $S(s_0) = 1$. Put $D = \det S$. Where $D(s) \neq 0$ the stack $[M(s); \pi]$ is injective, so π maps $\ker M(s)$ injectively into $\ker M(s_0)$. If moreover $\dim\ker M(s) \geq \dim\ker M(s_0)$, this map is onto, and some $x \in \ker M(s)$ has $\pi x = x_0$. Then $S(s)\,x = H\,[0; x_0] = S(s_0)\,x_0 = x_0$. By Cramer's rule $X(s) = \operatorname{adj} S(s)\,x_0$ equals $D(s)\,x$, so it lies in $\ker M(s)$. And $X(s_0) = x_0$, as $S(s_0) = 1$. □

The proof of Lemma 30.20 extends a height in $L(q_0)$, for a picture q_0, to heights at a general picture. For this it needs $\dim L(q) \geq \dim L(q_0)$ at a general picture q, which holds when $\dim L(q_0)$ is least.

Definition 30.18 (Main pictures)**.** Let ℓ_0 be the least value of $\dim L(q)$ over admissible pictures q of G. A *main picture* of G is an admissible picture q with $\dim L(q) = \ell_0$, and U denotes the set of main pictures.

Remark 30.19 (The geometry of the main component)**.** Over the set U of main pictures, $\dim L(q)$ is constant, equal to ℓ_0. Moreover $L(q)$ is the projection to the heights of the solutions of a linear system whose coefficients are polynomials in q (Definition 30.15). So the pairs (q, z) with $q \in U$ and $z \in L(q)$ form a vector bundle over U. Its closure X_0 is the *main component* of the pencil configuration space. It is irreducible of dimension $2|V(G)| + \ell_0$, and it contains every flat configuration $(q, 0)$ with $q \in U$. We never form X_0 itself. In our statements, "general" means off the zero set of a nonzero polynomial in the picture coordinates and, over each picture, off the zero set of a nonzero polynomial in the heights (Definition 30.8). In place of the irreducibility of X_0 we use that of the space of pictures and of each fibre $L(q)$ (Lemma 30.26).

Lemma 30.20 (One attaining configuration forces the general one)**.** *Let G have at least one body, let q_0 be a main picture of G (Definition 30.18) and $z_0 \in L(q_0)$, and suppose the configuration (q_0, z_0) has rank at least $6(|V(G)| - 1) - \operatorname{def}(\tilde{G})$ (Definition 30.8). Then the general configuration of G attains.*

Proof. The lifting system. Over an admissible picture q the solutions of the lifting system $M(q)$ form a space of dimension $\dim L(q)$ (Definition 30.15). As q_0 is a main picture, this dimension is at least $\dim L(q_0)$ at every admissible picture q.

The rank. The rank of a configuration is the rank of the panel-hinge framework whose panel normals are its points. Over an admissible picture its hinges are

nonzero (Definition 23.1), since two points $(x, y, z, 1)$ over distinct picture points are linearly independent (`IsAdmissiblePicture.supportExtensor_ne_zero`). So Lemma 23.120 gives a polynomial Q in the coordinates of the points, nonzero at (q_0, z_0), such that the rank is at least $6(|V(G)| - 1) - \mathrm{def}(\tilde{G})$ wherever Q is nonzero. Wherever every hinge is nonzero, the rank is at most this value (Lemma 20.16).

The polynomials P and R. Some polynomial A in the picture coordinates, nonzero at q_0, has only admissible non-roots. For instance, take the product of one coordinate difference for each pair of adjacent bodies, in a coordinate in which they differ at q_0, and, for each body, the determinant of the homogeneous picture points of three members of its closed neighbourhood that are independent at q_0 (`IsAdmissiblePicture.exists_mvPolynomial`). Lift z_0 to a solution x_0 of $M(q_0)$, and let D and X be as in Lemma 30.17, for M at q_0 and x_0, with $Z(q)$ the heights of $X(q)$. Let P be the product of A, D and Q at the configuration $(q, Z(q))$. Then $P(q_0) \neq 0$. A picture q with $P(q) \neq 0$ is admissible, and $Z(q) \in L(q)$. For such q, let R be Q at $(q, \cdot)$, a polynomial in the heights with $R(Z(q)) \neq 0$. At every $z \in L(q)$ with $R(z) \neq 0$ the two bounds above give (q, z) the rank $6(|V(G)| - 1) - \mathrm{def}(\tilde{G})$. □

Later steps use the solutions (z, h) of the lifting system $M(q)$ (Definition 30.15). The contraction step uses the same system with other vectors in place of the homogeneous picture points $(x_w, y_w, 1)$ (Definition 30.55).

Definition 30.21 (The lifting system with weights)**.** Let ω assign a vector $\omega(v, w) \in K^3$ to every ordered pair of bodies. The *lifting system with weights* ω of G is the homogeneous linear system in the heights z and one coefficient vector $h_v \in K^3$ per body v of G requiring $z_w = h_v \cdot \omega(v, w)$ for every body v and every w in the closed neighbourhood of v.

With $\omega(v, w) = (x_w, y_w, 1)$ this is the lifting system $M(q)$ of G at q (Definition 30.15), and the heights of its solutions form $L(q)$.

Lemma 30.22 (The lifting system reads only the picture at its bodies)**.** *The solutions of the lifting system of a graph H at a picture depend only on the picture at the bodies of H.*

Proof. Each condition of the lifting system of H reads the picture at a member of the closed neighbourhood of a body of H, and these are bodies of H (as in Lemma 30.5). □

Lemma 30.23 (The main pictures are nonempty and Zariski-open)**.** *There is a nonzero polynomial P in the picture coordinates such that every picture q with $P(q) \neq 0$ is a main picture. In particular the set U of main pictures is nonempty and Zariski-open.*

Proof. Admissible pictures exist (`Graph.exists_isAdmissiblePicture`): place the bodies at distinct points of the parabola $\{(t, t^2)\}$, as after Definition 30.1. So ℓ_0 exists, and it is attained at a main picture q_0.

At any main picture q_0, apply Lemma 30.17 to the lifting system $M(q)$ (Definition 30.15) at q_0 and the zero solution. It gives a polynomial D, nonzero at q_0, such that wherever $D(q) \neq 0$ the solutions of $M(q)$ form a space of dimension at

most $\dim L(q_0)$. Let P be the product of D and a polynomial nonzero at q_0 whose non-roots are admissible. It exists since the admissible pictures form a Zariski-open set, as noted after Definition 30.1 (`IsAdmissiblePicture.exists_mvPolynomial`). At a non-root q of P, q is admissible and $\dim L(q) \leq \dim L(q_0)$, so q is a main picture, giving the openness around q_0. Taking q_0 to be the main picture of the first paragraph, $\dim L(q_0) = \ell_0$, gives the claim. □

Formalization note 30.24. In place of the standing hypotheses on G, the formal statements assume the two consequences the construction uses: G has no loops, and every closed neighbourhood has at least three members. The third pin gives the polynomial P. The first gives a main picture, and the second, at any main picture q_0, a polynomial nonzero at q_0 whose non-roots are main pictures. These are the nonemptiness and the openness of U.

Lemma 30.25 (The general configuration attaining gives an adjacent-distinct realization)**.** *Let K be infinite. If the general configuration of G attains (Definition 30.8), then G has an adjacent-distinct pencil realization (Definition 29.26, at $d = 3$).*

Proof. As K is infinite, the nonzero polynomial P of Definition 30.8 has a non-root q. This picture is admissible, and some $z \in L(q)$ gives (q, z) the rank $6(|V(G)| - 1) - \mathrm{def}(\tilde{G})$. Apply Lemma 30.11 to (q, z). □

Lemma 30.26 (Two open conditions on a fibre meet)**.** *Let K be infinite and let L be a linear subspace of K^σ for an index set σ. If two polynomials in the coordinates of σ are each nonzero somewhere on L, some single point of L makes both nonzero.*

Proof. Let $z_1, z_2 \in L$ be points at which the first and the second polynomial are nonzero. On the line $z_1 + t\,(z_2 - z_1)$, $t \in K$, each polynomial restricts to a polynomial in t. The first restriction is nonzero at $t = 0$ and the second at $t = 1$, so their product is a nonzero polynomial in one variable. As K is infinite, it has a non-root t. The point $z_1 + t\,(z_2 - z_1)$ lies in L, and both polynomials are nonzero there. □

The next theorem states the main results of this subsection together. Later subsections cite the lemmas themselves.

Theorem 30.27 (The main component)**.** *Let K be infinite, and let G have at least one body, no loops, and at least three members in every closed neighbourhood. Then the main pictures of G (Definition 30.18) contain the non-roots of a nonzero polynomial in the picture coordinates. Over any picture q, two polynomials in the heights that are each nonzero somewhere on $L(q)$ are nonzero together somewhere on $L(q)$. Moreover, if a single configuration (q, z) with q a main picture and $z \in L(q)$ has rank at least $6(|V(G)| - 1) - \mathrm{def}(\tilde{G})$, then the general configuration of G attains (Definition 30.8), and G has an adjacent-distinct pencil realization (Definition 29.26, at $d = 3$).*

Proof. The first claim is Lemma 30.23. The second is Lemma 30.26, applied to the linear subspace $L(q)$ of the heights. For the third, the general configuration attains by Lemma 30.20, and then G has an adjacent-distinct pencil realization (Lemma 30.25). □

Formalization note 30.28. The four pins are, in order, the polynomial of the first claim, the common point of the second, the attainment of the third and the adjacent-distinct realization. Each is also a pin of the lemma that the proof cites for that claim.

30.2. **The flat rank.** This subsection computes the rank of a *flat* configuration $(q, 0)$, with all heights zero, over an admissible picture q. The rank is exactly $6|V(G)| - 3 - \dim L(q)$ (Theorem 30.35). With Jackson and Jordán's equality, the next subsection derives from it that, under the standing hypotheses, the general configuration attains whenever the two deficiencies agree (Corollary 30.44). This settles one case of the induction with no appeal to smaller graphs (Section 30.11). Throughout, def_2 and def_3 are the deficiencies of Definition 20.3 at $d = 2$ and $d = 3$ ($D = 3$ and $D = 6$).

The idea is that at zero heights every hinge $p_u \vee p_v$ is a line of the plane $z = 0$. We split the screw space into the screws of the lines of this plane and a complement (Lemma 30.31). At a hinge, the screws of the two bodies may differ only by a multiple of the hinge. So the hinges glue the complementary parts of the bodies' screws together, and on a connected graph these parts are all equal. They give three dimensions of motions. The screws of the lines of the plane correspond to the affine functions on K^2, a line to the functions vanishing on it. So the rest of a motion is a family (h_v) of affine functions such that $h_u - h_v$ vanishes at q_u and at q_v for every edge uv. These families are the lifting planes (Definition 30.29). Evaluating each h_v at q_v identifies them with the lifting space $L(q)$ (Lemma 30.30), and rank–nullity gives the flat rank.

Over the reals, h_v gives the velocities normal to the plane $z = 0$ of the points of that plane moving with the body v. The planes $\zeta = h_v(\xi, \eta)$ are then the faces of a polyhedron over the picture, a lifting as in Section 30.1. Crapo and Whiteley [CW82] describe this correspondence between the motions of a flat panel structure and the polyhedra over it for the tetrahedron, and note that it holds in great generality (Example 4.4). We prove what we need directly.

The lifting planes also form the motion space of a panel-hinge framework P_q in the plane, with panel normal $(x_v, y_v, 1)$ at each body v (Lemma 30.33). A partition count on P_q bounds $\dim L(q)$ below by $3 + \mathrm{def}_2(G)$ (Lemma 30.34). So the flat rank is at most $6(|V(G)|-1)-\mathrm{def}_2(G)$, with equality where the bound on $\dim L(q)$ is attained. Jackson and Jordán's equality says that, under the standing hypotheses, it is attained at a general picture (Theorem 30.43). Since $\mathrm{def}_3(G) \le \mathrm{def}_2(G)$ (Lemma 20.7), the flat configuration has the deficiency rank $6(|V(G)| - 1) - \mathrm{def}_3(G)$ only if the two deficiencies agree. When it has, it is a witness for Lemma 30.20 (Corollary 30.36). The last corollary is the case $\dim L(q) = 3$.

Definition 30.29 (The lifting planes)**.** Let q be a planar picture of G, and identify a vector $h \in K^3$ with the affine function $(x, y) \mapsto h \cdot (x, y, 1)$ on K^2. The space $F(q)$ of *lifting planes* is the set of families $(h_v)_{v \in V(G)}$ of affine functions with $(h_u - h_v)(q_u) = (h_u - h_v)(q_v) = 0$ for every edge uv.

Lemma 30.30 (The lifting planes and the lifting space)**.** *Let q be admissible. Then* $\dim F(q) = \dim L(q)$.

Proof. Define $\Phi(h)_w = h_w(q_w)$. Let v be a body. For a neighbour u of v, $h_u(q_u) = h_v(q_u)$, as $h_u - h_v$ vanishes at q_u. So on the closed neighbourhood of v, $\Phi(h)$ agrees with the affine function h_v, and $\Phi(h) \in L(q)$.

Conversely, let $z \in L(q)$. For each body v, let h_v be an affine function that agrees with z on the closed neighbourhood of v. Both ends of an edge uv lie in both closed neighbourhoods, so $h_u - h_v$ vanishes at q_u and q_v. Hence $(h_v)_v \in F(q)$, and Φ maps it to z.

If $\Phi(h) = 0$, then h_v vanishes at the picture point of every member of the closed neighbourhood of v. Three of these points are not collinear, as q is admissible, so $h_v = 0$. Hence Φ is an isomorphism. □

Lemma 30.31 (The motions of a flat configuration)**.** *Let G be connected, let q be admissible, and let $F_{(q,0)}$ be the body-hinge framework whose hinge at each edge uv is the line $p_u \vee p_v$ of the flat configuration $p = (q, 0)$ (Lemma 30.9). Then its infinitesimal motions form a space of dimension $3 + \dim F(q)$.*

Proof. Let $e_0, \dots, e_3$ be the standard basis of K^4, with e_2 the height direction, and write $e_{ij} = e_i \vee e_j$. The plane $z = 0$ is spanned by e_0, e_1 and e_3. So at $z = 0$ every hinge $p_u \vee p_v$ lies in $W_\Pi = \operatorname{span}(e_{01}, e_{03}, e_{13})$, the span of the lines of that plane. Split $\bigwedge^2 K^4 = W' \oplus W_\Pi$ with $W' = \operatorname{span}(e_{02}, e_{12}, e_{23})$, and write a screw as $m_v = (m'_v, m^\Pi_v)$. The hinge condition $m_u - m_v \in \operatorname{span}(p_u \vee p_v)$ (Definition 19.3) splits into $m'_u = m'_v$ and $m^\Pi_u - m^\Pi_v \in \operatorname{span}(p_u \vee p_v)$. As G is connected, the first condition makes m' constant, which gives three dimensions.

For the second, identify $W_\Pi = \bigwedge^2 \hat{\Pi}$, where $\hat{\Pi} = \operatorname{span}(e_0, e_1, e_3)$, with the affine functions on K^2 by $a \vee b \mapsto \det(a, b, \cdot)$, evaluated at $(x, y, 1)$. The line $p_u \vee p_v$ becomes a nonzero affine function vanishing at q_u and q_v. As $q_u \neq q_v$, its multiples are all the affine functions vanishing at these two points. So the second condition says that $h_u - h_v$ vanishes at q_u and q_v, where h_v is the affine function of m^Π_v. Its solutions are $F(q)$. □

Formalization note 30.32. The families of lifting planes range over the whole ambient set of bodies and are unconstrained off $V(G)$, as the infinitesimal motions of every framework are. Each body off $V(G)$ adds three to $\dim F(q)$ and six to the dimension of the motions in Lemma 30.31. These terms cancel in every rank.

Lemma 30.33 (The lifting planes are the motions of a plane framework)**.** *Let q be admissible, and let P_q be the panel-hinge framework in the plane on G whose panel normal at each body v is $(x_v, y_v, 1)$ (Definition 23.5, at $d = 2$). Then the infinitesimal motions of P_q form a space isomorphic to $F(q)$, and $\operatorname{rank} R(P_q) = 3|V(G)| - \dim L(q)$.*

Proof. The screw space of the plane is K^3. The hinge of P_q at an edge uv is the meet of the lines with normals $(x_u, y_u, 1)$ and $(x_v, y_v, 1)$ (Definition 23.1). It is a nonzero vector orthogonal to both normals, since $q_u \neq q_v$. So a relative screw is a multiple of the hinge exactly when it is orthogonal to both normals, which is the condition defining $F(q)$. The rank follows by rank–nullity and Lemma 30.30. □

Lemma 30.34 (The flat lower bound on the lifting space)**.** *Let G have at least one body and let q be admissible. Then* $\dim L(q) \geq 3 + \mathrm{def}_2(G)$*, where* $\mathrm{def}_2(G) = \max_P [3(|P| - 1) - 2d_G(P)]$*.*

Proof. By Lemma 30.33 and Lemma 30.30, it suffices to show that the motions of P_q have dimension at least $3 + \mathrm{def}_2(G)$. Each body of P_q has the screw space K^3, and each hinge is nonzero, as adjacent picture points differ. Let P be a partition of $V(G)$. The screw assignments that are constant on each part form a space of dimension $3|P|$. On it the edges inside a part impose no condition, and each crossing edge imposes at most two. So the motions have dimension at least $3|P| - 2d_G(P) = 3 + [3(|P| - 1) - 2d_G(P)]$ for every P, and hence at least $3 + \mathrm{def}_2(G)$. This is the relative deficiency bound at $D = 3$ (Lemma 20.16). □

Theorem 30.35 (The flat rank)**.** *Let G be connected and let q be admissible. Then the rank* $\operatorname{rank} R(q, 0)$ *of the flat configuration (Definition 30.8) is exactly* $6|V(G)| - 3 - \dim L(q)$*. Hence* $\operatorname{rank} R(q, 0) \leq 6(|V(G)| - 1) - \mathrm{def}_2(G)$*, with equality if and only if* $\dim L(q) = 3 + \mathrm{def}_2(G)$*.*

Proof. The rank of $(q, 0)$ is that of $F_{(q,0)}$ (Lemma 30.9). Its motions have dimension $3 + \dim F(q) = 3 + \dim L(q)$ (Lemma 30.31, Lemma 30.30). By rank–nullity in the $6|V(G)|$-dimensional space of screw assignments, the rank is $6|V(G)| - 3 - \dim L(q)$. The bound and its equality case follow from Lemma 30.34. □

Corollary 30.36 (When the flat configuration has the deficiency rank)**.** *Let G be connected and let q be admissible. The flat configuration $(q, 0)$ has rank $6(|V(G)| - 1) - \mathrm{def}_3(G)$ if and only if $\dim L(q) = 3 + \mathrm{def}_3(G)$, and this holds if and only if equality holds in Lemma 30.34 at q and $\mathrm{def}_2(G) = \mathrm{def}_3(G)$. If $\dim L(q) \leq 3 + \mathrm{def}_3(G)$, then the general configuration of G attains.*

Proof. The first equivalence is the exact rank of Theorem 30.35. By Lemma 30.34 and Lemma 20.7, every admissible picture q' has

$$\dim L(q') \;\geq\; 3 + \mathrm{def}_2(G) \;\geq\; 3 + \mathrm{def}_3(G).$$

At $q' = q$ this gives the second equivalence.

Now suppose that $\dim L(q) \leq 3 + \mathrm{def}_3(G)$. Then $\dim L(q') \geq \dim L(q)$ at every admissible picture q', so q is a main picture (Definition 30.18). By the exact rank, the flat configuration $(q, 0)$ has rank at least $6(|V(G)| - 1) - \mathrm{def}_3(G)$, and its height 0 lies in $L(q)$. So it is a witness for Lemma 30.20, and the general configuration of G attains. □

Corollary 30.37 (The general configuration when the lifting space has dimension three)**.** *Let G be connected, and suppose that $\dim L(q) = 3$ at some admissible picture q, so that $\ell_0 = 3$ (Lemma 30.3). Then $\mathrm{def}_2(G) = \mathrm{def}_3(G) = 0$, and the general configuration of G attains.*

Proof. By Lemma 30.34, $3 + \mathrm{def}_2(G) \leq 3$, so $\mathrm{def}_2(G) \leq 0$. Both deficiencies are nonnegative, and $\mathrm{def}_3(G) \leq \mathrm{def}_2(G)$ (Lemma 20.7), so both vanish. Then $\dim L(q) = 3 + \mathrm{def}_3(G)$, and Corollary 30.36 applies. □

30.3. **Jackson and Jordán's equality.** In this subsection we prove the equality of Jackson and Jordán (Theorem 30.43). Let G be simple, with at least three members in every closed neighbourhood. At a general picture the lower bound $\dim L(q) \geq 3 + \mathrm{def}_2(G)$ of Lemma 30.34 is then an equality. With the flat rank, the equality settles the case of equal deficiencies (Corollary 30.44). The contraction, ear and split-off steps use it at the smaller graphs they compare with G.

The equality is Jackson and Jordán's theorem on pin-collinear body-and-pin frameworks [JJ08b]. Such a framework consists of rigid bodies in the real plane, with each pair of adjacent bodies joined by a pin and the pins of each body on one line. In the plane a panel is a line and a hinge is a point, so these are Katoh and Tanigawa's panel-hinge frameworks at $d = 2$ [KT11, pp. 648–649]. Jackson and Jordán prove the following for a simple graph G without isolated vertices. Choose the line of each body generically, and pin adjacent bodies where their lines meet. Then the infinitesimal motions form a space of dimension $3 + \mathrm{def}_2(G)$. The plane framework P_q of Lemma 30.33 is a framework of this kind, in which the body v lies on the line with normal $(x_v, y_v, 1)$. Its motions are the lifting planes, and these form a space of dimension $\dim L(q)$ (Lemma 30.30). So over the reals, their theorem at a generic picture is our equality.

Jackson and Jordán's argument is specific to the reals. Their genericity is algebraic independence over $\mathbb{Q}$, and their proof moves points within small Euclidean neighbourhoods. We need the equality over every infinite field, and KT Theorem 5.6 holds over every infinite field (Theorem 23.54). So we derive the equality from its case $d = 2$. That case gives a panel-hinge framework on G in the plane of rank $3(|V(G)| - 1) - \mathrm{def}_2(G)$. The rank does not drop when the normals move to a general point. It does not change when each normal is divided by its third coordinate (Lemma 30.38). So at a general picture, P_q has at least this rank (Lemma 30.39). As $\mathrm{rank}\, R(P_q) = 3|V(G)| - \dim L(q)$, the bound reads $\dim L(q) \leq 3 + \mathrm{def}_2(G)$.

Lemma 30.38 (Rescaling the panel normals)**.** *Let (n_v) be panel normals and (c_v) nonzero scalars. Then at every edge e with ends u and v, the panel-hinge framework on G with normals $c_v n_v$ has the hinge $c_u c_v\, C(e)$, where $C(e)$ is its hinge at the normals n_v. In particular the two frameworks have the same infinitesimal motions, and their rigidity matrices have the same rank.*

Proof. The support extensor is linear in each of its two normals (Definition 23.1), in the second by antisymmetry (`panelSupportExtensor_smul_right`). A nonzero multiple of a hinge spans the same line, and the hinge constraint depends only on that line (Definition 19.3). Equal motion spaces give equal row spans (Lemma 19.25). □

Lemma 30.39 (The planar case of KT Theorem 5.6, in the chart)**.** *Let K be infinite and let G be simple with at least two bodies. There is a nonzero polynomial P in the picture coordinates such that at every picture q with $P(q) \neq 0$ the plane framework P_q of Lemma 30.33, with panel normal $(x_v, y_v, 1)$ at each body, has* $\mathrm{rank}\, R(P_q) \geq 3(|V(G)| - 1) - \mathrm{def}_2(G)$.

Proof. By Theorem 23.54 at $d = 2$ ($D = 3$), G has a panel-hinge realization of rank $3(|V(G)| - 1) - \mathrm{def}_2(G)$. As K is infinite, three nonzero polynomials in the normals have a common non-root n: the rank polynomial of this realization (Lemma 23.120),

a polynomial whose non-roots are in general position (Definition 23.9), and the product of the third coordinates of the normals. Dividing each normal n_v by its third coordinate gives the normals $(x_v, y_v, 1)$ of a picture q'. The framework at q' has the rank of the framework at n, at least $3(|V(G)|-1)-\mathrm{def}_2(G)$ (Lemma 30.38). Its hinges are nonzero (Lemma 23.11). So the rank polynomial of the framework at q', composed with the chart $q \mapsto (x_v, y_v, 1)$, is nonzero at q'. It is the required P. □

Formalization note 30.40. Lemma 30.39 assumes a supply of spare edge labels (Remark 23.40): it applies the case $d = 2$ of KT Theorem 5.6 in the form that assumes one, the second theorem pinned on Theorem 23.54. The equality mentions no edge label, so its formal proof first relabels the edges into a larger label set, which has spare labels (Definition 23.41), and the relabelling changes nothing that the equality reads (Lemma 23.42 and Lemma 30.41). The equality therefore assumes nothing about the labels.

Lemma 30.41 (Relabelling the edges keeps the lifting spaces and the pictures)**.** *Let G' be G with its edges relabelled injectively into a larger set of labels (Definition 23.41). Then G' has the lifting spaces $L(q)$ and the admissible and main pictures of G.*

Proof. The bodies and the pairs of adjacent bodies are unchanged, as in the proof of Lemma 23.42, and hence so are the closed neighbourhoods (`closedNbhd_embedEdges`) and the dimensions of the lifting spaces, which the main pictures read (`finrank_liftingSpace_embedEdges`). □

Formalization note 30.42. The three pins are the three clauses, in order: the lifting spaces, the admissible pictures and the main pictures.

Theorem 30.43 (Jackson and Jordán's equality)**.** *Let K be infinite and let G be simple, with at least one body and at least three members in every closed neighbourhood. There is a nonzero polynomial P in the picture coordinates such that every picture q with $P(q) \neq 0$ is a main picture with $\dim L(q) = 3+\mathrm{def}_2(G)$. Consequently $\ell_0 = 3 + \mathrm{def}_2(G)$: every main picture has $\dim L(q) = 3 + \mathrm{def}_2(G)$.*

Proof. Let P be the product of two nonzero polynomials. The non-roots of the first are main pictures (Lemma 30.23). The second is the polynomial of Lemma 30.39, which needs two bodies. Here G has at least three, since a closed neighbourhood of a body lies in $V(G)$ (`Graph.closedNbhd_subset_vertexSet`). (The formal proof applies Lemma 30.39 to G with its edges relabelled, Lemma 23.42 and Lemma 30.41.)

Let q be a non-root of P. The plane framework P_q has rank at least $3(|V(G)|-1)-\mathrm{def}_2(G)$, and exactly $3|V(G)|-\dim L(q)$ (Lemma 30.33). So $\dim L(q) \leq 3+\mathrm{def}_2(G)$. The reverse inequality is Lemma 30.34. As q is a main picture, $\ell_0 = \dim L(q) = 3 + \mathrm{def}_2(G)$ (Definition 30.18). □

Corollary 30.44 (The general configuration attains when the two deficiencies agree)**.** *Let K be infinite and let G be connected and simple, with at least three members in every closed neighbourhood. If $\mathrm{def}_2(G) = \mathrm{def}_3(G)$, then the general configuration of G attains (Definition 30.8).*

Proof. By Theorem 30.43, some main picture q has $\dim L(q) = 3 + \mathrm{def}_2(G) = 3 + \mathrm{def}_3(G)$. So the general configuration of G attains (Corollary 30.36). □

30.4. **Cut vertices and bridges.** This subsection and the five after it prove the steps of the induction, other than the case of equal deficiencies (Corollary 30.44). In each step we show that the general configuration of a graph G satisfying the standing hypotheses (Definition 30.45) attains, assuming the same of one or two smaller graphs (of none, for a cycle). This subsection proves the first two steps: at a cut vertex, and along a chain of bridges, a path of bridges whose interior bodies have degree two. The shape that the steps share is plainest in these two, and we describe it first.

Write

$$\mathrm{tgt}(G) \;=\; 6(|V(G)| - 1) - \mathrm{def}(\tilde{G})$$

for the deficiency rank of G (Definition 29.13, with $D = 6$), and call it the *target* of G. We end every step in the same way. We exhibit one configuration of G over a main picture of G, with its heights in the lifting space, whose rank is at least $\mathrm{tgt}(G)$. Then the general configuration of G attains (Lemma 30.20). At a cut vertex and along a chain of bridges we find this configuration (q, z) in three parts.

(1) *The picture.* As K is infinite, a product of nonzero polynomials has a non-root. So we can take q off the zero sets of finitely many nonzero polynomials in the picture coordinates. One is the polynomial of Lemma 30.23 for G, so that q is a main picture of G. The others are the polynomials of the attainment at the smaller graphs G' (Definition 30.8). Then for each G', the heights in $L_{G'}(q)$ at which G' has its target contain a nonempty Zariski-open subset of $L_{G'}(q)$.

(2) *The height.* Here each smaller graph G' is a subgraph of G. The rank of G' depends only on the points of its own bodies. So at a height z of G it equals the rank of G' at the restriction of z to $V(G')$ (Lemma 30.46, Lemma 30.47). If the restriction maps $L_G(q)$ onto $L_{G'}(q)$, the heights $z \in L_G(q)$ at which G' has its target therefore contain a nonempty Zariski-open subset of $L_G(q)$. Two such subsets meet (Lemma 30.26).

(3) *The count.* A rank-addition lemma bounds the rank of (q, z) below through the ranks of the smaller graphs. A deficiency count shows that this bound is at least $\mathrm{tgt}(G)$.

The later steps choose the picture and make the count in the same way, and vary how the height is found; each subsection says how. Often two open conditions still meet in one linear space, though not always in $L_G(q)$. One of them may ask that the hinges of an ear be independent. Some steps also move some bodies, with their heights, through a polynomial family of configurations of G, and we take a general member of the family. In two steps no two open conditions need to meet: the cycle (Theorem 30.71) and the split-off step (Theorem 30.102).

At a cut vertex and along a chain of bridges, the restriction maps $L_G(q)$ onto the lifting spaces of the two sides, at every picture q. Each side meets the rest of G at one body: the cut vertex, or the end of the path in that side. A height of the side extends to a height of G by the affine function with which it agrees on

the closed neighbourhood of that body (Lemma 30.48, Lemma 30.50). At a cut vertex the ranks of the two sides add (Lemma 19.26), and so do their deficiencies (Lemma 20.29). Along a chain of bridges, each bridge adds at least 5 to the rank (Lemma 19.24) and exactly 1 to the deficiency (Lemma 20.27).

We first record the hypotheses of the steps, and the two facts that let us read the rank of a subgraph at a height of G.

Definition 30.45 (The standing hypotheses)**.** A graph G satisfies the *standing hypotheses* if it is finite, simple and connected, with minimum degree at least two. Then every closed neighbourhood of G has at least three members: in a simple graph a body has as many distinct neighbours as its degree.

Lemma 30.46 (Heights restrict to a subgraph)**.** *Let $G' \leq G$ and let q be any picture, and write z' for the height that agrees with z on $V(G')$ and vanishes elsewhere.*

(1) If $z \in L_G(q)$, then $z' \in L_{G'}(q)$.
(2) The configuration points of z' agree with those of z at the bodies of G'.
(3) A polynomial in the heights of G', read at z', agrees with a polynomial in the heights of G, read at z.

Proof. z' agrees with z on $V(G')$. A closed neighbourhood in G' lies in the closed neighbourhood of the same body in G, where z agrees with an affine function of the picture. So z' agrees with that affine function on it too. The configuration points at a body of G' read only its picture point and its height. For the third, substituting zero for the heights off $V(G')$ in a polynomial and then evaluating at z is the same as evaluating the polynomial directly at z', which already vanishes there. □

Lemma 30.47 (The rank reads only the bodies of the graph)**.** *The rank of the rigidity matrix of the panel-hinge framework on G depends only on the panel normals at the bodies of G, and not on which end of each edge is recorded first.*

Proof. The hinge at an edge is built from the normals at its two ends, and swapping them negates it (Definition 23.1); the hinge constraint reads only the line it spans (Definition 19.3). So the motion spaces agree, and equal motion spaces give equal row spans (Lemma 19.25). □

Lemma 30.48 (The heights at a cut vertex)**.** *Let $G = G_1 \cup G_2$ with $V(G_1) \cap V(G_2) = \{v\}$ and disjoint edge sets. At every picture q, both restriction maps $L_G(q) \to L_{G_i}(q)$ (Lemma 30.46) are onto.*

Proof. Given $z^1 \in L_{G_1}(q)$, extend it across $V(G_2)$ by the affine function of its closed neighbourhood at v. Every closed neighbourhood of G other than $N_G[v]$ lies in one side, and $N_G[v] = N_1[v] \cup N_2[v]$ carries that affine function on both, so the extension lies in $L_G(q)$ and restricts to z^1. The other side is symmetric. □

Theorem 30.49 (A cut vertex)**.** *Let K be infinite and let G satisfy the standing hypotheses (Definition 30.45), with $V(G) = V_1 \cup V_2$, $V_1 \cap V_2 = \{v\}$ and every edge of G inside V_1 or inside V_2. If the general configurations of $G[V_1]$ and $G[V_2]$ attain, then the general configuration of G attains.*

Proof. We follow the three parts above. Take a picture q off the zero sets of the polynomials of the attainment at $G[V_1]$ and at $G[V_2]$, and of the main-picture polynomial of G (Lemma 30.23). For $i = 1, 2$ some polynomial R_i in the heights is nonzero somewhere on $L_{G[V_i]}(q)$, and $G[V_i]$ has its target at every height of $L_{G[V_i]}(q)$ off the zero set of R_i. Read at the restriction to V_i, R_i is a polynomial in the heights of G (Lemma 30.46). It is nonzero somewhere on $L_G(q)$, since the restriction is onto (Lemma 30.48). So some $z \in L_G(q)$ makes both nonzero (Lemma 30.26).

At (q, z) the rank of each side is its target (Lemma 30.47), and the rank of G is their sum (Lemma 19.26). The deficiencies add (Lemma 20.29), and $|V_1| + |V_2| = |V(G)| + 1$, so

$$\operatorname{tgt}(G[V_1]) + \operatorname{tgt}(G[V_2]) \;=\; \operatorname{tgt}(G).$$

Hence Lemma 30.20 applies. □

Lemma 30.50 (The heights along a chain of bridges)**.** *Let G be $G_1 \sqcup G_2$ plus a path $a - x_1 - \cdots - x_k - b$ ($k \geq 0$ new vertices), with $a \in V(G_1)$, $b \in V(G_2)$, and every other edge of G inside G_1 or inside G_2. At every picture q, both restriction maps $L_G(q) \to L_{G_i}(q)$ are onto.*

Proof. Given $z^1 \in L_{G_1}(q)$, take an affine function h agreeing with z^1 on the closed neighbourhood $N_{G_1}[a]$ (Definition 30.2), and extend z^1 by h on every body outside G_1. The body a is the only body of G_1 with an edge leaving G_1: the path's own bodies and b lie outside G_1, so only the path's first edge, at a, can cross. So the closed neighbourhood in G of a body of G_1 other than a lies in G_1, where the extension is z^1; the closed neighbourhood of a, or of a body outside G_1, meets G_1 only inside $N_{G_1}[a]$, so the extension equals h on all of it. The extension is therefore affine on every closed neighbourhood of G: it lies in $L_G(q)$ and restricts to z^1. The other restriction is symmetric. □

Formalization note 30.51. The formal statement gives the path by the sequence $a, x_1, \ldots, x_k, b$ of its bodies (`pathVertex`), whose consecutive terms its edges join in order. Every term but the first is b or some x_i (`pathVertex_eq_or_exists`), and so lies outside G_1.

Theorem 30.52 (A chain of bridges)**.** *Let K be infinite and let G satisfy the standing hypotheses. Let $V(G)$ be the disjoint union of V_1, V_2 and path bodies $x_1, \ldots, x_k$ ($k \geq 0$), with a path $a - x_1 - \cdots - x_k - b$ from $a \in V_1$ to $b \in V_2$, every other edge of G lying inside V_1 or inside V_2. If the general configurations of $G[V_1]$ and $G[V_2]$ attain, then the general configuration of G attains.*

Proof. We find q and z as in the proof of Theorem 30.49, with Lemma 30.50 for the restrictions. For $G[V_2]$ we apply it from b, along the reversed path. We count the ranks and the deficiencies directly, rather than apply the step one bridge at a time. That would ask a graph with one path body as its only body to attain, and no picture of such a graph is admissible.

Write $S_j = V_1 \cup \{x_1, \ldots, x_j\}$. Every edge off the path lies inside V_1 or inside V_2, so the only edge of G leaving S_j is the path edge from S_j to the next path body, or to b when $j = k$. The last of these edges is a bridge between $G[S_k]$ and $G[V_2]$. It

adds $D-1=5$ to the rank (Lemma 19.24; every hinge is nonzero at an admissible picture) and 1 to the deficiency (Lemma 20.27). Inside $G[S_{j+1}]$ the path body x_{j+1} is joined to S_j by a single edge. By the same two lemmas, it adds at least 5 to the rank and exactly 1 to the deficiency. Compared with the two sides together, the term $6(|V(G)|-1)$ is larger by $6(k+1)$ and the deficiency by $k+1$, so

$$\operatorname{tgt}(G) \;=\; \operatorname{tgt}(G[V_1]) + \operatorname{tgt}(G[V_2]) + 5(k+1).$$

The rank at (q,z) is at least this, so Lemma 30.20 applies. □

30.5. **Contraction at an additive core.** This subsection proves the contraction step. Let $W \subsetneq V(G)$ and $r \in W$. The *core* is the induced subgraph $H = G[W]$, and G/H is the contraction of H to the body r (Definition 21.3): it keeps every edge of G not inside W, with its end in W moved to r. A body outside W adjacent to a body of W is a neighbour of W. As G is simple, G/H is simple exactly when no neighbour of W is adjacent to two bodies of W. Suppose that the core is additive: it is rigid, and the planar deficiencies of H and G/H add up to at most that of G,

$$\operatorname{def}_2(H) + \operatorname{def}_2(G/H) \;\le\; \operatorname{def}_2(G).$$

We show that if the general configurations of both H and G/H attain, then so does that of G. This is Katoh and Tanigawa's contraction case [KT11, §6.2, Lemma 6.3], read on configurations. Their lemma contracts any proper rigid subgraph whose contraction is simple. Its proof takes realizations of the rigid subgraph and of the contracted graph from the induction, independently of each other. It joins them by a block-triangular rank bound [KT11, pp. 673–675]. Our proof has the same shape.

Both deficiencies of a core of planar deficiency zero vanish, so its general configuration attains by the case of equal deficiencies. Such a core is additive, as contracting it does not change the planar deficiency. At such a core the step therefore assumes only that the general configuration of G/H attains (Corollary 30.62). The structural coverage contracts both kinds of core (Section 30.11).

We follow the scheme of Section 30.4. The picture is chosen as there, general for H and for G/H and a main picture of G. The count is the block-triangular rank bound of KT eq. (6.3): the rank of G is at least the rank of the core plus the rank of the other edges with the columns of W deleted (Lemma 23.86). The targets of H and G/H add up to that of G, as contracting a rigid core does not change the deficiency (Lemma 29.22). The height is found differently. The graph G/H is not a subgraph of G, so its heights are not restrictions of heights of G. Instead we move the picture along a curve on which the core shrinks to the point of r, we move the heights with it, and we take a general member of this polynomial family of configurations of G.

On the curve, each body c of W moves from q_c at $t=1$ to q_r at $t=0$, and the other bodies stay at q (Definition 30.55). The heights along the curve are the solutions of a linear system $M(t)$ whose entries are polynomials in t, the rescaled lifting system. Its unknowns for the heights of the core are those heights magnified by $1/t$, which undoes the shrinking. So its rows between bodies of the core do not depend on t. At $t=0$ every body of W has the point $(q_r,0)$ of r, so on the edges outside the core the configuration of a solution of $M(0)$ is a configuration of G/H. In the panel-hinge framework whose normals are the points (the proof of Lemma 30.20),

every panel of the core is then the panel of r. Katoh and Tanigawa make the same choice in the proof of KT Claim 6.4 [KT11, p. 675]. There the choice shows that every realization of the contracted graph is among their placements of the edges outside the rigid subgraph, KT eq. (6.7).

The polynomial section of Lemma 30.17, with the parameter t, extends a solution of $M(0)$ to solutions of $M(t)$ at all but finitely many t, provided that $\dim\ker M(t) \geq \dim\ker M(0)$ there. We therefore need the kernel not to jump at $t = 0$. Below, $\ker M(t)$ is at least as large as the lifting space of G at a general point of the curve, whose dimension is at least $3 + \mathrm{def}_2(G)$. Above, $\ker M(0)$ is bounded through $L_{G/H}(q)$ and through the core heights of its solutions, which lie in $L_H(q)$ (Lemma 30.57). Jackson and Jordán's equality gives the dimensions of $L_H(q)$ and $L_{G/H}(q)$ (Theorem 30.43). With additivity, the upper bound is then at most $3 + \mathrm{def}_2(G)$ too. So the kernel does not jump. As the bounds meet, the core heights of the solutions of $M(0)$ are all of $L_H(q)$.

We need all of $L_H(q)$ for an open condition on $\ker M(0)$: that the core heights of the solution give H its target over q. It holds somewhere on $\ker M(0)$, as the heights in $L_H(q)$ at which H has its target contain a nonempty Zariski-open subset. The other condition is that at the end of the curve, the rows of the edges outside the core, with the columns of W deleted, have rank at least $\mathrm{tgt}(G/H)$ (Lemma 30.59). It holds at a solution of $M(0)$ whose heights agree off W with an attaining height of G/H (Lemma 30.58). So two open conditions meet in one linear space, as in the scheme, but in $\ker M(0)$ rather than in $L_G(q)$ (Lemma 30.26).

Last, we read the rank of the core along the curve over the fixed picture q. At $t \neq 0$ the configuration of H along the curve is the image of its configuration over q, with the core heights of the solution, under an invertible linear map of K^4 (Lemma 30.60). Along the polynomial section through a solution that satisfies both conditions, the first condition holds at all but finitely many t. There the core has its target $6(|W| - 1)$.

We first show that the core and the contraction satisfy the standing hypotheses. The first lemma holds at every d. The step uses it at $d = 3$, and its corollary at a core of planar deficiency zero uses it at $d = 2$.

Lemma 30.53 (A rigid core satisfies the standing hypotheses)**.** *Let G be simple and $W \subseteq V(G)$ with $|W| \geq 2$, and suppose $G[W]$ has deficiency zero at some $d \geq 1$ (Definition 20.3). Then $G[W]$ satisfies the standing hypotheses.*

Proof. $G[W]$ is simple, and it is connected and of minimum degree at least two by Lemma 20.23. □

Lemma 30.54 (The contraction satisfies the standing hypotheses)**.** *Let G satisfy the standing hypotheses and be 2-edge-connected. Let $W \subsetneq V(G)$ with $|W| \geq 2$, such that no body outside W is adjacent to two bodies of W, and let G/H be the contraction of $H = G[W]$ to a body $r \in W$. Then G/H is simple and satisfies the standing hypotheses.*

Proof. An edge of G/H is an edge of G not inside W. Two such edges with the same ends in G/H would give a body outside W adjacent to two bodies of W, or two parallel edges of G, so G/H is simple. A path of G maps to a walk of G/H, so

G/H is connected. At r, the 2-edge-connectivity of G gives two edges leaving W. Their ends outside W are distinct: a body at the end of both would be adjacent to two bodies of W, or joined to one by two parallel edges. At a body outside W the contraction is injective on the closed neighbourhood. So every closed neighbourhood of G/H has at least three members, and G/H has minimum degree at least two. □

The curve and the rescaled system are defined for every W and every $r \in W$.

Definition 30.55 (The contraction curve and the rescaled lifting system)**.** Let $W \subseteq V(G)$, $r \in W$, and let q be a planar picture. For $t \in K$ the *contraction curve* $q(t)$ keeps each body outside W at q and puts each $c \in W$ at $(1-t)\,q_r + t\,q_c$. The *rescaled lifting system* $M(t)$ is the lifting system with weights (Definition 30.21) whose weight $\omega(v,w)$ is $(x_w - (1-t)\,x_r,\ y_w - (1-t)\,y_r,\ t)$ when $w \notin W$ and the closed neighbourhood of v meets W, and $(x_w, y_w, 1)$ otherwise. The *heights* $z(t)$ of a solution (z,h) of $M(t)$ are $t\,z_w$ at $w \in W$ and z_w elsewhere.

So $q(1) = q$, and $q(0)$ puts all of W at q_r. The entries of $M(t)$ are polynomials in t. We call the graph $\zeta = h_v \cdot (\xi, \eta, 1)$ of h_v the plane of the body v. The bodies whose closed neighbourhoods meet W are the bodies of W and the neighbours of W. Let $t \neq 0$. At a body c of W, the lifted point $((1-t)\,q_r + t\,q_c,\ t\,z_c)$ is the image of (q_c, z_c) under the homothety of ratio t centred at $(q_r, 0)$. At the bodies of W and the neighbours of W, $M(t)$ writes the planes in the frame magnified by $1/t$ about $(q_r, 0)$, in which the core is drawn over q with the heights z. So the rows at a body of W and a member of W of its closed neighbourhood keep the weight $(x_w, y_w, 1)$ at every t: they are the rows of the lifting system of H at q.

At $t = 0$ the other rows of these bodies read as follows. The rows at a body c of W and a neighbour u of c outside W say that the point (q_u, z_u) lies on the plane through $(q_r, 0)$ parallel to the plane of c. The rows at a neighbour u of W and a member w of its closed neighbourhood outside W say that (q_w, z_w) lies on the plane through $(q_r, 0)$ parallel to the plane of u. The constant coefficient of h_u occurs only in the rows of u at its neighbours in W.

Lemma 30.56 (The rescaled system off the end of the curve)**.** *At every t, the heights of a solution of $M(t)$ lie in $L_G(q(t))$. At $t \neq 0$ every height of $L_G(q(t))$ is the heights of a solution of $M(t)$, so* $\dim \ker M(t) \geq \dim L_G(q(t))$.

Proof. At a body v whose closed neighbourhood meets W, the heights $z(t)$ on that neighbourhood agree with the affine function of the picture $q(t)$ with coefficients $(h_{v,1},\ h_{v,2},\ t\,h_{v,3} - (1-t)(h_{v,1}x_r + h_{v,2}y_r))$; at the other bodies they agree with h_v. At $t \neq 0$ the substitution inverts: divide the core heights, and the corrected constant coefficients, by t. □

Lemma 30.57 (The kernel at the end of the curve, through the core heights)**.** *Let $r \in W \subseteq V(G)$ and $H = G[W]$, let q be a picture, and let ρ send a vector (z,h) of unknowns of the rescaled lifting system to the height that agrees with z on W and vanishes elsewhere.*

(1) At every t, ρ maps the solutions of $M(t)$ into $L_H(q)$.
(2) If q is admissible for H and for G/H, then $\dim \ker M(0) + 3 \leq \dim \rho(\ker M(0)) + \dim L_{G/H}(q)$.

Proof. (1) As H is induced, the closed neighbourhood of a core body c in H is its closed neighbourhood in G met with W. The rows of $M(t)$ there keep the weight $(x_w, y_w, 1)$ at every t, so ρ sends the heights of a solution into $L_H(q)$.

(2) By rank–nullity it suffices that the solutions of $M(0)$ with $\rho = 0$ form a space N with $\dim N + 3 \leq \dim L_{G/H}(q)$. A solution in N has zero heights on W, so by the rows of (1) its plane at every core body is zero: three members of each closed neighbourhood in H have independent picture points. Send a solution in N to the height of G/H that agrees with it outside W and vanishes at r. This height lies in $L_{G/H}(q)$, by the rows at $t = 0$ (read after Definition 30.55). At a neighbour u of W, the members of its closed neighbourhood outside W lie on a plane through $(q_r, 0)$, which is the point of r in the image. The planes of the core are zero, so the rows at the core put every neighbour of W at height zero, and the height vanishes on the closed neighbourhood of r in G/H.

The map is injective. If the image of a solution vanishes, every plane outside W vanishes, since three members of its closed neighbourhood in G/H have independent picture points; and the constant coefficient of a neighbour of W vanishes by its row at a neighbour in W. The image vanishes on the closed neighbourhood of r, three members of which have independent picture points, so the only affine height in the image is zero. As $\mathrm{Aff}(q)$ is three-dimensional and lies in $L_{G/H}(q)$ (Lemma 30.3), this gives the bound. □

Lemma 30.58 (The rescaled system at the end of the curve)**.** *Let $r \in W \subseteq V(G)$ and $H = G[W]$, and let q be a picture. Suppose that no body outside W is adjacent to two bodies of W. Then every $z \in L_{G/H}(q)$ with $z_r = 0$ agrees off W with the heights of a solution of $M(0)$.*

Proof. For each body v of G/H, let a_v be an affine function with which z agrees on the closed neighbourhood of v in G/H. As $z_r = 0$, the plane of a_r passes through $(q_r, 0)$, and it passes through the point of every neighbour of W. Put the core on this plane: give each body c of W the height $a_r(q_c)$, and give h_c the coefficients of a_r. Give each body v outside W the height z_v, and give h_v the coefficients of a_v if v is not a neighbour of W. At a neighbour u of W, the closed neighbourhood of u in G/H contains r, so the plane of a_u passes through $(q_r, 0)$; give h_u the linear coefficients of a_u. Then the rows at $t = 0$ hold, as read after Definition 30.55. The constant coefficient of h_u is fixed by the row of u at its one neighbour in W. □

The next lemma bounds the rank of the rows of the edges outside the core, with the columns of W deleted, near the end of the curve. Katoh and Tanigawa obtain the corresponding bound, KT eq. (6.9), from the generic choice of their panels (KT Claim 6.4). Here it comes from the lower semicontinuity of the rank, so the step works over every infinite field.

Lemma 30.59 (The surviving rows near the collapsed placement)**.** *Let $r \in W \subseteq V(G)$ and $H = G[W]$, and let n be panel normals at which every hinge of G/H is nonzero and $\operatorname{rank} R(G/H, n) \geq N$. There is a polynomial Q in the panel normals of G, nonzero at the* collapsed placement *(every body of W at the normal n_r, every other body at its own), such that at every p with $Q(p) \neq 0$ the rows of $G \setminus E(H)$ at p, with the columns of W deleted, have rank at least N.*

Proof. At the collapsed placement each row of $G \setminus E(H)$, with the columns of W deleted, is the corresponding row of G/H with the column of r deleted (as in Lemma 23.98), and deleting the one column of r loses no rank (Lemma 19.35). So there are N such rows, independent at the collapsed placement, and Lemma 23.93 gives a polynomial nonzero there off whose zero set they stay independent. This is the comparison of KT eqs. (6.5) and (6.9) [KT11, pp. 674–675]. □

Lemma 30.60 (The rank of the core along the curve)**.** *Let $r \in W$ and $H = G[W]$, let q be a picture and $t \in K$ nonzero, and let (z, h) be any vector of unknowns of the rescaled lifting system. Then the rank of H at the configuration $(q(t), z(t))$ equals its rank at (q, z).*

Proof. At a body c of W the point of $(q(t), z(t))$ is $((1-t)\,x_r + t\,x_c,\ (1-t)\,y_r + t\,y_c,\ t\,z_c,\ 1)$, the image of $(x_c, y_c, z_c, 1)$ under the linear map

$$(\xi, \eta, \zeta, \omega) \mapsto (t\,\xi + (1-t)\,x_r\,\omega,\ t\,\eta + (1-t)\,y_r\,\omega,\ t\,\zeta,\ \omega)$$

of K^4, which is invertible as $t \neq 0$. So Lemma 30.13 applies. □

Theorem 30.61 (Contraction at an additive core)**.** *Let K be infinite and let G satisfy the standing hypotheses and be 2-edge-connected. Let $W \subsetneq V(G)$ with $|W| \geq 2$ and $\operatorname{def}(\tilde{G}[W]) = 0$, such that no body outside W is adjacent to two bodies of W, and let G/H be the contraction of $H = G[W]$ to a body of W. Suppose $\operatorname{def}_2(H) + \operatorname{def}_2(G/H) \leq \operatorname{def}_2(G)$. If the general configurations of H and of G/H attain, then the general configuration of G attains.*

Proof. The pieces. H satisfies the standing hypotheses (Lemma 30.53, at $d = 3$). By Lemma 30.54, G/H is simple and satisfies them too. As H is rigid, G/H has the deficiency of G (Lemma 29.22, at $d = 3$).

The picture. As H and G/H satisfy the standing hypotheses, Jackson and Jordán's equality applies to both (Theorem 30.43). Take a picture q off the zero sets of five nonzero polynomials: those of the attainment and of the equality at H and at G/H, and the main-picture polynomial of G (Lemma 30.23). Then q is admissible for H and for G/H, and both have attaining heights over q. Also $\dim L_H(q) = 3 + \operatorname{def}_2(H)$ and $\dim L_{G/H}(q) = 3 + \operatorname{def}_2(G/H)$. Subtracting the constant height at r does not change the rank of G/H (Lemma 30.14). So G/H has an attaining height z' with $z'_r = 0$.

The kernel at the end of the curve. Let ρ take a vector of unknowns to its core heights, as in Lemma 30.57. The polynomial section through the zero solution at $t = 0$ (Lemma 30.17, with the parameter t) gives $\dim\ker M(t) \leq \dim\ker M(0)$ for all but finitely many t. For all but finitely many t, also, $q(t)$ is a main picture of G, as the main-picture polynomial of G along the curve is nonzero at $t = 1$, where $q(1) = q$. At a $t \neq 0$ with both properties,

$$3 + \operatorname{def}_2(G) \leq \dim L_G(q(t)) \leq \dim\ker M(t) \leq \dim\ker M(0)$$

(Lemma 30.34, Lemma 30.56). By Lemma 30.57, Jackson and Jordán's equality at H and G/H, and additivity,

$$\dim\ker M(0) + 3 \leq \dim \rho(\ker M(0)) + \dim L_{G/H}(q) \leq \dim L_H(q) + \dim L_{G/H}(q) \leq 6 + \operatorname{def}_2(G).$$

So $\dim\ker M(0) = 3+\mathrm{def}_2(G)$, and the second chain is an equality. Hence $\rho(\ker M(0))$, which lies in $L_H(q)$, has the dimension of $L_H(q)$: the core heights of the solutions of $M(0)$ are all of $L_H(q)$.

Two open conditions. The polynomial of the attainment at H over q is nonzero somewhere on $L_H(q)$. Composed with ρ, it is therefore nonzero somewhere on $\ker M(0)$. Let Q be the polynomial of Lemma 30.59 for the rank $\mathrm{tgt}(G/H)$ of G/H at (q, z'), and read it at the configuration $(q(0), z(0))$ of a solution. It is nonzero at the solution that agrees with z' off W (Lemma 30.58), as the configuration of that solution at $t = 0$ is the collapsed placement of (q, z'). So some $x_0 \in \ker M(0)$ makes both polynomials nonzero (Lemma 30.26).

The curve. Apply Lemma 30.17 to $M(t)$, at $t = 0$ and x_0. It gives a polynomial D in t with $D(0) \neq 0$, and a vector $X(t)$ of polynomials in t with $X(0) = x_0$. At every $t \neq 0$ with $q(t)$ a main picture of G the kernel does not jump: $\dim\ker M(0) = 3 + \mathrm{def}_2(G) \leq \dim L_G(q(t)) \leq \dim\ker M(t)$. So $X(t)$ solves $M(t)$ at every such t with $D(t) \neq 0$, and its heights $z(t)$ lie in $L_G(q(t))$ (Lemma 30.56).

A good parameter. Take t off the zero sets of t, of D, of the main-picture polynomial of G along the curve, and of the two polynomials of the open conditions along $X(t)$, which are nonzero at $t = 0$. Then $\rho(X(t))$ lies in $L_H(q)$ (Lemma 30.57(1)), and it is an attaining height of H over q. So H has rank $6(|W|-1)$ at $(q, \rho(X(t)))$, and hence at $(q(t), z(t))$ (Lemma 30.60, Lemma 30.47). As Q is nonzero at $(q(t), z(t))$, the rows of the other edges there, with the columns of W deleted, have rank at least $\mathrm{tgt}(G/H)$. By Lemma 23.86 the rank of G at $(q(t), z(t))$ is at least $6(|W| - 1) + \mathrm{tgt}(G/H) = \mathrm{tgt}(G)$, since $|V(G/H)| = |V(G)| - |W| + 1$ (Lemma 21.51) and G/H has the deficiency of G. So Lemma 30.20 applies. □

Corollary 30.62 (Contraction at a core of planar deficiency zero)**.** *Let K be infinite and let G satisfy the standing hypotheses and be 2-edge-connected. Let $W \subsetneq V(G)$ with $|W| \geq 2$ and $\mathrm{def}_2(G[W]) = 0$, such that no body outside W is adjacent to two bodies of W, and let G/H be the contraction of $H = G[W]$ to a body of W. If the general configuration of G/H attains, then the general configuration of G attains.*

Proof. The core H satisfies the standing hypotheses (Lemma 30.53, at $d = 2$). Since $\mathrm{def}_3(H) \leq \mathrm{def}_2(H) = 0$ (Lemma 20.7), both deficiencies of H vanish. So H is rigid, and its general configuration attains (Corollary 30.44). Contracting a core of planar deficiency zero does not change the planar deficiency (Lemma 29.22, at $d = 2$). So $\mathrm{def}_2(H) + \mathrm{def}_2(G/H) = \mathrm{def}_2(G)$, and Theorem 30.61 applies. □

30.6. **Ears and cycles.** This subsection proves two steps of the induction, at a cycle and at an open ear with at least five interior bodies. It also proves a third result, at a closed ear, which the induction does not use.

Definition 30.63 (Open ears)**.** Let $V_1 \subseteq V(G)$. An *open ear* of G on V_1 is a path $a - x_1 - \cdots - x_k - b$ ($k \geq 0$) with distinct ends $a, b \in V_1$ and distinct *interior bodies* $x_1, \ldots, x_k$ outside V_1, such that $V(G) = V_1 \cup \{x_1, \ldots, x_k\}$ and every edge of G off the path lies inside V_1.

The ends a and b may be adjacent, by an edge off the path. An open ear with $k = 0$ is an edge ab, on $V_1 = V(G)$; the steps of the induction take $k \geq 1$. A *closed*

ear $c - x_1 - \cdots - x_k - c$ on V_1 ($k \geq 1$) meets the same conditions with its two ends equal to one body $c \in V_1$.

The rank along an open ear has an exact formula. Let p be a configuration of G whose hinges on the ear are nonzero. Write $R(H, p)$ for the rigidity matrix of a subgraph H of G with the hinges of p (Definition 30.8). Let Λ be the span of the hinges of the ear. Let ρ be the space of relative screws of a and b in $G[V_1]$: the differences $S(b) - S(a)$ over the infinitesimal motions S of $G[V_1]$ with these hinges (Definition 19.27). Then

$$\operatorname{rank} R(G, p) \;=\; \operatorname{rank} R(G[V_1], p) + 5(k+1) + \dim(\rho + \Lambda) - 6$$

(Lemma 19.32, at $D = 6$). Here $5(k+1)$ is the rank of the ear's path on its own, and Λ is the space of its relative screws at a and b. The path meets $G[V_1]$ only at a and b. So the two row spaces meet in the span of the joint rows that confine the relative screw of a and b to $\rho + \Lambda$ (Lemma 19.28). That span has dimension $6 - \dim(\rho + \Lambda)$. The ear adds the full $5(k+1)$ to the rank exactly when $\rho + \Lambda$ is the whole screw space.

We follow the scheme of Section 30.4, with this formula and Lemma 20.31 for the count. The ear adds k bodies, and the deficiency grows by at least $k - 5$. So the target grows by at most $6k - (k-5) = 5(k+1)$. Hence G has rank at least $\operatorname{tgt}(G)$ at a configuration where $G[V_1]$ has its target and $\rho + \Lambda$ is the whole screw space. An ear with at least five interior bodies has at least six hinges, and we show that at a general configuration they span the screw space. With fewer interior bodies they cannot, and the next two subsections bound $\dim(\rho + \Lambda)$ below under further hypotheses (Section 30.7, Section 30.8).

At an open ear the height must meet two open conditions in $L_G(q)$. As at a cut vertex, $G[V_1]$ must have its target at the restriction. In addition, six hinges of the ear must be linearly independent. Under the polarity, the hinge at an edge uw corresponds to the join $p_u \vee p_w$ (Lemma 30.9). The independence of six joins is a polynomial condition, in the picture at fixed heights and in the heights at a fixed picture (Lemma 30.67). So it suffices that it hold at one configuration. That configuration may lie over a non-admissible picture, provided that its heights are heights of G at a general picture. We place six edges of the ear on the edges of the closed hexagon of Lemma 30.68, whose joins are independent over every field. This placement puts a and b at one point, and its picture is not admissible. Its heights are nonzero only at x_2 and x_3, which lie in no closed neighbourhood of a body of V_1. Each interior body has a closed neighbourhood of three members, which impose no condition on the heights (Lemma 30.64). So these heights are heights of G at every admissible picture (Lemma 30.65). This needs $k \geq 4$; six distinct edges need $k \geq 5$.

A cycle is an edge ab together with an open ear on $\{a, b\}$, and the same formula gives its rank. Each closed neighbourhood of a cycle has three members, so at an admissible picture every height is a height of the cycle. We therefore keep the heights of a closed polygon of Lemma 30.68 and choose only the picture. No two open conditions need to meet. A closed ear makes its end a cut vertex whose other side is a cycle. So the closed ear follows from the cut-vertex step and the cycle. The induction does not use it: the structural coverage reduces every graph with a cut vertex at a cut vertex or along a chain of bridges (Lemma 30.118).

We begin with the heights.

Lemma 30.64 (Three points impose nothing)**.** *Let q be admissible for G, and let v be a body of G whose closed neighbourhood has at most three members. Then every height agrees with an affine function of the picture on the closed neighbourhood of v.*

Proof. By admissibility three members of the closed neighbourhood of v have linearly independent homogeneous picture points, so they are all its members, and any three values at them are taken by exactly one affine function. □

Lemma 30.65 (The heights of an open ear)**.** *Let $a - x_1 - \cdots - x_k - b$ be an open ear of G on V_1 with $k \geq 2$, and let q be admissible for G.*

(1) A height lies in $L_G(q)$ as soon as it agrees with an affine function of the picture on the closed neighbourhood in G of every body of V_1.
(2) Every $z^1 \in L_{G[V_1]}(q)$ is the restriction to V_1 (Lemma 30.46) of a height $z \in L_G(q)$ taking any prescribed values at $x_2, \dots, x_{k-1}$.

Proof. The closed neighbourhood of an interior body consists of the body and its two neighbours on the path, and every height is affine on it (Lemma 30.64). This gives (1). For (2), the closed neighbourhood in G of a body of V_1 is its closed neighbourhood in $G[V_1]$, together with x_1 at a and x_k at b. Take affine functions h_a and h_b agreeing with z^1 on the closed neighbourhoods of a and b in $G[V_1]$. Set $z_{x_1} = h_a(q_{x_1})$ and $z_{x_k} = h_b(q_{x_k})$, which is possible since $k \geq 2$ makes $x_1 \neq x_k$. Then (1) applies. □

The next four lemmas give the independence of the hinges, starting from the coordinates of a join.

Lemma 30.66 (Joins in flat coordinates)**.** *For $p = (p_x, p_y, p_z, p_w) \in K^4$ write $\bar{p} = (p_x, p_y, p_w) \in K^3$. There is a linear isomorphism of the screw space $\bigwedge^2 K^4$ onto $K^3 \times K^3$, adapted to the splitting of Lemma 30.31, that sends the join $p \vee p'$ of $p, p' \in K^4$ to $(p_z \bar{p}' - p'_z \bar{p},\ \bar{p} \times \bar{p}')$. In particular, a family of joins is linearly independent as soon as the family of these vectors is.*

Proof. Expand $p \vee p'$ in the basis of W' and W_Π used in the proof of Lemma 30.31: the coefficients on W_Π are the 2×2 minors of $\bar{p}$ and $\bar{p}'$, and those on W' pair the heights with the planar parts. □

Lemma 30.67 (Independence of joins is an open condition)**.** *Let $(u_i, w_i)_{i \in I}$ be a finite family of pairs of bodies, and write p for the points of a configuration (q, z). If the joins $p_{u_i} \vee p_{w_i}$ are linearly independent at (q_0, z), there is a polynomial P in the picture, nonzero at q_0, such that they are linearly independent at (q, z) whenever $P(q) \neq 0$. Likewise, if they are independent at (q, z_0), there is a polynomial R in the heights, nonzero at z_0, such that they are independent at (q, z) whenever $R(z) \neq 0$.*

Proof. By Lemma 30.66 the coordinates of each join are polynomials in the picture at fixed heights, and in the heights at a fixed picture. A finite family of vectors is linearly independent exactly when some maximal minor of its coordinate matrix is nonzero; take for P, or R, one such minor. □

Lemma 30.68 (Four closed polygons with independent joins)**.** *Let $Y_0, \dots, Y_5 \in K^4$ be the points $(1,0,0,1)$, $(0,-1,0,1)$, $(0,0,1,1)$, $(-1,0,1,1)$, $(0,0,0,1)$ and $(-1,-1,0,1)$. For $n = 3,4,5,6$, the n joins $Y_0 \vee Y_1, \dots, Y_{n-2} \vee Y_{n-1}, Y_{n-1} \vee Y_0$ of the closed polygon $Y_0 Y_1 \cdots Y_{n-1}$ are linearly independent, over every field.*

Proof. By Lemma 30.66 the joins have integer coordinates, and in each of the four cases their coordinate matrix has a maximal minor equal to ± 1, which is nonzero in every characteristic. □

Formalization note 30.69. The four pins are the cases $n = 3$, 4, 5 and 6, in this order.

Lemma 30.70 (Independent joins span the hinges)**.** *Let p be the points of a configuration of G, and let $g_1, \dots, g_m$ be edges of G, with ends u_i, w_i, whose joins $p_{u_i} \vee p_{w_i}$ are linearly independent. Then the hinges at $g_1, \dots, g_m$ of the panel-hinge framework defining the rank of p (Definition 30.8) span a space of dimension at least m. In particular, if $m = 6$, the hinges at any set of edges containing $g_1, \dots, g_6$ span the whole screw space.*

Proof. The hinge at an edge uw spans the line of the meet of the planes $p_u^\perp$ and $p_w^\perp$. The polarity, an invertible linear map of the screw space, maps the join $p_u \vee p_w$ to that meet (Lemma 30.9). So the meets at $g_1, \dots, g_m$ are independent and lie in the span of the hinges there. The screw space has dimension six. □

Theorem 30.71 (A cycle)**.** *Let K be infinite and let G be a cycle on $m \geq 3$ bodies: $V(G) = \{a, x_1, \dots, x_k, b\}$ with these bodies distinct and $k = m - 2$, and the edges of G are those of the path $a - x_1 - \cdots - x_k - b$ and one edge ab. Then the general configuration of G attains.*

Proof. The configuration. Every closed neighbourhood of G has three members, so at an admissible picture q every height lies in $L_G(q)$ (Lemma 30.64). Put the bodies at the points of Lemma 30.68. For $m \leq 5$, put $a, x_1, \dots, x_k, b$ at $Y_0, \dots, Y_{m-1}$, so that the joins at the m edges are those of the closed m-gon. For $m \geq 6$, put a, x_1, x_2, x_3 at $Y_0, \dots, Y_3$, the bodies $x_4, \dots, x_k$ at Y_4 and b at Y_5. Then the joins at the edges ax_1, x_1x_2, x_2x_3, x_3x_4, x_kb and ba are those of the closed hexagon. Let z be the heights of these points, and let P be the polynomial in the picture that Lemma 30.67 gives for these $\min(m, 6)$ joins at the heights z. Take a picture q off the zero sets of P and of the main-picture polynomial of G (Lemma 30.23). Then q is a main picture of G, and the joins are independent at (q, z).

The count. The cycle is the edge ab with the open ear $a - x_1 - \cdots - x_k - b$ on $\{a, b\}$, and every hinge is nonzero at the admissible picture q. The edge has rank 5 and relative screws spanned by its hinge (Lemma 19.30 and Lemma 19.31, at $k = 0$). So by Lemma 19.32 the rank at (q, z) is $5 + 5(k+1) + \lambda - 6 = 5m - 6 + \lambda$, where λ is the dimension of the span of all the hinges. By Lemma 30.70, $\lambda \geq \min(m, 6)$. The partition into single bodies has value $6(m-1) - 5m = m - 6$ (Definition 20.3), so $\mathrm{def}(\tilde{G}) \geq \max(0, m-6)$. Hence $\mathrm{tgt}(G) \leq \min(6m - 6, 5m) \leq 5m - 6 + \lambda$, and by Lemma 30.20 the general configuration of G attains. □

Theorem 30.72 (An open ear with at least five interior bodies)**.** *Let K be infinite and let G satisfy the standing hypotheses (Definition 30.45), with an open ear $a - x_1 - \cdots - x_k - b$ on V_1 with $k \geq 5$, whose ends a and b may be adjacent. If the general configuration of $G[V_1]$ attains, then the general configuration of G attains.*

Proof. The hexagon. Let q^* put every body of V_1 at $(1, 0)$, the bodies x_1, x_2, x_3 at $(0, -1)$, $(0, 0)$, $(-1, 0)$, the bodies $x_4, \ldots, x_{k-1}$ at $(0, 0)$ and x_k at $(-1, -1)$. Let z^* be 1 at x_2 and x_3 and 0 elsewhere. The points of $a, x_1, x_2, x_3, x_4, x_k$ are then $Y_0, \ldots, Y_5$ (Lemma 30.68), and x_{k-1} is at Y_4 too. So the joins at the six ear edges ax_1, x_1x_2, x_2x_3, x_3x_4, $x_{k-1}x_k$ and x_kb, distinct since $k \geq 5$, are those of the closed hexagon. Let P be the polynomial in the picture that Lemma 30.67 gives for these joins at the heights z^*; it is nonzero at q^*.

The picture and the height. Take a picture q off the zero sets of P, of the polynomial of the attainment at $G[V_1]$, and of the main-picture polynomial of G (Lemma 30.23). The height z^* vanishes on V_1 and at x_1 and x_k. So it lies in $L_G(q)$ (Lemma 30.65(1)), and the six joins are independent at (q, z^*). The restriction maps $L_G(q)$ onto $L_{G[V_1]}(q)$ (Lemma 30.65(2)). So the heights of $L_G(q)$ at which $G[V_1]$ has its target contain a nonempty Zariski-open subset (Lemma 30.46). So do the heights at which the six joins stay independent, as z^* is one of them (Lemma 30.67, in the heights). The two subsets meet (Lemma 30.26); let z be a height in both.

The count. At (q, z) the rank of $G[V_1]$ is its target (Lemma 30.47). The hinges of the ear span the screw space (Lemma 30.70), so $\rho + \Lambda$ is the whole screw space. Every hinge is nonzero at the admissible picture q, so the rank of G exceeds that of $G[V_1]$ by $5(k + 1)$ (Lemma 19.32). The target grows by at most $5(k + 1)$, as in the opening of this subsection (Lemma 20.31). So by Lemma 30.20 the general configuration of G attains. □

Theorem 30.73 (A closed ear)**.** *Let K be infinite and let G satisfy the standing hypotheses, with a closed ear $c - x_1 - \cdots - x_k - c$ on V_1 with $k \geq 2$. If the general configuration of $G[V_1]$ attains, then the general configuration of G attains.*

Proof. The body c is a cut vertex: the sides V_1 and $V_2 = \{c, x_1, \ldots, x_k\}$ meet in c, and every edge of G lies inside one of them. As G has no loop, $G[V_2]$ is the cycle $c - x_1 - \cdots - x_k - c$ on $k+1 \geq 3$ bodies. It attains by Theorem 30.71, with the edge x_1c in the role of ab. So the general configuration of G attains by Theorem 30.49. □

Remark 30.74 (Graphs built from a cycle by ears)**.** By induction on the number of ears, the three theorems above show that the general configuration attains at every graph built from a cycle by adding, one at a time, closed ears with at least two interior bodies and open ears with at least five. Each graph along the way satisfies the standing hypotheses, and removing its last ear leaves the subgraph induced on the other bodies. The statement is not formalized separately: it is a special case of Theorem 30.124.

30.7. **Open ears with two, three or four interior bodies.** This subsection proves the ear steps for an open ear $a - x_1 - \cdots - x_k - b$ on V_1 with $k = 2, 3, 4$, whose ends may be adjacent. We follow the scheme of Section 30.4, and we count with the ear rank formula of Section 30.6. The target of G exceeds that of $G[V_1]$ by

$6k - \mathrm{def}(\tilde{G}) + \mathrm{def}(\tilde{G}[V_1])$. So a configuration of G over a main picture of G, at which $G[V_1]$ has its target, has rank at least $\mathrm{tgt}(G)$ as soon as

$$\dim(\rho + \Lambda) \;\geq\; k + 1 + \mathrm{def}(\tilde{G}[V_1]) - \mathrm{def}(\tilde{G}). \qquad (\star)$$

With two interior bodies we assume $\mathrm{def}(\tilde{G}[V_1]) \leq \mathrm{def}(\tilde{G})$. Then $(\star)$ holds once the three hinges of the ear are linearly independent, and the height is found as in the step at five or more interior bodies (Theorem 30.72). The three joins of the ear are independent at one flat configuration, whose picture need not be admissible, and the heights 0 lie in every lifting space.

With three or four interior bodies no bound on the deficiencies is assumed. Instead the step also uses the graph G'' obtained from G by suppressing x_2: the splitting-off $G''^{x_1x_3}_{x_2}$ whose new edge x_1x_3 carries the label of x_1x_2 (Definition 21.10). It has the same $G[V_1]$ and an open ear on V_1 with $k-1$ interior bodies. Its deficiency is at most that of G (Lemma 21.34).

We count against G''. Let W be the span of ρ and the hinges of the ear of G''. Where $G[V_1]$ and G'' have their targets, the ear rank formula at G'' bounds $\dim W$ below, and $(\star)$ asks for at most one dimension more. To gain it we put x_2 back on a line through the point of x_1 or of x_3. This raises the span by one unless W already contains every line through that point (Lemma 30.81). It therefore suffices that W be the whole screw space whenever it contains every line through the points of x_1 and of x_3. Then x_2 is put at the point of x_1, and the span still contains W. With four interior bodies, such a W contains the six lines joining the points of x_1, x_3, x_4 and b. These span the screw space when the four points are linearly independent (Lemma 30.77). With three, the lines through the points of x_1 and of x_3 span at least five dimensions (Lemma 30.78). At a general ear, either some relative screw lies outside their span, and then such a W is the whole screw space, or W has dimension at most three and cannot contain them (Lemma 30.80).

The configuration is chosen in two stages. First, the picture is chosen as in the scheme: general for $G[V_1]$ and for G'', and a main picture of G. At it we take one height of G'' at which both smaller graphs have their targets. Its restriction to V_1 fixes the data on V_1, and with them ρ and the rank of $G[V_1]$. Second, the bodies of the ear move with their heights through a polynomial family of configurations of G and of G'', the ear data (Definition 30.82), and we take a general member of this family. Each condition that the step needs is the non-vanishing of a polynomial in the ear data that is nonzero at one member (Lemma 30.84, Lemma 30.85).

We first collect the geometry of lines that the count uses, then the ear data, and then prove the three steps.

The steps with three and four interior bodies compute ranks with the joins of the points as hinges (Lemma 30.9), in the screw space $\bigwedge^2 K^4$. For $c \in \bigwedge^2 K^4$ write $(\sigma(c), \tau(c)) \in K^3 \times K^3$ for its flat coordinates (Lemma 30.66), so that $\sigma(p \vee p') = p_z\,\bar{p}' - p'_z\,\bar{p}$ and $\tau(p \vee p') = \bar{p} \times \bar{p}'$.

Definition 30.75 (The pairing of lines)**.** The *pairing* of $c, c' \in \bigwedge^2 K^4$ is

$$\langle c, c' \rangle = \sigma(c) \cdot \tau(c') + \sigma(c') \cdot \tau(c).$$

The pairing is bilinear and symmetric.

Lemma 30.76 (The pairing of two joins)**.** *For $p, q, r, s \in K^4$, $\langle p \vee q, r \vee s\rangle = \det(p, q, r, s)$; in particular two lines through a common point pair to zero.*

Proof. Expand both sides in the coordinates of the four points, with the flat coordinates of Lemma 30.66. □

Two lines therefore pair to zero exactly when their four points are linearly dependent, that is, when the lines meet. The pairing is the classical bilinear form of line geometry. In statics, Crapo and Whiteley use it as the product of two wrenches, which on two forces is the same determinant [CW82, Examples 2.4–2.5, pp. 49–50].

The next lemma is the case $d = 3$ of KT Lemma 2.1 [KT11, pp. 651–652] (Lemma 18.9), for any four linearly independent vectors of K^4. Its proof is that of KT, with the pairing in place of the join with the opposite edge.

Lemma 30.77 (The joins of four independent points)**.** *If $p_0, p_1, p_2, p_3 \in K^4$ are linearly independent, the six joins $p_i \vee p_j$, $i < j$, are linearly independent, so they span $\bigwedge^2 K^4$.*

Proof. Pair a vanishing combination $\sum c_{ij}\, p_i \vee p_j = 0$ with the opposite edge $p_k \vee p_l$. By Lemma 30.76 the pairing of two edges is $\pm\det(p_0, p_1, p_2, p_3) \neq 0$ when they are opposite and 0 when they share a point, so every c_{ij} vanishes. The space $\bigwedge^2 K^4$ has dimension six. □

The *star* of a point $y \in K^4$ is the span $\mathrm{star}(y)$ of the joins $y \vee u$, $u \in K^4$: the lines through y.

Lemma 30.78 (The lines through two points)**.** *For $y, y' \in K^4$, every element c of $\mathrm{star}(y) + \mathrm{star}(y')$ has $\langle c, y \vee y'\rangle = 0$. If y and y' are linearly independent, then $\dim(\mathrm{star}(y) + \mathrm{star}(y')) \geq 5$.*

Proof. A line through y or through y' meets the line yy', so the pairing vanishes (Lemma 30.76). Extend y, y' to a basis y, y', e, e' of K^4. Five of the six joins of Lemma 30.77, all but $e \vee e'$, lie in the two stars. □

At a height of a graph, the points of the closed neighbourhood of a body lie on one plane, the plane of that body: the graph of the affine function with which the height agrees there (Definition 30.2). In the step with three interior bodies, the points of a and x_1 lie on the plane of a, and those of x_3 and b on the plane of b. For $h \in K^3$ write $\hat{\pi}_h = \{(u_1, u_2, h \cdot u, u_3) : u \in K^3\} \subseteq K^4$ for the homogeneous points of the plane $z = h_1x + h_2y + h_3$, and $\bigwedge^2 \hat{\pi}_h$ for the space of screws c with $\sigma(c) = -\,h \times \tau(c)$.

Lemma 30.79 (The lines of a plane)**.** *For $h \in K^3$, the join of any two points of $\hat{\pi}_h$ lies in $\bigwedge^2 \hat{\pi}_h$, and $\dim \bigwedge^2 \hat{\pi}_h \leq 3$.*

Proof. The first claim is a computation in the flat coordinates of Lemma 30.66. On $\bigwedge^2 \hat{\pi}_h$ the coordinate σ is determined by τ, so τ is injective there. □

Lemma 30.80 (Lines joining two planes)**.** *Let $h, h' \in K^3$, and let $c \in \bigwedge^2 K^4$ have $\langle c, u \vee v\rangle = 0$ for every $u \in \hat{\pi}_h$ and every $v \in \hat{\pi}_{h'}$. If $h \neq h'$, then $c = 0$; if $h = h'$, then*

$c \in \bigwedge^2 \hat{\pi}_h$. *Consequently, if every element of a subspace* $\rho \subseteq \bigwedge^2 K^4$ *has this property, and* $f_1, f_2, f_3 \in \bigwedge^2 K^4$ *lie in* $\bigwedge^2 \hat{\pi}_h$ *when* $h = h'$, *then* $\dim(\rho + \operatorname{span}(f_1, f_2, f_3)) \leq 3$.

Proof. For $u, v \in K^3$ let $\hat{u} \in \hat{\pi}_h$ and $\hat{v} \in \hat{\pi}_{h'}$ be the points over them. In the flat coordinates of Lemma 30.66,

$$\langle c, \hat{u} \vee \hat{v} \rangle = \sigma(c) \cdot (u \times v) + (h \cdot u)\,(v \cdot \tau(c)) - (h' \cdot v)\,(u \cdot \tau(c)).$$

We test it on the unit vectors e_1, e_2, e_3 of K^3, and write $\delta = h - h'$ and $\tau = \tau(c)$.

Suppose $h \neq h'$, say $\delta_i \neq 0$. At $u = v = e_i$ the pairing is $\delta_i \tau_i$, so $\tau_i = 0$. For $j \neq i$, the sum of the pairings at (e_i, e_j) and at (e_j, e_i) is $\delta_i \tau_j + \delta_j \tau_i$, so $\tau_j = 0$. So $\tau = 0$, and the pairing at (e_j, e_l) is $\sigma(c) \cdot (e_j \times e_l)$. These cross products are the unit vectors up to sign, so $\sigma(c) = 0$.

Suppose $h = h'$. As $(h \cdot u)(\tau \cdot v) - (h \cdot v)(\tau \cdot u) = (h \times \tau) \cdot (u \times v)$, the pairing is $(\sigma(c) + h \times \tau) \cdot (u \times v)$, and the cross products $u \times v$ span K^3. So $\sigma(c) = -\,h \times \tau(c)$.

For the consequence, if $h \neq h'$ then $\rho = 0$. If $h = h'$ then ρ and the f_i lie in $\bigwedge^2 \hat{\pi}_h$, of dimension at most three (Lemma 30.79). □

In the steps, the points y and y' of the next lemma are those of x_1 and x_3, in either order. Then W is the span for G'', with the join $y \vee y'$ at its edge $x_1 x_3$, and W_t is the span for G with x_2 put back at $y + tu$.

Lemma 30.81 (Putting a body back next to a neighbour)**.** *Let* K *be infinite, let* $\rho \subseteq \bigwedge^2 K^4$ *be a subspace and* F *a finite family in* $\bigwedge^2 K^4$, *let* $y, y', u \in K^4$, *and let* $W = \rho + \operatorname{span} F + K\,(y \vee y')$. *For* $t \in K$ *write* $W_t = \rho + \operatorname{span} F + \operatorname{span}(y \vee (y + tu),\ (y + tu) \vee y')$. *If* $y \vee u \notin W$, *then* $\dim W_t \geq \dim W + 1$ *for some* $t \neq 0$.

Proof. Write $R = \rho + \operatorname{span} F$. For $t \neq 0$, $y \vee (y + tu) = t\,(y \vee u)$ and $(y + tu) \vee y' = v_t$ with $v_t = y \vee y' + t\,(u \vee y')$. So W_t contains R, $y \vee u$ and v_t. A vector v outside a subspace S has $v + tw \notin S$ for one of any two distinct nonzero values of t, since otherwise S contains w and then v. As K is infinite, two such values exist.

If $y \vee y' \in R$, then $W = R$, and $\dim W_t \geq \dim(R + K\,(y \vee u)) = \dim W + 1$ for every $t \neq 0$.

Otherwise $\dim W = \dim R + 1$, and $y \vee y' \notin R + K\,(y \vee u)$, since a relation $y \vee y' = r + c\,(y \vee u)$ with $r \in R$ has $c \neq 0$ and puts $y \vee u$ in W. So some $t \neq 0$ has $v_t \notin R + K\,(y \vee u)$, and $\dim W_t \geq \dim R + 2 = \dim W + 1$. □

The ear data below parametrize the second stage of the configuration. They serve the steps with three and four interior bodies, and the step with two interior bodies at non-adjacent ends (Theorem 30.96). In the steps with three and four interior bodies, the same ear data also give configurations of G'', whose ear has the same first and last interior bodies.

Definition 30.82 (The ear data over fixed base data)**.** Let $a - x_1 - \cdots - x_k - b$ be an open ear of G on V_1 with $k \geq 2$. Fix *base data*: a picture q_1, heights $z_1 \in L_{G[V_1]}(q_1)$, and $h_a, h_b \in K^3$ with $z_1(w) = h_a \cdot (q_1(w), 1)$ on the closed neighbourhood of a in $G[V_1]$, and likewise h_b at b. The *ear data* are a point s giving every body a point of K^2 and a value in K. They give the picture $q(s)$, equal to q_1 on V_1 and to the points of s elsewhere, and the height $z(s)$, equal to z_1 on V_1, to $h_a \cdot (q(s)_{x_1}, 1)$ at

x_1, to $h_b \cdot (q(s)_{x_k}, 1)$ at x_k, to the value of s at the other interior bodies, and to 0 elsewhere. The configuration of s is $(q(s), z(s))$.

Lemma 30.83 (An open ear over fixed base data)**.** *If $q(s)$ is admissible for G, then $z(s) \in L_G(q(s))$. The points of the configuration $(q(s), z(s))$ are polynomial in s, and on V_1 they are those of (q_1, z_1).*

Proof. The lifting space of $G[V_1]$ and the closed neighbourhoods of a and b in $G[V_1]$ read the picture only on V_1, where $q(s)$ is q_1 (Lemma 30.5). So $z(s)$ agrees with an affine function of the picture on the closed neighbourhood in G of every body of V_1. At a this neighbourhood is the one in $G[V_1]$ together with x_1, and $z(s)$ agrees with h_a on it. At b it is the one in $G[V_1]$ together with x_k, which differs from x_1 as $k \geq 2$, and $z(s)$ agrees with h_b on it. At any other body of V_1 it lies inside V_1, where $z(s)$ is z_1. As $z(s)$ vanishes off $V(G)$, Lemma 30.65(1) applies.

Every coordinate of a point of the configuration is a constant, a coordinate of s, or, at x_1 and x_k, affine in the picture coordinates of s. This proves the polynomial claim. On V_1 the coordinates are those of the base data. □

The next two lemmas make each condition of the steps a polynomial condition on the ear data.

Lemma 30.84 (Pictures along the ear data)**.** *If P is a polynomial in the picture, $s \mapsto P(q(s))$ is a polynomial in s.*

Proof. Every coordinate of $q(s)$ is a constant or a coordinate of s (Definition 30.82), so substituting them for the picture's coordinates in P gives a polynomial in s. □

Lemma 30.85 (Openness along a polynomial parametrization)**.** *Let the points of a configuration be polynomial in a parameter s. If, at one parameter, every hinge of a graph H is nonzero and the rank of H is at least r, there is a polynomial in s, nonzero there, such that the rank of H is at least r wherever it does not vanish. If, at one parameter, $\dim(\rho + \operatorname{span} F) \geq n$ for a fixed subspace $\rho \subseteq \bigwedge^2 K^4$ and a family F of joins of points of the configuration, the same holds for this bound.*

Proof. For the rank bound, substitute the points into the polynomial of Lemma 23.120, which asks for the nonzero hinges. For the span, take a basis of ρ together with F. The coordinates of its members are polynomial in s (Lemma 30.66), and n of them are linearly independent at the given parameter. They stay independent wherever a maximal minor of their coordinate matrix, nonzero there, does not vanish. □

Theorem 30.86 (An open ear with two interior bodies)**.** *Let K be infinite and let G satisfy the standing hypotheses (Definition 30.45), with an open ear $a - x_1 - x_2 - b$ on V_1 whose ends a and b may be adjacent, and suppose $\operatorname{def}(\tilde{G}[V_1]) \leq \operatorname{def}(\tilde{G})$ (Definition 20.3, at $d = 3$). If the general configuration of $G[V_1]$ attains, then the general configuration of G attains.*

Proof. The hexagon. Let q^* put a, x_1, x_2 and b at $(0,0)$, $(-1,-1)$, $(1,0)$ and $(0,-1)$, and every other body at $(0,0)$. At the heights 0 the points of a, x_1, x_2, b are then Y_4, Y_5, Y_0, Y_1 (Lemma 30.68). So the joins at the three ear edges ax_1, x_1x_2 and x_2b are $Y_4 \vee Y_5$, $Y_5 \vee Y_0$ and $Y_0 \vee Y_1$. These are three joins of the closed hexagon, and they

are linearly independent. Let P be the polynomial in the picture that Lemma 30.67 gives for these joins at the heights 0; it is nonzero at q^*. The picture q^* need not be admissible.

The picture and the height. Take a picture q off the zero sets of P, of the polynomial of the attainment at $G[V_1]$, and of the main-picture polynomial of G (Lemma 30.23). The heights 0 lie in the linear space $L_G(q)$, and the three joins are independent at $(q, 0)$. We find the height as in the proof of Theorem 30.72. The heights of $L_G(q)$ whose restriction to V_1 gives $G[V_1]$ its target contain a nonempty Zariski-open subset (Lemma 30.65(2), Lemma 30.46). So do the heights at which the three joins stay independent (Lemma 30.67, in the heights). The two subsets meet (Lemma 30.26); let z be a height in both.

The count. At (q, z) the rank of $G[V_1]$ is its target (Lemma 30.47), and every hinge is nonzero at the admissible picture q. The hinges of the ear span a space of dimension at least three (Lemma 30.70), so $\dim(\rho+\Lambda) \geq 3$. As $\operatorname{def}(\tilde{G}[V_1]) \leq \operatorname{def}(\tilde{G})$, this gives $(\star)$ at $k = 2$. So Lemma 19.32 and Lemma 30.20 apply. □

Remark 30.87. At an admissible picture the three joins are independent at the heights 0 already. Admissibility at x_1 and at x_2 keeps a, x_1, x_2 and x_1, x_2, b off a line. So the lines ax_1, x_1x_2 and x_2b of the plane $z = 0$ have no common point, and three such lines are linearly independent. The proof needs the joins independent at one configuration only, and takes the hexagon's, as at five or more interior bodies.

Theorem 30.88 (An open ear with four interior bodies)**.** *Let K be infinite and let G satisfy the standing hypotheses, with an open ear $a - x_1 - x_2 - x_3 - x_4 - b$ on V_1 whose ends a and b may be adjacent. Let G'' be G with x_2 suppressed: the splitting-off $G''^{x_1x_3}_{x_2}$ (Definition 21.10) whose new edge x_1x_3 carries the label of the deleted edge x_1x_2. If the general configurations of $G[V_1]$ and of G'' attain, then the general configuration of G attains.*

Proof. Write $f = \operatorname{def}(\tilde{G}[V_1])$. The graph G'' has the open ear $a - x_1 - x_3 - x_4 - b$ on V_1, and the same $G[V_1]$. We choose the configuration in the two stages of the opening of this subsection. The second stage takes a general ear datum twice: once for G'', and once for G after x_2 is put back. The stages and the count are the same in the proof of Theorem 30.89 (`Graph.XOAttains.of_openEar_splitOff`). Only the third condition on the ear data, and its use in putting x_2 back, are specific to four interior bodies.

The first stage: base data. Take a picture q^* off the zero sets of the main-picture polynomial of G (Lemma 30.23) and of the polynomials of the attainment at $G[V_1]$ and at G''. In $L_{G''}(q^*)$ the heights at which G'' has its target contain a nonempty Zariski-open subset. So do the heights whose restriction to V_1 gives $G[V_1]$ its target (Lemma 30.65(2) for the ear of G'', Lemma 30.46). The two subsets meet (Lemma 30.26); let z^* be a height in both. The base data (Definition 30.82) are the restriction (q_1, z_1) of (q^*, z^*) to V_1, and the affine functions h_a and h_b with which z^* agrees on the closed neighbourhoods of a and b in G''. These contain the closed neighbourhoods of a and b in $G[V_1]$, as Definition 30.82 asks, and also x_1 and x_4. So (q^*, z^*) is the configuration of one ear datum s^*. The rank of $G[V_1]$ and its relative screws ρ depend only on the hinges at the edges of $G[V_1]$, and the ear data do not

move the points on V_1 (Lemma 30.47, Lemma 30.5). So at every ear datum, in the frameworks of G and of G'' alike, $G[V_1]$ has its target and the relative screws ρ (`Graph.exists_earBase_splitOff`).

The second stage, for G''. We ask three conditions of the ear data. Each holds off the zero set of a polynomial that is nonzero at one ear datum (Lemma 30.84, Lemma 30.85). The first two are that G'' has rank at least its target and that the picture is admissible for G''. Both hold at s^*, where the hinges of G'' are nonzero. The third is that the six joins of the points of x_1, x_3, x_4 and b span $\bigwedge^2 K^4$. Let q_b and z_b be the picture and the height of b. Put x_1 over $q_b + (0,1)$ on the plane of a, x_4 over $q_b + (1,0)$ on the plane of b, and x_3 over q_b at the height $z_b + 1$. The points of x_1, x_3, x_4 and b are then linearly independent, so the third condition holds there (Lemma 30.77). Take ear data y off all three zero sets, and let y_1, y_3 and y_4 be the points of x_1, x_3 and x_4. Then G'' has rank at least its target and $G[V_1]$ has its target. So by the ear rank formula at G'', with the joins as hinges (Lemma 19.32, Lemma 30.9),

$$W = \rho + \operatorname{span}(p_a \vee y_1,\ y_1 \vee y_3,\ y_3 \vee y_4,\ y_4 \vee p_b)$$

has $\dim W \geq 4 + f - \operatorname{def}(\tilde{G''})$.

Putting x_2 back. Keep the other bodies of y and give x_2 an affine point x. As x_2 is a middle body of the ear, this is an ear datum of G. Let Λ be the span of the joins of its ear. Every line through an affine point y' is $y' \vee u$ with u of last coordinate 0, since $y' \vee v = y' \vee (v - v_4 y')$. If W does not contain every line through y_1, take such a u with $y_1 \vee u \notin W$. Then $y_1 + tu$ is affine for every t, and for some $t \neq 0$ the point $x = y_1 + tu$ has $\dim(\rho + \Lambda) \geq \dim W + 1$ (Lemma 30.81). The same holds at y_3. Suppose that W contains every line through y_1 and every line through y_3. Then it contains $y_1 \vee y_3$, $y_1 \vee y_4$, $y_1 \vee p_b$, $y_3 \vee y_4$ and $y_3 \vee p_b$, and also $y_4 \vee p_b$, a join of the ear of G''. These are the six joins of y_1, y_3, y_4, p_b, which span $\bigwedge^2 K^4$ by the choice of y. So $W = \bigwedge^2 K^4$, and we take $x = y_1$: the join $y_1 \vee x$ vanishes, but $\rho + \Lambda$ contains W. Either way $\dim(\rho + \Lambda) \geq \min(\dim W + 1, 6)$ (`exists_insertion_four`).

The configuration of G. The ear datum just built may have a vanishing hinge, but the span bound of Lemma 30.85 asks for none. So $\dim(\rho + \Lambda) \geq \min(\dim W + 1, 6)$ holds off the zero set of a polynomial in the ear data of G that is nonzero there. The condition that the picture be a main picture of G is polynomial too, and nonzero at s^* (Lemma 30.84). Take an ear datum off both zero sets. Its picture is a main picture of G, so the hinges are nonzero, and its heights lie in L_G (Lemma 30.83). As at every ear datum, $G[V_1]$ has its target and the relative screws ρ.

The count. $\operatorname{def}(\tilde{G''}) \leq \operatorname{def}(\tilde{G})$ (Lemma 21.34), and $\operatorname{def}(\tilde{G}) \geq f + k - 5 = f - 1$ (Lemma 20.31). So $\min(\dim W + 1, 6) \geq 5 + f - \operatorname{def}(\tilde{G})$, which is $(\star)$ at $k = 4$. By Lemma 19.32 and Lemma 30.20 the general configuration of G attains. □

Theorem 30.89 (An open ear with three interior bodies)**.** *Let K be infinite and let G satisfy the standing hypotheses, with an open ear $a - x_1 - x_2 - x_3 - b$ on V_1 whose ends a and b may be adjacent. Let G'' be G with x_2 suppressed: the splitting-off G''_{x_2} (Definition 21.10) whose new edge x_1x_3 carries the label of the deleted edge x_1x_2. If*

the general configurations of $G[V_1]$ and of G'' attain, then the general configuration of G attains.

Proof. We follow the proof of Theorem 30.88 (`Graph.X0Attains.of_openEar_splitOff`), with the open ear $a{-}x_1{-}x_3{-}b$ of G''. Write $f = \mathrm{def}(\tilde{G}[V_1])$. The first stage gives the base data, the ear datum s^* and the relative screws ρ (`Graph.exists_earBase_splitOff`). Write $\hat{\pi}_a = \hat{\pi}_{h_a}$ and $\hat{\pi}_b = \hat{\pi}_{h_b}$ for the homogeneous points of the planes of a and b. At every ear datum, p_a and the point y_1 of x_1 lie in $\hat{\pi}_a$, and the point y_3 of x_3 and p_b lie in $\hat{\pi}_b$.

The case split on ρ, made before the ear data are chosen. In the first case $\langle c, u \vee v\rangle = 0$ for every $c \in \rho$, $u \in \hat{\pi}_a$ and $v \in \hat{\pi}_b$. In the second, some $c \in \rho$ pairs to a nonzero value with some $u_0 \vee v_0$. We then ask one more condition of the ear data, $\langle c, y_1 \vee y_3\rangle \neq 0$. For $s, s' \in K^3$ let $\hat{s} \in \hat{\pi}_a$ and $\hat{s}' \in \hat{\pi}_b$ be the points over them. The map $(s, s') \mapsto \langle c, \hat{s} \vee \hat{s}'\rangle$ is bilinear and not zero. The vectors $(0,0,1)$, $(1,0,1)$ and $(0,1,1)$ span K^3, so it is nonzero at a pair of them. The points over them are affine, so some ear datum puts x_1 and x_3 there (`exists_klein_liftPlane_affine_ne_zero`). Let $c^\perp$ be the hyperplane of screws pairing to zero with c. The condition says $\dim(c^\perp + K\,(y_1 \vee y_3)) = 6$. So it holds off the zero set of a polynomial in the ear data that is nonzero at that ear datum (Lemma 30.85).

The second stage, for G''. Take ear data y off the zero sets of the polynomials that ask the rank of G'' to be at least its target and the picture to be admissible for G''. In the second case, take y also off the zero set for $\langle c, y_1 \vee y_3\rangle$. As in the four-body step, the ear rank formula at G'' (Lemma 19.32) shows that

$$W = \rho + \mathrm{span}(p_a \vee y_1,\ y_1 \vee y_3,\ y_3 \vee p_b)$$

has $\dim W \geq 3 + f - \mathrm{def}(\tilde{G''})$. As x_1x_3 is an edge of G'' and the picture is admissible for G'', the pictures of x_1 and x_3 differ. So y_1 and y_3 are linearly independent, and $\mathrm{star}(y_1) + \mathrm{star}(y_3)$ has dimension at least five and pairs to zero with $y_1 \vee y_3$ (Lemma 30.78).

A span containing both stars. We show that W contains both stars only if $W = \bigwedge^2 K^4$. In the first case $\dim W \leq 3$ (Lemma 30.80). When $h_a = h_b$ this uses that the three joins of the ear of G'' are lines of the one plane $\hat{\pi}_a = \hat{\pi}_b$ (Lemma 30.79). So W does not contain both stars (`not_star_sup_star_le_of_klein`). In the second case c lies in W, and outside $\mathrm{star}(y_1) + \mathrm{star}(y_3)$, as it pairs to a nonzero value with $y_1 \vee y_3$. So a W containing both stars has dimension six (`eq_top_of_star_sup_star_le`).

Putting x_2 back, and the configuration of G. We argue as in the proof of Theorem 30.88. Some affine point x for x_2 has $\dim(\rho + \Lambda) \geq \min(\dim W + 1, 6)$, where $\Lambda = \mathrm{span}(p_a \vee y_1,\ y_1 \vee x,\ x \vee y_3,\ y_3 \vee p_b)$ (`exists_insertion_three`). It lies on a line through y_1 or y_3 that is not in W (Lemma 30.81), or at y_1 when $W = \bigwedge^2 K^4$. This bound and the main-picture condition are polynomial conditions on the ear data of G (Lemma 30.84, Lemma 30.85). At a common non-root the picture is a main picture of G, so the hinges are nonzero, and the heights lie in L_G (Lemma 30.83); $G[V_1]$ has its target and the relative screws ρ.

The count. $\operatorname{def}(\tilde{G}'') \le \operatorname{def}(\tilde{G})$ (Lemma 21.34), and $\operatorname{def}(\tilde{G}) \ge f + k - 5 = f - 2$ (Lemma 20.31). So $\min(\dim W + 1, 6) \ge 4 + f - \operatorname{def}(\tilde{G})$, which is $(\star)$ at $k = 3$. By Lemma 19.32 and Lemma 30.20 the general configuration of G attains. □

Remark 30.90 (θ-graphs)**.** A θ-graph has two branch bodies joined by three internally disjoint paths, of $p_1 \le p_2 \le p_3$ edges. It satisfies the standing hypotheses when $p_2 \ge 2$. Induction on the number of bodies then shows that its general configuration attains, by the ear steps along its longest path. The two shorter paths form a cycle C on $s = p_1 + p_2$ bodies, which attains (Theorem 30.71). The longest path is an open ear on $V(C)$ with $k = p_3 - 1$ interior bodies, whose ends are adjacent when $p_1 = 1$. For $k \ge 5$, Theorem 30.72 applies. For $k = 3, 4$, Theorem 30.89 and Theorem 30.88 apply, since suppressing x_2 leaves the θ-graph with paths of p_1, p_2 and $p_3 - 1$ edges. For $k = 2$, $s \le 6$. A partition of C into $r \ge 2$ parts has at least r edges between its parts, so its value is at most $r - 6 \le 0$ (Definition 20.3). Hence $\operatorname{def}(\tilde{C}) = 0$, and Theorem 30.86 applies. The remaining θ-graphs, $K_4 - e$ and $K_{2,3}$, have both deficiencies zero, and Corollary 30.44 applies.

The structural coverage (Theorem 30.124) reaches a θ-graph through its general cases: by Corollary 30.44 when its two deficiencies agree, by contraction when it has a planar-rigid set (Section 30.10), and otherwise at a usable chain (Definition 30.119).

30.8. **Open ears with one or two interior bodies at non-adjacent ends.** This subsection proves the ear steps for an open ear on V_1 with one or two interior bodies, $a - x_1 - b$ or $a - x_1 - x_2 - b$, whose ends a and b are not adjacent. In both steps we assume

$$\operatorname{def}_2(G[V_1]; a, b) + 2 \;\le\; \operatorname{def}_2(G[V_1])$$

(Definition 20.33, at $d = 2$): every partition of V_1 with a and b in one part falls at least two below the planar deficiency of $G[V_1]$. With one interior body we also assume $\operatorname{def}(\tilde{G}[V_1]) \le \operatorname{def}(\tilde{G})$. With two we assume no bound on $\operatorname{def}(\tilde{G})$, unlike Theorem 30.86. Instead we assume that the general configuration of G_1 attains, where G_1 is G with x_2 suppressed. We follow the scheme of Section 30.4. For the count, with ρ and Λ as in Section 30.6, we prove the inequality $(\star)$ of Section 30.7 at $k = 1$ and at $k = 2$. Its derivation there holds for every k.

The hypothesis on the merged deficiency places the planes of the two ends. At a solution (z, h) of the lifting system of $G[V_1]$ (Definition 30.15), let π_a and π_b be the graphs of h_a and h_b, the planes of a and b (Section 30.7). As $z_b = h_b \cdot (q_b, 1)$, the point p_b lies on π_a exactly when $h_a - h_b$ vanishes at $(q_b, 1)$, and likewise for p_a and π_b. If both held, z would be a height of $G[V_1] + ab$, the graph $G[V_1]$ with the edge ab added. Adding ab lowers the planar deficiency by at least two (Lemma 21.36). So at a general picture the lifting space of $G[V_1] + ab$ is at least two smaller than that of $G[V_1]$ (Theorem 30.43, Lemma 30.34). The two conditions are therefore independent on the solutions for $G[V_1]$ (Lemma 30.91). At a general solution, $p_b \notin \pi_a$ and $p_a \notin \pi_b$.

One interior body. Here $(\star)$ at $k = 1$ asks only that $\dim(\rho + \Lambda) \ge 2$, as $\operatorname{def}(\tilde{G}[V_1]) \le \operatorname{def}(\tilde{G})$. The two joins of the ear give this at every admissible picture, where the picture points of a, x_1 and b are not collinear. It remains to find the height. A solution for G restricts to one for $G[V_1]$ whose planes π_a and π_b both contain the point of x_1, so $h_a - h_b$ vanishes at $(q_{x_1}, 1)$. At a fixed picture this is a linear condition

on the solutions for $G[V_1]$, which the open set where $G[V_1]$ has its target need not meet. So we choose the picture of x_1 together with the solution, as a point of their incidence. Such a point can be found off the zero sets of a polynomial in the solution and of one in the picture of x_1 (Lemma 30.92). For this, $h_a - h_b$ must range over a space of affine functions of dimension at least two, as the independence above gives. The result is the configuration of the step (Lemma 30.93). It also has $p_b \notin \pi_a$ and $p_a \notin \pi_b$, which the step with two interior bodies uses.

Two interior bodies. As with three and four interior bodies, we count against a smaller graph, here G_1. It has the one-body ear $a - x_1 - b$ on V_1, and its deficiency is at most that of G (Lemma 21.34). Let W be the span of ρ and the two joins of its ear. Where $G[V_1]$ and G_1 have their targets, the ear rank formula at G_1 (Section 30.6) gives $\dim W$, and $(\star)$ asks for at most one dimension more. We take the configuration of G_1 from the one-body step, with G_1 at its target too. There the point y of x_1 lies on both π_a and π_b, and each end's point lies off the other end's plane. So the two planes meet in a line through y. With u_0 a direction of this line, of last coordinate 0, the points p_a, y, u_0 and p_b are linearly independent. Their six joins span the screw space (Lemma 30.77), and W contains two of them, $p_a \vee y$ and $y \vee p_b$. Unless W is the whole screw space, one of the other four lies outside W. We then place x_1 on π_a and x_2 on π_b, on lines through y chosen for that join, and the span gains a dimension (Lemma 30.95). The heights follow the points through the ear data (Definition 30.82). A general ear datum gives the configuration of G, as in the second stage of Section 30.7.

We prove the independence and the incidence first, then the configuration and the step with one interior body, and then the insertion and the step with two.

In the steps, the graphs G' and H of the next lemma are $G[V_1]$ and $G[V_1] + ab$.

Lemma 30.91 (The incidence functionals at the two ends are independent)**.** *Let G' and H be graphs on the same bodies, and a, b two bodies of G', such that every closed neighbourhood in H lies in the closed neighbourhood of the same body in G', except that the one of a may also contain b and the one of b may also contain a. Let q be admissible for G', with $\dim L_H(q) + 2 \leq \dim L_{G'}(q)$. Then on the solutions (z, h) of the lifting system of G' at q (Definition 30.15), the two linear functionals*

$$(z, h) \mapsto (h_a - h_b) \cdot (q_a, 1), \qquad (z, h) \mapsto (h_a - h_b) \cdot (q_b, 1)$$

are linearly independent.

Proof. At a common zero (z, h) of the two functionals, h_a and h_b take the same values at q_a and at q_b. As $z_a = h_a \cdot (q_a, 1)$ and $z_b = h_b \cdot (q_b, 1)$, the height z agrees with h_a on the closed neighbourhood of a in G' together with b, and with h_b on that of b together with a. At every other body v, the closed neighbourhood in H lies in the one in G', on which z agrees with h_v. So $z \in L_H(q)$. At a picture admissible for G' a solution is determined by its heights, and the solutions form a space of dimension $\dim L_{G'}(q)$ (Definition 30.15). So the common zeros of the two functionals form a space of dimension at most $\dim L_H(q) \leq \dim L_{G'}(q) - 2$, and the two are independent. □

When we apply the next lemma, L is the space of solutions of the lifting system of $G[V_1]$ at a picture, and D maps a solution (z, h) to $h_a - h_b$.

Lemma 30.92 (An incidence meets two open conditions)**.** *Let K be infinite, let L be a linear subspace of K^σ for a finite index set σ, and let $D\colon K^\sigma \to K^3$ be linear with $\dim D(L) \geq 2$; read $D(z)$ as the affine function $q \mapsto D(z) \cdot (q, 1)$ on K^2. Let A be a polynomial in the coordinates of σ, nonzero somewhere on L, and B a nonzero polynomial on K^2. Then there are $z \in L$ and $q \in K^2$ with*

$$D(z) \cdot (q, 1) = 0, \qquad A(z) \neq 0, \qquad B(q) \neq 0.$$

Proof. As $\dim D(L) \geq 2$ and the constant affine functions form a line, some $z_1 \in L$ has $D(z_1)$ nonconstant. One of the first two coefficients of $D(z)$ is a linear polynomial in z nonzero at z_1. So some $z_0 \in L$ has both $A(z_0) \neq 0$ and $D(z_0)$ nonconstant (Lemma 30.26). As $\dim D(L) \geq 2$, some $u \in L$ has $D(u)$ not a multiple of $D(z_0)$.

For $q \in K^2$ put $w(q) = D(u)(q)\, z_0 - D(z_0)(q)\, u \in L$; then $D(w(q))$ vanishes at q. An affine function vanishing on the zero line of the nonconstant $D(z_0)$ is a multiple of it, so some q_0 on that line has $D(u)(q_0) \neq 0$, and there $w(q_0) = D(u)(q_0)\, z_0$. So $(q, s) \mapsto A(s\, w(q))$ is a polynomial, nonzero at q_0 and $s = D(u)(q_0)^{-1}$. Its product with B is a nonzero polynomial, and at a non-root (q, s) the point $z = s\, w(q)$ of L has $D(z) \cdot (q, 1) = 0$, $A(z) \neq 0$ and $B(q) \neq 0$. □

We state the configuration of the one-body step for any simple graph H with such an ear and any polynomial B, so that the step with two interior bodies can apply it to G_1 (Theorem 30.96).

Lemma 30.93 (The base configuration of a one-body ear)**.** *Let K be infinite and let H be simple, with an open ear $a - x - b$ on V_1 with one interior body, whose ends are not adjacent. Let $G' = H[V_1]$, suppose that the general configuration of G' attains and that $\operatorname{def}_2(G'; a, b) + 2 \leq \operatorname{def}_2(G')$ (Definition 20.33, at $d = 2$), and let B be a nonzero polynomial in the picture coordinates whose non-roots are admissible for H. Then there are a picture q with $B(q) \neq 0$ and a polynomial A in the coordinates of the lifting system of H at q, nonzero at one of its solutions, such that every solution (z, h) with $A(z, h) \neq 0$ has*

(1) $(h_a - h_b) \cdot (q_a, 1) \neq 0$ and $(h_a - h_b) \cdot (q_b, 1) \neq 0$: the point of b lies off the plane of a, and the point of a off the plane of b; and

(2) rank $6(|V_1| - 1) - \operatorname{def}(\tilde{G}')$ at (q, z) as a configuration of G' (Definition 30.8).

Proof. The smaller graph. Let H' be H with x suppressed: the splitting-off H^{ab}_x (Definition 21.10) whose new edge ab carries the label of the edge ax. It is G' together with the edge ab, simple since a and b are not adjacent, and $\operatorname{def}_2(H') + 2 \leq \operatorname{def}_2(G')$ (Lemma 21.36, at $D = 3$).

The incidence. Take a picture q' off the zero sets of three nonzero polynomials: B, the polynomial of the attainment at G', and the polynomial of Jackson and Jordán's equality at H' (Theorem 30.43). The equality applies to H', as its closed neighbourhoods contain those of G', which have three members at a picture admissible for G'. Then $\dim L_{H'}(q') = 3 + \operatorname{def}_2(H') \leq \dim L_{G'}(q') - 2$ (Lemma 30.34). So on the solutions of the lifting system of G' at q', the two functionals of Lemma 30.91

are independent, and the linear map $(z,h) \mapsto h_a - h_b$ has rank at least two. Apply Lemma 30.92 to this map. As A take the product of the two functionals and the polynomial in the heights of the attainment at G'. Each factor is nonzero somewhere on the solutions, so the product is too (Lemma 30.26). For the polynomial on K^2 take B as a polynomial in the picture point of x, with the picture fixed at q' elsewhere; it is nonzero at q'_x. This gives a solution (z',h') for G' with $A(z',h') \neq 0$, and a picture point q_x with $B \neq 0$ at which $h'_a - h'_b$ vanishes.

The solution for H. Let q be q' with x moved to q_x. Then $B(q) \neq 0$, and G' reads the same picture at q as at q' (Lemma 30.5). Extend (z',h') to x. The height at x is $h'_a \cdot (q_x, 1)$, which equals $h'_b \cdot (q_x, 1)$. The coefficient vector at x is that of the affine function through the three points of its closed neighbourhood $\{a,x,b\}$ (Lemma 30.64). This is a solution of the lifting system of H at q, as the closed neighbourhoods of a and b in H are those in G' with x added. Read A on the solutions for H through their restriction to V_1 (Lemma 30.46). It is nonzero at this solution. Wherever it is nonzero, both functionals are nonzero, and the heights give G' its target (Lemma 30.47). □

Theorem 30.94 (An open ear with one interior body)**.** *Let K be infinite and let G satisfy the standing hypotheses (Definition 30.45), with an open ear $a - x_1 - b$ on V_1 whose ends a and b are not adjacent. Suppose* $\mathrm{def}(\tilde{G}[V_1]) \leq \mathrm{def}(\tilde{G})$ *(Definition 20.3, at $d = 3$) and* $\mathrm{def}_2(G[V_1]; a, b) + 2 \leq \mathrm{def}_2(G[V_1])$ *(Definition 20.33, at $d = 2$). If the general configuration of $G[V_1]$ attains, then the general configuration of G attains.*

Proof. The configuration. Apply Lemma 30.93 to G, with B the main-picture polynomial of G (Lemma 30.23). It gives a main picture q of G, and at q a solution of the lifting system of G with $A \neq 0$. Its heights $z \in L_G(q)$ give $G[V_1]$ its target (Lemma 30.47).

The count. The closed neighbourhood of x_1 is $\{a, x_1, b\}$, and its picture points are not collinear, as q is admissible. So p_a, p_{x_1} and p_b are linearly independent, and so are the two joins $p_a \vee p_{x_1}$ and $p_{x_1} \vee p_b$. Every hinge is nonzero at q, and $\dim(\rho + \Lambda) \geq 2$ (Lemma 30.70). By the hypothesis on the deficiencies this is $(\star)$ at $k = 1$, so Lemma 19.32 and Lemma 30.20 apply. □

In the step with two interior bodies, π_a and π_b are the planes of a and b at the configuration of G_1, and y is the point of x_1 there. The points $x_1 \in \pi_a$ and $x_2 \in \pi_b$ of the next lemma are the new points of the two interior bodies.

Lemma 30.95 (Putting the second body back between the two end planes)**.** *Let K be infinite, and let π_a and π_b be the planes $p_z = \ell_a \cdot \bar{p}$ and $p_z = \ell_b \cdot \bar{p}$ of K^4, for $\ell_a, \ell_b \in K^3$ (with $\bar{p}$ as in Lemma 30.66). Let $p_a \in \pi_a$, $p_b \in \pi_b$ and $y \in \pi_a \cap \pi_b$ be affine points, of last coordinate 1, with $p_a \notin \pi_b$ and $p_b \notin \pi_a$. Then for every subspace $\rho \subseteq \bigwedge^2 K^4$ there are affine points $x_1 \in \pi_a$ and $x_2 \in \pi_b$ with*

$$\dim(\rho + \mathrm{span}(p_a \vee x_1,\ x_1 \vee x_2,\ x_2 \vee p_b)) \ \geq\ \min(\dim(\rho + \mathrm{span}(p_a \vee y,\ y \vee p_b)) + 1,\ 6).$$

Proof. Write $W = \rho + \mathrm{span}(p_a \vee y, y \vee p_b)$. If $W = \bigwedge^2 K^4$, take $x_1 = x_2 = y$. Otherwise note that $\pi_a \neq \pi_b$, as $p_b \notin \pi_a$, so the two planes meet in a line through y; let u_0 be a direction of it, of last coordinate 0. The points p_a, y, u_0, p_b are linearly

independent: both plane equations vanish at y and u_0, the equation of π_a vanishes at p_a but not at p_b and that of π_b the other way round, and y is affine while u_0 is not. Their six joins span $\bigwedge^2 K^4$ (Lemma 30.77), and W contains $p_a \vee y$ and $y \vee p_b$, so one of $y \vee u_0$, $u_0 \vee p_b$, $p_a \vee u_0$, $p_a \vee p_b$ lies outside W.

Put $x_1 = y + tu$ and $x_2 = y + \sigma(p_b - y)$, with $\sigma \neq 1$ and u a direction in π_a of last coordinate 0. For $t \neq 0$, the span of ρ and the three joins contains $y \vee p_b$ (a nonzero multiple of $x_2 \vee p_b$), $u \vee x_2$, and $p_a \vee y + t\,(p_a \vee u)$. So if $u \vee x_2 \notin W$, some $t \neq 0$ gives dimension at least $\dim W + 1$, since a vector outside a subspace stays outside it after adding t times another for some $t \neq 0$ (as in the proof of Lemma 30.81). It remains to choose u and σ with $u \vee x_2 \notin W$. If $y \vee u_0 \notin W$, take $u = u_0$ and $\sigma = 0$. Otherwise, for $\sigma \notin \{0, 1\}$, the join $u \vee x_2$ is modulo W a nonzero multiple of $u_0 \vee p_b$ when $u = u_0$, and of $p_a \vee u_0$ for the same curve with the two planes exchanged. For $u = p_a - y$ and $\sigma \notin \{0, 1\}$ it is modulo W a nonzero multiple of $p_a \vee p_b$. □

Theorem 30.96 (An open ear with two interior bodies at non-adjacent ends)**.** *Let K be infinite and let G satisfy the standing hypotheses, with an open ear $a - x_1 - x_2 - b$ on V_1 whose ends a and b are not adjacent, and suppose* $\mathrm{def}_2(G[V_1]; a, b) + 2 \leq \mathrm{def}_2(G[V_1])$ *(Definition 20.33, at $d = 2$). Let G_1 be G with x_2 suppressed: the splitting-off $G_{x_2}^{x_1 b}$ (Definition 21.10) whose new edge $x_1 b$ carries the label of the deleted edge $x_1 x_2$. If the general configurations of $G[V_1]$ and of G_1 attain, then the general configuration of G attains.*

Proof. Write $f = \mathrm{def}(\tilde{G}[V_1])$, and compute ranks with the joins as hinges (Lemma 30.9). The graph G_1 has the open ear $a - x_1 - b$ on V_1 and the same $G[V_1]$. It is simple, and its ends are not adjacent.

The configuration of G_1. Apply Lemma 30.93 to G_1. As B take the product of the polynomial of the attainment at G_1 and the main-picture polynomial of G (Lemma 30.23). At the picture q that the lemma gives, the non-roots of A form a nonempty Zariski-open set of solutions of the lifting system of G_1. As q is off the zero set of the polynomial of the attainment at G_1, the solutions whose heights give G_1 its target contain another such set. The two sets meet (Lemma 30.26). Take a solution in both, with heights z. Let π_a and π_b be the planes of its coefficient vectors at a and b, and y the point of x_1. Then $y \in \pi_a \cap \pi_b$, as x_1 is adjacent to a and to b in G_1. Also $p_b \notin \pi_a$ and $p_a \notin \pi_b$, and $G[V_1]$ and G_1 have their targets. So the ear rank formula at G_1 (Lemma 19.32) gives

$$s \;=\; \dim(\rho + \mathrm{span}(p_a \vee y,\; y \vee p_b)) \;=\; 2 + f - \mathrm{def}(\tilde{G}_1).$$

Putting x_2 back. By Lemma 30.95 there are affine points $y_1 \in \pi_a$ and $y_2 \in \pi_b$ with $\dim(\rho + \Lambda) \geq \min(s + 1, 6)$, where $\Lambda = \mathrm{span}(p_a \vee y_1,\; y_1 \vee y_2,\; y_2 \vee p_b)$.

The configuration of G. Take as base data the restriction of (q, z) to V_1, with the affine functions of π_a and π_b (Definition 30.82). The points y_1 and y_2 at x_1 and x_2 are the configuration of one ear datum of G. So the bound $\dim(\rho + \Lambda) \geq \min(s + 1, 6)$ holds off the zero set of a polynomial in the ear data that is nonzero there (Lemma 30.85). The main-picture condition of G is a polynomial condition on the ear data too, nonzero at the ear datum with picture q (Lemma 30.84). At a common non-root the picture is a main picture of G, so the hinges are nonzero, and

the heights lie in L_G (Lemma 30.83). The rank of $G[V_1]$ and ρ are those of the base data (Lemma 30.47, Lemma 30.5).

The count. $\operatorname{def}(\tilde{G}_1) \leq \operatorname{def}(\tilde{G})$ (Lemma 21.34), and $\operatorname{def}(\tilde{G}) \geq f + k - 5 = f - 3$ (Lemma 20.31). So $\min(s+1, 6) \geq 3 + f - \operatorname{def}(\tilde{G})$, which is $(\star)$ at $k = 2$. By Lemma 19.32 and Lemma 30.20 the general configuration of G attains. □

30.9. **Splitting off a body of degree two.** This subsection proves the split-off step. Let x be a body of degree two whose neighbours a and b are not adjacent, and let $V_1 = V(G) \setminus \{x\}$, so that $a - x - b$ is an open ear on V_1 (Definition 30.63). Splitting off x gives the graph $G'' = G_x^{ab}$ (Definition 21.10): it is $G[V_1]$ together with a new edge ab, which carries the label of ax. We show that if the general configuration of G'' attains, then so does that of G, provided that

$$\delta \;=\; \operatorname{def}(\tilde{G}[V_1]) - \operatorname{def}(\tilde{G}[V_1]; a, b) \;\geq\; 5$$

(Definition 20.33, at $d = 3$). Here δ is how far the deficiency of $G[V_1]$ drops when only partitions with a and b in one part are counted. The structural coverage applies this step to a chain $a - x - b$ with $\delta \geq 5$ (Definition 30.119).

The idea is to put x back on the line through a and b. Take a configuration of G'' at its target rank, and place the point of x on the line through the points of a and b, distinct from both. We call this the *special position* of x. There the hinges at ax and xb are nonzero multiples of the hinge at ab, and adding x raises the rank by exactly five (Lemma 30.98). The hypothesis $\delta \geq 5$ is what makes five enough: it implies that the target of G is at most the target of G'' plus five.

The special position does not finish the proof. There a, x and b are pictured on one line, so the picture is not a main picture of G, and Lemma 30.20 needs one. We therefore move the picture of x along a line to a general point, and move the heights with it so that each closed neighbourhood stays in a plane (Lemma 30.99, Lemma 30.100). Along this curve the rank falls below its starting value at only finitely many points (Lemma 30.97), and the picture fails to be main at only finitely many. Any other point of the curve gives the configuration that Lemma 30.20 needs.

Jackson and Jordán's proof of their pin-collinear theorem [JJ08b] handles the same situation: a vertex of degree two whose neighbours are not adjacent, where joining the neighbours by an edge lowers the deficiency of the rest of the graph. They realize the smaller graph with the new edge by induction, move the pin of that edge slightly, and then add back the vertex by a 1-extension and a 0-extension. We reverse the order: we put x back first, at the special position, where the rank is known exactly, and then move it. Since we move along a polynomial curve rather than within a small Euclidean neighbourhood, the argument works over every infinite field.

We begin with the lower semicontinuity of the rank (Lemma 23.120) along a polynomial curve of normals.

Lemma 30.97 (The rank along a polynomial curve of normals)**.** *Let G be a finite graph, and let $t \mapsto n(t) = (n_v(t))$ be panel normals whose coordinates are polynomials in $t \in K$, such that the hinge at every edge of G is nonzero at $t = 0$. Then for all but finitely many $t \in K$, the rank of the rigidity matrix of the panel-hinge framework on G with normals $n(t)$ is at least its value at $t = 0$.*

Proof. By Lemma 23.120 at $n(0)$ there is a polynomial Q in the normal coordinates, nonzero at $n(0)$, such that the rank is at least its value at $n(0)$ wherever Q is nonzero. The polynomial $t \mapsto Q(n(t))$ in one variable is nonzero at $t = 0$, so it has finitely many roots, and off them the bound holds. □

The next lemma computes the rank at the special position. It is stated for panel normals in K^{k+2}; configurations are the case $k = 2$, where $D - 1 = 5$.

Lemma 30.98 (The rank at the special position)**.** *Let G have an open ear $a - x - b$ on V_1 with one interior body, and let G'' be the splitting-off G_x^{ab} (Definition 21.10) whose new edge ab carries the label of the edge ax. Let (n_v) be panel normals in K^{k+2} with n_a and n_b linearly independent and*

$$n_x = (1 - s)\, n_a + s\, n_b, \qquad s \neq 0, 1.$$

Then the rank of the rigidity matrix of the panel-hinge framework on G with these normals is that of G'' plus $D - 1$, where $D = \binom{k+2}{2}$ is the dimension of the screw space.

Proof. The panel support extensor is bilinear and alternating (Definition 23.1), so up to sign the hinges at ax and xb are $s\,C$ and $(1 - s)\,C$, where $C \neq 0$ is the hinge at ab in G''; every other edge has the same hinge in G and in G''. Since $S_x - S_b = (S_x - S_a) + (S_a - S_b)$, a screw assignment S is therefore a motion of G if and only if it is a motion of G'' with $S_x - S_a \in KC$. As x is not a body of G'', the map $S \mapsto S_x - S_a$ takes the motions of G'' onto the screw space, so the motions of G have codimension $D - 1$ in those of G''. The row spaces are the annihilators of the motion spaces (Lemma 19.25), so the ranks differ by $D - 1$. □

To choose the special position we need one more lemma. A solution of the lifting system is a pair (z, h): heights z, and for each body v an affine function h_v that agrees with z on the closed neighbourhood of v (Definition 30.15). The curve needs a special point at which $h_a - h_b$ is nonzero on some solution for $G[V_1]$, unless $h_a = h_b$ on every solution (Lemma 30.100). Such a point exists unless every solution has $h_a - h_b$ vanishing on the line through q_a and q_b. The next lemma shows that in that case $h_a = h_b$ on every solution. It is stated for pairs (z, h), not for heights alone, because at a body of degree one in $G[V_1]$ the heights do not determine h_v.

Lemma 30.99 (End planes that all meet over the line of the ends coincide)**.** *Let G have an open ear $a - x - b$ on V_1 with one interior body, let $G' = G[V_1]$, and let G'' be the splitting-off G_x^{ab} whose new edge ab carries the label of the edge ax. Let q be admissible for G and for G'', with $\dim L_{G''}(q) \leq \dim L_G(q)$. If every solution (z, h) of the lifting system of G' at q (Definition 30.15) has*

$$(h_a - h_b) \cdot (q_a, 1) = 0 \qquad \text{and} \qquad (h_a - h_b) \cdot (q_b, 1) = 0,$$

then every such solution has $h_a = h_b$.

Proof. Suppose some solution y_0 has $h_a \neq h_b$. The affine function $h_a - h_b$ vanishes at q_a and q_b, and q_a, q_b, q_x are not collinear, as q is admissible for G. So the functional $\varphi(z, h) = (h_a - h_b) \cdot (q_x, 1)$ is nonzero at y_0.

We compare the solution spaces of the three lifting systems at q. First, every solution for G' is one for G''. Indeed, the closed neighbourhoods of a and b in G'' gain b and a respectively, and by hypothesis $h_a \cdot (q_b, 1) = h_b \cdot (q_b, 1) = z_b$ and $h_b \cdot (q_a, 1) = z_a$. Second, restriction to V_1 maps the solutions for G injectively into the solutions for G' on which φ vanishes. A restricted solution is a solution for G', since each closed neighbourhood in G' lies inside the one in G. And φ vanishes on it, since x lies in the closed neighbourhoods of both a and b in G, so h_a and h_b both take the value z_x at q_x. The map is injective: a solution for G that vanishes on V_1 has $z_x = h_a \cdot (q_x, 1) = 0$, so h_x vanishes at the three non-collinear points q_a, q_x, q_b, and $h_x = 0$ (as in Lemma 30.64). Since $\varphi(y_0) \neq 0$, the solutions for G form a space of smaller dimension than the solutions for G', and hence than those for G''. At a picture admissible for a graph, the solutions form a space of dimension $\dim L(q)$ (Definition 30.15). So $\dim L_G(q) < \dim L_{G''}(q)$, a contradiction. □

The next lemma is the pure linear-algebra fact behind the curve.

Lemma 30.100 (A line of solutions through the special position)**.** *Let L be a subspace of a vector space over K, let φ_0 and ψ be linear functionals, and let $y_0 \in L$ with $\varphi_0(y_0) = 0$. Suppose that φ_0 is nonzero somewhere on L, or that φ_0 and ψ both vanish on L. Then some $w \in L$ has $\varphi_0(y_0 + tw) + t\,\psi(y_0 + tw) = 0$ for every $t \in K$.*

Proof. If $\varphi_0(W) \neq 0$ for some $W \in L$, take $w = \varphi_0(W)^{-1}(\psi(W)\, y_0 - \psi(y_0)\, W)$. Then $\varphi_0(y_0 + tw) = -t\,\psi(y_0)$ and $\psi(y_0 + tw) = \psi(y_0)$. (One finds this w by solving the condition on the line $y_0 + KW$ at each t. The solution is rational in t, and as the condition is homogeneous it can be rescaled to clear the denominator.) Otherwise φ_0 and ψ vanish on L, and $w = 0$ will do. □

The curve also needs the height that the ear body x contributes.

Lemma 30.101 (A solution for the ear body extends to a height)**.** *Let G have an open ear $a - x - b$ on V_1 with one interior body, and let q be admissible for G. If a solution (z, h) of the lifting system of $G[V_1]$ at q has $(h_a - h_b) \cdot (q_x, 1) = 0$, then the height that agrees with z away from x and takes the value $h_a \cdot (q_x, 1)$ at x lies in $L_G(q)$.*

Proof. At a body v of V_1, the closed neighbourhood in G is the one in $G[V_1]$, with x added when v is a or b. The height agrees with h_v on it: at x, because h_a and h_b agree at $(q_x, 1)$. The closed neighbourhood of x is $\{a, x, b\}$, and every height agrees with an affine function on three points (Lemma 30.64). □

Theorem 30.102 (Splitting off a body of degree two)**.** *Let K be infinite and let G satisfy the standing hypotheses (Definition 30.45), with an open ear $a - x - b$ on V_1 with one interior body whose ends a and b are not adjacent. Let G'' be G with x suppressed: the splitting-off G_x^{ab} (Definition 21.10) whose new edge ab carries the label of the edge ax. Suppose* $\operatorname{def}(\tilde{G}[V_1]; a, b) + 5 \leq \operatorname{def}(\tilde{G}[V_1])$ *(Definition 20.33, Definition 20.3, at $d = 3$). If the general configuration of G'' attains, then the general configuration of G attains.*

Proof. The counts. The new edge ab lowers by five the value of every partition of V_1 that separates a and b, and by hypothesis the other partitions are already five

below $\operatorname{def}(\tilde{G}[V_1])$. So $\operatorname{def}(\tilde{G}'') + 5 \leq \operatorname{def}(\tilde{G}[V_1])$ (Lemma 21.36, at $D = 6$). With $\operatorname{def}(\tilde{G}[V_1]) \leq \operatorname{def}(\tilde{G}) + 4$ (Lemma 20.31, at $k = 1$) this gives $\operatorname{def}(\tilde{G}'') + 1 \leq \operatorname{def}(\tilde{G})$. Also $\operatorname{def}_2(G'') \leq \operatorname{def}_2(G)$ (Lemma 21.34, at $d = 2$).

The picture. As a and b are not adjacent, G'' is simple and each of its closed neighbourhoods has as many members as in G, at least three. So Jackson and Jordán's equality applies to G'' (Theorem 30.43). Take a picture q off the zero sets of three nonzero polynomials: that of the attainment at G'', that of the equality at G'', and the main-picture polynomial of G (Lemma 30.23). Then q is admissible for G'' and a main picture of G, some $z'' \in L_{G''}(q)$ gives G'' its target rank, and

$$\dim L_{G''}(q) \;=\; 3 + \operatorname{def}_2(G'') \;\leq\; 3 + \operatorname{def}_2(G) \;\leq\; \dim L_G(q)$$

(Lemma 30.34). Let $y_0 = (z'', h)$ be a solution of the lifting system of G'' at q with heights z''. It is a solution for $G[V_1]$, and $h_a - h_b$ vanishes at $(q_a, 1)$ and $(q_b, 1)$, since ab is an edge of G''.

The special position. If some solution for $G[V_1]$ has $h_a - h_b$ nonzero at $(q_a, 1)$ or at $(q_b, 1)$, then for all but at most one s it is nonzero at $(1-s)(q_a, 1) + s\,(q_b, 1)$. Otherwise every solution for $G[V_1]$ has $h_a = h_b$ (Lemma 30.99). Fix $s \neq 0, 1$ accordingly, put $q_x^0 = (1 - s)q_a + s\,q_b$, and let $\varphi_0(z, h) = (h_a - h_b) \cdot (q_x^0, 1)$ and $\psi(z, h) = (h_a - h_b) \cdot (q_x - q_x^0, 0)$. By Lemma 30.100 there is a solution w for $G[V_1]$ such that $y(t) = y_0 + tw$ satisfies $\varphi_0(y(t)) + t\,\psi(y(t)) = 0$ for every t.

The curve. Let $q(t)$ be q with the picture of x moved to $q_x^0 + t\,(q_x - q_x^0)$, so that $q(1) = q$. As $q(t)$ agrees with q on V_1, $y(t)$ is a solution for $G[V_1]$ at $q(t)$ (Lemma 30.22), and the identity above says that its $h_a - h_b$ vanishes at $(q(t)_x, 1)$. Let $z(t)$ be the height of $y(t)$ on V_1, extended by $h_a \cdot (q(t)_x, 1)$ at x. The points of the configuration $(q(t), z(t))$ are polynomials in t. At $t = 0$ they are at the special position: $p_x = (1 - s)p_a + s\,p_b$, because h_a takes the values z_a'' and z_b'' at q_a and q_b. Also p_a and p_b are independent, as $q_a \neq q_b$. Every hinge of G is nonzero at $t = 0$: q_x^0 differs from q_a and q_b as $s \neq 0, 1$, and every other edge is an edge of G''. So at $t = 0$ the rank of G is that of G'' at (q, z'') plus five (Lemma 30.98, Lemma 30.47).

A good parameter. By Lemma 30.97, the rank of G at $(q(t), z(t))$ is at least its value at $t = 0$ for all but finitely many t. The main-picture polynomial of G at $q(t)$ is a polynomial in t, nonzero at $t = 1$, so $q(t)$ is a main picture of G for all but finitely many t. Take t outside both finite sets. Then $z(t) \in L_G(q(t))$ (Lemma 30.101), and $(q(t), z(t))$ has rank at least

$$6(|V_1| - 1) - \operatorname{def}(\tilde{G}'') + 5 \;\geq\; 6(|V(G)| - 1) - \operatorname{def}(\tilde{G}),$$

as $|V(G)| = |V_1| + 1$. By Lemma 30.20 the general configuration of G attains. □

30.10. **The deficiency of a sparse graph.** This subsection proves the estimates on partitions that the structural coverage uses (Section 30.11). The proof of the generic statement uses some of them too (Section 30.12). Fix a dimension d, and let $D = \binom{d+1}{2}$, as in Definition 20.3. We call the D-deficiency $\operatorname{def}_{\tilde{G}}(P)$ of a partition P its *value*. For $X \subseteq V(G)$ let $e(X)$ be the number of edges of G joining two distinct bodies of X, and let

$$s_D(X) \;=\; D\,(|X| - 1) \;-\; (D - 1)\,e(X),$$

the value of the partition of $G[X]$ into singletons. We call $s_D(X)$ the *singleton value* of X, and write $s = s_3$, the case $d = 2$. A set W of at least two bodies is *rigid* if $G[W]$ is a rigid subgraph at $d = 3$ (Definition 20.17), that is, if $\mathrm{def}_3(G[W]) = 0$. It is *planar-rigid* if $\mathrm{def}_2(G[W]) = 0$. For bodies a and b, $\mathrm{def}_d(G; a, b)$ is the merged deficiency of Definition 20.33.

The graph G is *sparse* if $s(X) \geq 1$ for every set X of at least two bodies, that is, if every such set spans at most $\frac{1}{2}(3|X| - 4)$ edges. Sparsity is the absence of planar-rigid sets. Indeed, $\mathrm{def}_2(G[X])$ is at least the value $s(X)$ of the singletons, which is positive in a sparse graph. Conversely, a graph with no planar-rigid set is sparse (Lemma 30.105). For a graph without loops, sparsity says that the multigraph $2G$, with every edge of G doubled, is $(3, 4)$-sparse (Definition 16.1). A set X of at least three bodies is *tight* if $G[X]$ is sparse and $s(X) = 1$, so that X spans as many edges as sparsity allows. For example, a set inducing a 4-cycle is tight.

In a sparse graph, a partition with a part Z has value at most $s(V(G)) - s(Z)$ at $d = 2$ (Lemma 30.103). As $s(Z) \geq 0$, the partition into singletons attains the planar deficiency: $\mathrm{def}_2(G) = s(V(G))$. The estimates below compare singleton values in this way. We use three further facts. Adding a body with t edges into a set changes its singleton value by $D-(D-1)t$ (Lemma 30.104). Merging the parts of a partition that meet a set of deficiency zero does not lower its value (Lemma 30.106). The third relates the two dimensions. Let a partition P have value $v_2(P)$ at $d = 2$ and $v_3(P)$ at $d = 3$, and let $d_G(P)$ edges of G join distinct parts of P (Definition 20.3). Then

$$v_3(P) \;=\; 6\,(|P| - 1) - 5\,d_G(P) \;=\; 2\,v_2(P) - d_G(P).$$

So a partition whose value at $d = 2$ is at most zero has value at most zero at $d = 3$. Planar counts can therefore show that a set is rigid.

From these facts we prove that a tight set is rigid, and that in a sparse graph a maximal rigid set has the least singleton value among the sets containing it. At a set W of least singleton value among the sets containing it, the planar deficiencies of $G[W]$ and of $G/G[W]$ add up to at most that of G. The contraction step at an additive core assumes this inequality (Section 30.5). Last, we bound the merged deficiencies at the ends of an open ear with one or two interior bodies. The split-off step assumes such a bound for one interior body (Section 30.9). So do the ear steps at non-adjacent ends, for one or two (Section 30.8).

Lemma 30.103 (The singletons bound the partitions of a sparse graph)**.** *Let P be a partition of $V(G)$ with a part Z, and suppose $s_D(Y) \geq 0$ for every nonempty $Y \subseteq V(G) \setminus Z$. Then*

$$\mathrm{def}_{\tilde{G}}(P) + s_D(Z) \;\leq\; s_D(V(G)).$$

In particular, if $s_D(Y) \geq 0$ for every nonempty $Y \subseteq V(G)$, then $\mathrm{def}_{\tilde{G}}(P) \leq s_D(V(G))$ for every partition P of $V(G)$.

Proof. Both P and the partition into singletons are separated by Z, so each splits at Z with the same cut term (Lemma 20.26). On $G[Z]$ the partition P has one part and value zero, and the singletons have value $s_D(Z)$. The second statement follows by induction on the number of bodies, since $s_D(Z) \geq 0$ and the rest is the statement

at $G[V(G) \setminus Z]$. The first follows from one such split and the second statement at $G[V(G) \setminus Z]$. □

Lemma 30.104 (Adding one body)**.** *Let $X \subseteq V(G)$ and $u \notin X$, and let P be a partition of $X \cup \{u\}$. Let t be the number of edges of G from u to a body of X in a part of P other than the part of u, and let $\epsilon = 1$ if u is alone in its part and $\epsilon = 0$ otherwise. Then*

$$\operatorname{def}_{\tilde{G}[X\cup\{u\}]}(P) \;=\; \operatorname{def}_{\tilde{G}[X]}(P|_X) + \epsilon D - (D-1)\,t.$$

Consequently, if $D \geq 3$, $\operatorname{def}(\tilde{G}[W]) = 0$ and a body $u \notin W$ is adjacent to two distinct bodies of W, then $\operatorname{def}(\tilde{G}[W \cup \{u\}]) = 0$.

Proof. The number of parts grows by one exactly when u is alone, and the crossing edges gain the t edges at u. For the consequence, let P partition $W \cup \{u\}$. If u shares a part, the value of P is at most that of $P|_W$, which is at most zero. If u is alone, both of its edges into W cross, and the value is at most $0 + D - 2(D-1) < 0$. □

Lemma 30.105 (A set of nonpositive singleton value contains a set of deficiency zero)**.** *Let $X \subseteq V(G)$ with $|X| \geq 2$ and $s_D(X) \leq 0$. Then some $Y \subseteq X$ with $|Y| \geq 2$ has $\operatorname{def}(\tilde{G}[Y]) = 0$. Hence, if no set of at least two bodies has deficiency zero, then $s_D(X) \geq 1$ for every X with $|X| \geq 2$; at $d = 2$, a graph with no planar-rigid set is sparse.*

Proof. Take $Y \subseteq X$ minimal with $|Y| \geq 2$ and $s_D(Y) \leq 0$. Every nonempty proper subset of Y then has nonnegative singleton value, a singleton having value zero. A partition of Y other than $\{Y\}$ has a part $Z \subsetneq Y$, and its value is at most $s_D(Y) - s_D(Z) \leq 0$ (Lemma 30.103); the partition $\{Y\}$ has value zero. □

Lemma 30.106 (Merging along a set of deficiency zero)**.** *Let $D \geq 1$ and let $W \subseteq V(G)$ be nonempty with $\operatorname{def}(\tilde{G}[W]) = 0$.*

(1) For every partition P of $V(G)$, some partition P' coarser than P, with W inside one of its parts, has $\operatorname{def}_{\tilde{G}}(P) \leq \operatorname{def}_{\tilde{G}}(P')$.

(2) For all bodies a and b, $\operatorname{def}(\tilde{G}; a, b) \leq \operatorname{def}(\tilde{G})$, with equality if $a, b \in W$.

Proof. (1) Let P' merge the parts of P that meet W into one part. If m parts of P meet W and e edges join distinct ones among them, merging them lowers the value by $D(m-1) - (D-1)e$ (Lemma 27.7). The e edges include the edges of $G[W]$ crossing $P|_W$, so this quantity is at most the value of $P|_W$ in $G[W]$, which is at most zero.

(2) The merged deficiency is a maximum over fewer partitions. For the equality, merge a partition attaining $\operatorname{def}(\tilde{G})$ along W by (1); the result has a and b in one part. □

Lemma 30.107 (A tight set is rigid)**.** *Let $X \subseteq V(G)$ with $|X| \geq 3$, and suppose $s(Z) \geq 1$ for every $Z \subseteq X$ with $|Z| \geq 2$. If $s(X) \leq 1$, then $\operatorname{def}_3(G[X]) = 0$.*

Proof. Let P be a partition of X, with values $v_2(P)$ and $v_3(P)$ in $G[X]$. If P has a part Z with $2 \leq |Z| < |X|$, then $v_2(P) \leq s(X) - s(Z) \leq 0$ (Lemma 30.103), so $v_3(P) \leq 0$. The partition $\{X\}$ has $v_3 = 0$. For the partition into singletons, $v_2 = s(X) \leq 1$ gives $2e(X) \geq 3|X| - 4 \geq 5$, so $e(X) \geq 3$ and $v_3 = 2s(X) - e(X) < 0$. □

The structural coverage takes either a maximal rigid set, or a set maximal among the rigid sets that avoid one body x of degree two (Theorem 30.123). The next lemma treats both, with X_0 empty or $X_0 = \{x\}$.

Lemma 30.108 (A maximal rigid set bounds the singleton values above it)**.** *Let G be sparse.*

(1) Let $W \subseteq Y \subseteq V(G)$ with W nonempty and $\mathrm{def}_3(G[W]) = 0$. If $s(Y) \leq s(W)$, and $s(W) \leq s(Q)$ for every Q with $W \subseteq Q \subsetneq Y$, then $\mathrm{def}_3(G[Y]) = 0$.

(2) Let $W \subseteq V(G)$ be nonempty with $\mathrm{def}_3(G[W]) = 0$, and let X_0 be empty or a single body outside W that sends at most two edges into every set avoiding it and at most one into W. Suppose W is maximal among the sets $W' \supseteq W$ in $V(G) \setminus X_0$ with $\mathrm{def}_3(G[W']) = 0$. Then $s(W) \leq s(X)$ for every X with $W \subseteq X \subseteq V(G)$.

Proof. (1) Let P be a partition of Y, with values taken in $G[Y]$. Let P' be a partition coarser than P with W inside one part Q and $v_3(P) \leq v_3(P')$ (Lemma 30.106). If $Q = Y$ then $v_3(P') = 0$. Otherwise $v_2(P') \leq s(Y) - s(Q) \leq 0$ (Lemma 30.103), so $v_3(P') \leq 2v_2(P') \leq 0$.

(2) Suppose not, and take $X \subseteq V(G)$ with $W \subseteq X$ and $s(X) < s(W)$, of least size. Then $X \neq W$, and $s(W) \leq s(Q)$ for every $Q \supseteq W$ with fewer bodies than X. If X avoids X_0, then (1) at $Y = X$ makes $G[X]$ rigid, against the maximality of W. Otherwise $X_0 = \{x\}$ with $x \in X$; put $Y = X \setminus \{x\}$, which contains W. Adding x to Y changes the singleton value by $3 - 2t$, where $t \leq 2$ is the number of edges from x into Y (Lemma 30.104), and $t \leq 1$ if $Y = W$. So $Y \neq W$, since otherwise $s(X) \geq s(W) + 1$. Then $s(Y) \geq s(W)$, as Y has fewer bodies than X. Also $s(X) \geq s(Y) - 1$, so $s(X) = s(W) - 1$ and $s(Y) = s(W)$. Now (1) at Y makes $G[Y]$ rigid, a set avoiding X_0 that strictly contains W, against the maximality of W. □

Lemma 30.109 (The planar deficiencies of a core and its contraction add up to at most that of the graph)**.**

(1) For a map c of the bodies, the value of a partition of the image graph $c(G)$ equals the value of its pull-back along c to a partition of $V(G)$.

(2) Let G be sparse, let $r \in W \subseteq V(G)$, and suppose $s(W) \leq s(X)$ for every X with $W \subseteq X \subseteq V(G)$. Let $G/G[W]$ collapse W to r (Definition 21.3). Then

$$\mathrm{def}_2(G[W]) + \mathrm{def}_2(G/G[W]) \;\leq\; \mathrm{def}_2(G).$$

Proof. (1) The parts and the crossing edges correspond under c.

(2) By Lemma 30.103 and sparsity, $\mathrm{def}_2(G[W]) \leq s(W)$. Also $s(V(G)) \leq \mathrm{def}_2(G)$, as $s(V(G))$ is the value of the singletons. A partition of $G/G[W]$ pulls back by (1) to a partition of $V(G)$ with W inside one part Z. The deleted edges of $G[W]$ lie inside Z, so the pull-back has the same value in G. That value is at most $s(V(G)) - s(Z) \leq s(V(G)) - s(W)$ (Lemma 30.103). Adding the three bounds gives the claim. □

Lemma 30.110 (The deficiencies at a body of degree two)**.** *Let $V_1 \subseteq V(G)$ and $x \in V(G) \setminus V_1$, and let $a \neq b$ in V_1 be joined to x by edges.*

(1) If G is sparse, then a and b are not adjacent, and $\mathrm{def}_2(G[V_1]; a, b) + 2 \leq \mathrm{def}_2(G[V_1])$.
(2) If no set of at least two bodies containing x is rigid, then $\mathrm{def}_3(G[V_1]; a, b)+5 \leq \mathrm{def}_3(G[V_1])$.

Proof. (1) An edge ab would give the triangle $\{x, a, b\}$ singleton value zero. Let a partition of V_1 with a and b in one part Z attain the merged deficiency. Its value plus $s(Z)$ is at most $s(V_1) \leq \mathrm{def}_2(G[V_1])$ (Lemma 30.103), so it suffices that $s(Z) \geq 2$. Adding x to Z lowers the singleton value by at least one, as x has at least two edges into Z (Lemma 30.104). As $s(Z \cup \{x\}) \geq 1$ by sparsity, $s(Z) \geq 2$.

(2) Suppose otherwise, and let P attain the merged deficiency m, with Q the part of a and b. Refining Q by a partition R of Q, relabelled apart from the other parts of P, changes the value of P by the value of R in $G[Q]$ (Lemma 20.26). Since $\mathrm{def}_3(G[V_1]) \leq m + 4$, every such R has value at most 4 in $G[Q]$, and at most 0 if it keeps a and b together.

We show that $G[Q \cup \{x\}]$ is rigid, against the hypothesis. Let R' be a partition of $Q \cup \{x\}$, and compare its value with that of its restriction to Q (Lemma 30.104). If x is alone, at least its two edges to a and b cross, and the value drops by at least $10 - 6 = 4$, to at most 0. If x shares a part, each edge from x to another part costs 5. When a and b are apart, at least one such edge exists, and the value is at most $4 - 5 < 0$. When they are together, the value is at most 0. □

Lemma 30.111 (A non-adjacent pair outside every rigid set)**.** *Let G be sparse, and let $a \neq b$ be bodies of G that are not adjacent, with* $\mathrm{def}_3(G; a, b) \neq \mathrm{def}_3(G)$. *Then* $\mathrm{def}_2(G; a, b) + 2 \leq \mathrm{def}_2(G)$.

Proof. Otherwise the part Z of a and b in a partition attaining the planar merged deficiency has $s(Z) \leq 1$ (Lemma 30.103). The pair $\{a, b\}$ has singleton value 3, as a and b are not adjacent (Lemma 30.104), so $|Z| \geq 3$, and Z is rigid (Lemma 30.107). Then $\mathrm{def}_3(G; a, b) = \mathrm{def}_3(G)$ (Lemma 30.106). □

30.11. **The structural coverage.** This subsection completes the induction on the number of bodies. Call a graph *smaller than* G if it satisfies the standing hypotheses (Definition 30.45) and has fewer bodies than G. Each step of the preceding subsections applies to a graph G that satisfies the standing hypotheses and some further conditions. It assumes that the general configurations of none, one or two graphs with fewer bodies attain. We say that G *reduces* to a property $\mathcal{P}$ of graphs if G meets the further conditions of some step, and the graphs that this step assumes to attain satisfy $\mathcal{P}$ (Definition 30.112). If every graph satisfying the standing hypotheses reduces to the graphs smaller than it, then the general configuration of every such graph attains, by strong induction on the number of bodies (Theorem 30.113). The structural coverage proves that every such graph does reduce, by a case analysis (Theorem 30.124).

The table below lists the cases. The proof tests the first five in order, so each of them assumes that those above it fail. Once all five fail, G is 2-connected, sparse (Section 30.10) and not a cycle. Its bodies of degree two then lie on *chains*: paths of bodies of degree two between bodies of degree at least three (Definition 30.114). We write a chain as an open ear $a - x_1 - \cdots - x_k - b$ on V_1 (Definition 30.63). A

chain is *usable* if it meets the condition of one of the chain rows of the table (Definition 30.119), and then the step of that row applies (Lemma 30.120). If no chain is usable, the last row applies. For each case the table names the step that settles it, and the smaller graphs whose attainment that step assumes. In the first two rows V_1 and V_2 are the two sides of a cut vertex or of a chain of bridges (Section 30.4), and W is the set contracted.

Case	*Step*	*Smaller graphs*
G is not 2-edge-connected	a chain of bridges (Theorem 30.52)	$G[V_1]$, $G[V_2]$
$G - v$ is disconnected for some body v	a cut vertex (Theorem 30.49)	$G[V_1]$, $G[V_2]$
every body has degree two	a cycle (Theorem 30.71)	none
$\mathrm{def}_2(G) = \mathrm{def}_3(G)$	equal deficiencies (Corollary 30.44)	none
a planar-rigid set	contraction at a maximal planar-rigid set W (Corollary 30.62)	$G/G[W]$
a usable chain with $k = 3$	an open ear (Theorem 30.89)	$G[V_1]$, and G with x_2 suppressed
a usable chain with $k = 4$	an open ear (Theorem 30.88)	$G[V_1]$, and G with x_2 suppressed
a usable chain with $k \geq 5$	an open ear (Theorem 30.72)	$G[V_1]$
a usable chain with $k = 2$ and $\mathrm{def}_3(G[V_1]; a, b) = \mathrm{def}_3(G[V_1])$	an open ear (Theorem 30.86)	$G[V_1]$
a usable chain with $k = 2$, a and b not adjacent, and $\mathrm{def}_2(G[V_1]; a, b) + 2 \leq \mathrm{def}_2(G[V_1])$	an open ear (Theorem 30.96)	$G[V_1]$, and G with x_2 suppressed
a usable chain with $k = 1$ and $\mathrm{def}_3(G[V_1]; a, b) = \mathrm{def}_3(G[V_1])$	an open ear (Theorem 30.94)	$G[V_1]$
a usable chain with $k = 1$ and $\mathrm{def}_3(G[V_1]; a, b) + 5 \leq \mathrm{def}_3(G[V_1])$	splitting off x_1 (Theorem 30.102)	G with x_1 suppressed
no usable chain	contraction at an additive core W (Theorem 30.61)	$G[W]$, $G/G[W]$

The chain rows are the conditions of Definition 30.119, with $k \geq 3$ split by the step. A chain may meet two of them, and then either step applies. Remark 30.90 follows these cases through the θ-graphs.

In the last row, Theorem 30.123 finds the core. Its proof has three steps. First, every chain has one or two interior bodies, as a chain with more is usable. Second, each chain gives a rigid set. With two interior bodies its ends are adjacent, and the chain and its ends form a 4-cycle, which is rigid. With one interior body, that body

lies in a rigid set. Third, a maximal rigid set containing the set found is the core. In a sparse graph a maximal rigid set has the least singleton value among the sets containing it. So the planar deficiencies of the core and of the contraction add up to at most that of G (Section 30.10). When G itself is rigid, a maximal rigid set is all of $V(G)$, and we take the core maximal among the rigid sets that avoid one body of degree two.

Definition 30.112 (One step of the induction)**.** Let $\mathcal{P}$ be a property of graphs, and let open ears be as in Definition 30.63. The graph G *reduces to* $\mathcal{P}$ if one of the following holds.

(1) $\mathrm{def}_2(G) = \mathrm{def}_3(G)$.
(2) G has a cut vertex with sides V_1 and V_2 as in Theorem 30.49, and $G[V_1]$ and $G[V_2]$ satisfy $\mathcal{P}$.
(3) G has a chain of bridges between V_1 and V_2 as in Theorem 30.52, and $G[V_1]$ and $G[V_2]$ satisfy $\mathcal{P}$.
(4) G is a cycle, given as an edge ab and a path from a to b through its other bodies, as in Theorem 30.71.
(5) G has an open ear $a - x_1 - \cdots - x_k - b$ on V_1, $G[V_1]$ satisfies $\mathcal{P}$, and one of the following holds: $k \geq 5$; $k \in \{3, 4\}$ and G with x_2 suppressed satisfies $\mathcal{P}$; $k = 2$ and $\mathrm{def}_3(G[V_1]) \leq \mathrm{def}_3(G)$; $k = 2$, a and b are not adjacent, $\mathrm{def}_2(G[V_1]; a, b) + 2 \leq \mathrm{def}_2(G[V_1])$ and G with x_2 suppressed satisfies $\mathcal{P}$; or $k = 1$, a and b are not adjacent, $\mathrm{def}_3(G[V_1]) \leq \mathrm{def}_3(G)$ and $\mathrm{def}_2(G[V_1]; a, b) + 2 \leq \mathrm{def}_2(G[V_1])$.
(6) G has an open ear $a{-}x{-}b$ on V_1 whose ends are not adjacent, $\mathrm{def}_3(G[V_1]; a, b) + 5 \leq \mathrm{def}_3(G[V_1])$, and G with x suppressed satisfies $\mathcal{P}$.
(7) G is 2-edge-connected, and there are $W \subsetneq V(G)$ with $|W| \geq 2$, no body outside W adjacent to two bodies of W, and $r \in W$, such that $G/G[W]$ (collapsing W to r) satisfies $\mathcal{P}$ and either $\mathrm{def}_2(G[W]) = 0$, or $\mathrm{def}_3(G[W]) = 0$, $\mathrm{def}_2(G[W]) + \mathrm{def}_2(G/G[W]) \leq \mathrm{def}_2(G)$ and $G[W]$ satisfies $\mathcal{P}$.

Here suppressing an interior body x_i is the splitting-off $G_{x_i}^{x_{i-1}x_{i+1}}$ (Definition 21.10, with $x_0 = a$ and $x_{k+1} = b$) whose new edge carries the label of the edge $x_{i-1}x_i$.

Theorem 30.113 (The induction on the number of bodies)**.** *Let K be infinite.*

(1) Let G satisfy the standing hypotheses and reduce to $\mathcal{P}$. If the general configuration of every graph satisfying $\mathcal{P}$ attains, then the general configuration of G attains.
(2) If every graph satisfying the standing hypotheses reduces to the graphs smaller than it, then the general configuration of every graph satisfying the standing hypotheses attains.

Proof. (1) Each clause of Definition 30.112, and each alternative in clauses (5) and (7), states the further conditions of one step, with $\mathcal{P}$ at the graphs that the step assumes to attain. The steps are equal deficiencies (Corollary 30.44), the cut vertex and the chain of bridges (Theorem 30.49, Theorem 30.52), the cycle (Theorem 30.71), the open ears (Theorem 30.72, Theorem 30.88, Theorem 30.89, Theorem 30.86, Theorem 30.96, Theorem 30.94), the split-off (Theorem 30.102) and the two contractions

(Corollary 30.62, Theorem 30.61). As every graph satisfying $\mathcal{P}$ attains, the step applies.

(2) Strong induction on the number of bodies, by (1) with $\mathcal{P}$ the property of being smaller than G. □

The rest of the subsection shows that G reduces in each case of the table. We begin with the chains, and with the standing hypotheses at the smaller graphs that a chain gives.

Definition 30.114 (Chains)**.** A *chain* of G is an open ear $a - x_1 - \cdots - x_k - b$ on V_1 with $k \geq 1$ whose interior bodies all have degree two in G and whose ends have degree at least three.

Lemma 30.115 (Every body of degree two lies on a chain)**.** *Let G satisfy the standing hypotheses, with $G - v$ connected for every body v and some body of degree other than two. Then every body of degree two is an interior body of a chain.*

Proof. Extend a path through the body along bodies of degree two as far as possible (`Graph.IsOpenEar.exists_maximal`). An interior body has both of its neighbours on the path. So if an end had degree two, its other neighbour would either extend the path or be the other end. In the second case the path and that edge form a cycle. Either the cycle is all of G, and every body of G has degree two, or the cycle meets the rest of G only at the other end, which is then a cut vertex. So both ends have degree at least three. □

Lemma 30.116 (The standing hypotheses at the smaller graphs of a chain)**.** *Let G satisfy the standing hypotheses.*

1. *If $G - v$ is connected for every body v, then $G[V_1]$ satisfies the standing hypotheses for every chain on V_1.*
2. *Let the edges at a body v be one to u and one to w, where u and w are distinct and not adjacent. Then the splitting-off G_v^{uw}, whose new edge carries the label of the edge vu, satisfies the standing hypotheses.*

Proof. (1) The ends of the chain keep at least two neighbours in V_1, and every other body of V_1 keeps all of them. In $G - x_1$ no body of V_1 other than b has a neighbour outside V_1. So if $G[V_1]$ were disconnected, a part of it that avoids b would be left by no edge of $G - x_1$, which is connected (`Graph.Connected.induce_of_gate`).

(2) The new edge is not parallel to an old one, since u and w are not adjacent, and u and w each exchange the edge to v for the new edge. A set of bodies left by no edge of G_v^{uw} contains both or neither of u and w, and then, with v added in the first case, it is left by no edge of G. □

The next two lemmas settle the first three cases of the table.

Lemma 30.117 (A graph of degree two everywhere is a cycle)**.** *Let G satisfy the standing hypotheses, with every body of degree two. Then G reduces to the graphs smaller than it.*

Proof. By the cycle clause. Extend a path through a body as far as possible along bodies of degree two (`Graph.IsOpenEar.exists_maximal`). As every body has degree two, the path stops only when its ends are adjacent. The path with that edge is then closed under adjacency, and so it is all of G, as G is connected. □

Lemma 30.118 (Graphs that are not 2-connected)**.** *Let G satisfy the standing hypotheses.*

(1) If G is not 2-edge-connected, then G reduces to the graphs smaller than it.
(2) If G is 2-edge-connected and $G - v$ is disconnected for some body v, then G reduces to the graphs smaller than it.

Proof. (1) By the clause of a chain of bridges. As G is connected but not 2-edge-connected, some nonempty proper set V' of bodies is left by exactly one edge; its sides are V' and $V(G) \setminus V'$. Grow a path from that edge, keeping the two sides: an end of degree two has its other edge inside its own side, since that edge is not the one leaving V', so the path extends into that side, which loses the end. The path stops when both ends have degree at least three. Each side is then connected, since its end is its only body with a neighbour outside it. The end loses exactly one edge, so it keeps degree at least two, and every other body keeps all its neighbours (`Graph.IsX0Graph.induce_of_gate`).

(2) By the cut-vertex clause at v. As $G - v$ is disconnected, it has a nonempty set C of bodies, closed under adjacency in $G - v$ and not all of its bodies. The sides are $C \cup \{v\}$ and $V(G) \setminus C$. The body v sends at least two edges into each side, since a single one would be a bridge, and every other body keeps all its neighbours. Each side is connected, since v is its only body with a neighbour outside it (`Graph.IsX0Graph.induce_of_gate`). □

Definition 30.119 (Usable chains)**.** A chain $a{-}x_1{-}\cdots{-}x_k{-}b$ on V_1 is *usable* if $k \geq 3$; or $k = 2$ and either $\mathrm{def}_3(G[V_1]; a, b) = \mathrm{def}_3(G[V_1])$, or a and b are not adjacent and $\mathrm{def}_2(G[V_1]; a, b) + 2 \leq \mathrm{def}_2(G[V_1])$; or $k = 1$ and either $\mathrm{def}_3(G[V_1]; a, b) = \mathrm{def}_3(G[V_1])$ or $\mathrm{def}_3(G[V_1]; a, b) + 5 \leq \mathrm{def}_3(G[V_1])$.

Lemma 30.120 (A usable chain gives a reduction)**.** *Let G satisfy the standing hypotheses and be sparse, with $G - v$ connected for every body v. If some chain of G is usable, then G reduces to the graphs smaller than it.*

Proof. By the ear clause or the split-off clause of Definition 30.112 at the usable chain, on V_1. The graphs $G[V_1]$ and G with an interior body suppressed have fewer bodies than G. Both satisfy the standing hypotheses (Lemma 30.116). For the second, the two neighbours of the suppressed body must not be adjacent. When $k \geq 2$, one of them is an interior body, adjacent only to the bodies next to it on the chain. When $k = 1$, they are a and b, which are not adjacent as G is sparse (Lemma 30.110).

Where $\mathrm{def}_3(G[V_1]; a, b) = \mathrm{def}_3(G[V_1])$, some partition attaining $\mathrm{def}_3(G[V_1])$ has a and b in one part. So $\mathrm{def}_3(G[V_1]) \leq \mathrm{def}_3(G)$ (Lemma 20.32), the bound that the ear steps with one or two interior bodies assume. At $k = 1$ sparsity also gives $\mathrm{def}_2(G[V_1]; a, b) + 2 \leq \mathrm{def}_2(G[V_1])$ (Lemma 30.110). □

Lemma 30.121 (Contraction at a maximal planar-rigid set)**.** *Let G satisfy the standing hypotheses and be 2-edge-connected, with $\mathrm{def}_2(G) > 0$ and a planar-rigid set. Then G reduces to the graphs smaller than it.*

Proof. By the contraction clause at a maximal planar-rigid set W. It is proper, since $\mathrm{def}_2(G) > 0$. A body outside W with two neighbours in W would enlarge it (Lemma 30.104, at $d = 2$). So no such body exists, and $G/G[W]$ is smaller than G (Lemma 30.54, Lemma 21.51). □

The proof of Theorem 30.123 counts edges at the bodies of degree two. The count finds such bodies, and, when G is rigid and no two of them are adjacent, a rigid set that avoids one of them.

Lemma 30.122 (Counting bodies of degree two)**.** *Let G satisfy the standing hypotheses.*

(1) If G is sparse, then at least four bodies of G have degree two.
(2) If no two bodies of degree two are adjacent, then $s_6(V(G) \setminus \{x\}) \leq 0$ for every body x of degree two.

Proof. (1) Sparsity at $V(G)$ gives $2|E(G)| \leq 3|V(G)| - 4$, so the sum of $3 - \deg v$ over the bodies is at least 4; each term is at most one, and positive only at degree two.

(2) With m bodies of degree two, their $2m$ edges are distinct, so $|E(G)| \geq 2m$, while $2|E(G)| \geq 2m + 3(|V(G)| - m)$. Hence $5|E(G)| \geq 6|V(G)|$, and removing x removes one body and two edges. □

Theorem 30.123 (A core when no chain is usable)**.** *Let G satisfy the standing hypotheses, with $G - v$ connected for every body v, some body of degree other than two, and no planar-rigid set. If no chain of G is usable, then there are $W \subsetneq V(G)$ with $|W| \geq 2$ and $\mathrm{def}_3(G[W]) = 0$, with no body outside W adjacent to two bodies of W, and $r \in W$ such that*

$$\mathrm{def}_2(G[W]) + \mathrm{def}_2(G/G[W]) \;\leq\; \mathrm{def}_2(G).$$

Proof. The chains. As G has no planar-rigid set, it is sparse (Lemma 30.105). So it has bodies of degree two (Lemma 30.122), and each of them lies on a chain (Lemma 30.115). No chain has three or more interior bodies, as such a chain is usable. Let a chain $a - x_1 - x_2 - b$ on V_1 have two. Its ends are adjacent. Otherwise, as the chain is not usable, $\mathrm{def}_3(G[V_1]; a, b) \neq \mathrm{def}_3(G[V_1])$, and Lemma 30.111 at $G[V_1]$ would make it usable. So a, x_1, x_2 and b induce a 4-cycle, which is tight and so rigid (Lemma 30.107). The body of a chain with one interior body lies in a rigid set, since otherwise the chain would be usable (Lemma 30.110).

When $\mathrm{def}_3(G) > 0$. Take a rigid set given by a chain, and a maximal rigid set W containing it. Then $W \neq V(G)$. A body outside W has at most one neighbour in W, as otherwise it would enlarge W (Lemma 30.104). By Lemma 30.108 with X_0 empty, $s(W) \leq s(X)$ for every $X \supseteq W$, and Lemma 30.109 gives the inequality.

When $\mathrm{def}_3(G) = 0$. We take a body x of degree two and a rigid set that avoids it. If two bodies of degree two are adjacent, they lie on a chain with two interior bodies, whose 4-cycle is rigid. The 4-cycle contains two bodies of degree two, and at

least four bodies have degree two (Lemma 30.122). So some body x of degree two lies off it. Otherwise let x be any body of degree two. Then $s_6(V(G) \setminus \{x\}) \leq 0$ (Lemma 30.122), so $V(G) \setminus \{x\}$ contains a rigid set (Lemma 30.105).

Let W be maximal among the rigid sets that avoid x and contain the one taken. The body x has at most one neighbour in W. A neighbour of degree two would have only one neighbour in W, while every body of a rigid set has at least two neighbours in it (Lemma 20.23). Two neighbours a and b of degree at least three would make $a - x - b$ a chain on $V(G) \setminus \{x\}$. As W is rigid and contains a and b, the merged deficiency of $G - x$ at a and b would equal its deficiency (Lemma 30.106), and the chain would be usable. Every other body outside W has at most one neighbour in W, by the maximality of W (Lemma 30.104). Lemma 30.108 with $X_0 = \{x\}$ and Lemma 30.109 give the inequality. □

Theorem 30.124 (The structural coverage)**.** *Every graph satisfying the standing hypotheses reduces to the graphs smaller than it. Consequently, for K infinite, the general configuration of every graph satisfying the standing hypotheses attains, and in particular that of every simple 2-edge-connected graph on at least three bodies.*

Proof. Let G satisfy the standing hypotheses. We take the cases of the table in order. If G is not 2-edge-connected, or $G - v$ is disconnected for some body v, then G reduces by Lemma 30.118. If every body has degree two, it reduces by Lemma 30.117. If $\mathrm{def}_2(G) = \mathrm{def}_3(G)$, it reduces by clause (1) of Definition 30.112. Otherwise $\mathrm{def}_2(G) > 0$, as $\mathrm{def}_3(G) \leq \mathrm{def}_2(G)$ (Lemma 20.7) and a deficiency is never negative. Then a planar-rigid set gives a reduction (Lemma 30.121).

In the remaining case G is sparse (Lemma 30.105), 2-connected, and has a body of degree other than two. A usable chain gives a reduction (Lemma 30.120). If there is none, Theorem 30.123 gives a core W. Then $G[W]$ and $G/G[W]$ are smaller than G (Lemma 30.53, Lemma 30.54, Lemma 21.51), and G reduces by clause (7) at W.

The rest is Theorem 30.113. A simple 2-edge-connected graph on at least three bodies is connected (Definition 20.20), since no edge leaves the vertices of a component. Its minimum degree is at least two (Lemma 20.21), so it satisfies the standing hypotheses. □

30.12. **The two main-component statements.** This subsection proves the two main-component statements (Definition 29.31), the hypotheses of Theorem 29.33. The distinct statement is immediate (Lemma 30.125): the general configuration of a simple two-edge-connected graph on at least three bodies attains, and one attaining configuration is an adjacent-distinct realization. The generic statement does not follow in this way. When two pencil hubs have three common neighbours, as in $K_{2,3}$, no configuration over a general picture is nondegenerate (Lemma 30.127).

We prove the generic statement instead inside the induction of the pencil reduction (Theorem 29.15). There the step at a graph G may assume the conditioned pair at every graph with fewer bodies, at least one (Theorem 30.137). The generic conjunct of the conditioned pair asks for a realization only at a simple graph. A graph that is not two-edge-connected is treated as in the cut-edge case of the reduction. So the step may assume that G is simple and two-edge-connected, on at least three

bodies. If G has no planar-rigid set (Section 30.10), a general configuration is nondegenerate and attains (Theorem 30.132). On three bodies G is a triangle, which has a direct construction. Otherwise a minimal planar-rigid set gives one of two structures (Lemma 30.136). The first is a body of degree two whose neighbours are pencil hubs and whose removal does not raise the deficiency. The second is a pendant triangle: two adjacent bodies of degree two with a common neighbour. We remove this body, or these two, and the induction hypothesis gives the smaller graph the conditioned pair. We show that the smaller graph is nondegeneracy-feasible, and extend its generic realization across the removed bodies (Lemma 30.134, Lemma 30.135).

Lemma 30.125 (The distinct main-component statement)**.** *Let K be an infinite field. Every simple two-edge-connected multigraph on at least three bodies has a pencil realization at the deficiency rank with adjacent concurrency points distinct: the distinct main-component statement (Definition 29.31) holds.*

Proof. A simple two-edge-connected multigraph on at least three bodies is connected (Definition 20.20), since no edge leaves the vertices of a component. Its minimum degree is at least two (Lemma 20.21), so it satisfies the standing hypotheses (Definition 30.45). Its general configuration attains (Theorem 30.124). One attaining configuration is such a realization (Lemma 30.25). No hypothesis on the number of edge labels is used. □

We record two lemmas for the proof of the generic statement. The first part of the first is the bound noted after Definition 29.23.

Lemma 30.126 (Feasibility bounds the hub neighbourhoods)**.** *If a finite multigraph is nondegeneracy-feasible (Definition 29.23), then every closed hub-neighbourhood has at most three members, and no triangle contains two pencil hubs.*

Proof. At a nondegenerate realization the normals of a closed hub-neighbourhood of v are independent and all orthogonal to the nonzero point of v, so there are at most three of them.

Let x, y, z be a triangle with hubs y and z. At a nondegenerate realization the normals of y and z are independent. The points of x, y and z are orthogonal to both, so they lie in a two-dimensional space W, which the independent points of y and z span. If x is not a hub, its closed neighbourhood is $\{x, y, z\}$, and its three points cannot be independent. If x is a hub, its normal is orthogonal to W, so it lies in the span of the normals of y and z, and the closed hub-neighbourhood of x, which contains x, y and z, has dependent normals. □

Lemma 30.127 (Two hubs with three common neighbours)**.** *Let $a \neq b$ be pencil hubs of G with three common neighbours x, y, z. No pencil realization at which the points of x, y and z are linearly independent is nondegenerate. In particular, over a planar picture at which q_x, q_y, q_z are not collinear, no height gives a nondegenerate realization with the configuration points as concurrency points.*

Proof. At a nondegenerate realization the normals of a and b are linearly independent, since both lie in the closed hub-neighbourhood of x. The point of a body lies in the panel of each of its neighbours, as noted after Definition 29.23. So the points

of x, y and z are orthogonal to both normals, and three independent points would lie in a two-dimensional space. Over a picture at which q_x, q_y and q_z are not collinear, the configuration points of x, y and z are independent whatever the heights, since dropping the height coordinate maps them to independent homogeneous picture points. □

The lemma is the reason the generic statement is not proved through the main component. Let G be simple, with pencil hubs $a \neq b$ and three common neighbours x, y and z. The determinant of the homogeneous picture points of x, y and z is a nonzero polynomial in the picture coordinates, as it is nonzero at the picture with $q_x = (0,0)$, $q_y = (1,0)$ and $q_z = (0,1)$. Multiplied by it, a polynomial that witnesses that the general configuration of G attains (Definition 30.8) is still a witness. At each non-root of the product, q_x, q_y and q_z are not collinear. So the attaining configurations over these general pictures are never nondegenerate. Such a graph G has a planar-rigid set. The five bodies a, b, x, y, z span at least six edges, so their singleton value is at most $3 \cdot 4 - 2 \cdot 6 = 0$. Hence some set of at least two of them is planar-rigid (Lemma 30.105). The obstruction therefore does not arise in the case without a planar-rigid set, treated below.

At $K_{2,3}$, let a and b be the bodies of degree three, so that x, y and z are those of degree two. The generic statement asks for a generic realization of $K_{2,3}$, as $K_{2,3}$ is nondegeneracy-feasible. To see this, take linearly independent normals n_a and n_b. Put the points of x, y and z at three distinct points of the line $n_a^\perp \cap n_b^\perp$, and put $p_a \in n_a^\perp$ and $p_b \in n_b^\perp$ off that line. Then $p_a \cdot n_b \neq 0$ and $p_b \cdot n_a \neq 0$, so the points of x, a and b are linearly independent. Give x the plane through them, and y and z likewise, and give each edge the line through the points of its ends. This is a nondegenerate pencil realization, and its points of x, y and z are linearly dependent, as the lemma requires.

By the lemma, no configuration over a main picture of $K_{2,3}$ is nondegenerate, since at every main picture q_x, q_y and q_z are not collinear. Indeed, at an admissible picture where they are not collinear, the planes of a and b are both the plane through the points of x, y and z. So $L(q)$ consists of the globally affine heights, and has dimension three (Lemma 30.3). Where they are collinear, the heights of x, y and z need only agree with an affine function along their line, and the heights of a and b are then free, so $\dim L(q) = 4$. The induction obtains a generic realization of $K_{2,3}$ instead from one of the 4-cycle left when a body of degree two, say x, is removed. It puts the point of x on the line where the panels of a and b meet (Lemma 30.134).

We turn to the case without a planar-rigid set (Theorem 30.132). At a configuration (q, z) with q admissible and $z \in L(q)$, the plane normal of a body is determined up to a scalar by its plane, the plane through the points of its closed neighbourhood (Lemma 30.12). So over a picture in general position, the conditions of nondegeneracy on the normals reduce to conditions that certain planes differ (Lemma 30.131). The planes of two bodies u and w differ at a general configuration by Jackson and Jordán's equality, applied to G and to the graph G_{uw} of the next definition. Without a planar-rigid set, G_{uw} has the smaller planar deficiency, so at a general picture its lifting space has the smaller dimension. The heights of G whose planes at u and

w agree extend, injectively, to heights of G_{uw}. So they form a proper subspace of $L(q)$ (Lemma 30.130).

Definition 30.128 (A new body joined to two bodies)**.** For distinct bodies u, w of G, the graph G_{uw} is G together with one new body x and two new edges, joining x to u and to w.

Formalization note 30.129. The new body and the two new edge labels of G_{uw} lie outside the sets of bodies and edge labels of G, which are enlarged for them, as the set of edge labels is in Definition 23.41. So G_{uw} needs no unused body or edge label of G.

Lemma 30.130 (The planes of two bodies separate without a planar-rigid set)**.** *Let K be infinite and G simple, with at least three members in every closed neighbourhood, and with $\operatorname{def}_2(G[Y]) \neq 0$ for every set Y of at least two bodies. Let $u \neq w$ be bodies of G. There is a nonzero polynomial P in the picture coordinates such that every picture q with $P(q) \neq 0$ has some height $z \in L(q)$ and, at every body, an affine function that agrees with z on its closed neighbourhood, with the functions at u and w different.*

Proof. Equal planes. Give the new body x of G_{uw} (Definition 30.128) any picture point. A height of G whose planes at u and w agree extends to G_{uw}: the height of x is the value of the common plane at its picture point. The closed neighbourhoods of u and w gain x, which lies on that plane, and the plane agrees with the heights on the closed neighbourhood $\{x, u, w\}$ of x. This extension is injective.

The count. We show that $\operatorname{def}_2(G_{uw}) \leq \operatorname{def}_2(G) - 1$. Compare the value of a partition of the bodies of G_{uw} with that of its restriction to $V(G)$, which is at most $\operatorname{def}_2(G)$. If x is alone in its part, there is one more part and there are two more crossing edges, so the value is lower by $2 \cdot 2 - 3 = 1$. Otherwise the number of parts is the same, and the value is lower by 2 for each of u and w outside the part of x. It remains that x, u and w share a part. Then the value is that of the restriction, in which u and w share a part X. As G has no planar-rigid set, every set of at least two bodies has singleton value at least one (Lemma 30.105). So the restriction has value at most $s(V(G)) - s(X) \leq \operatorname{def}_2(G) - 1$ (Lemma 30.103).

The equality. At a picture general for both graphs, Jackson and Jordán's equality (Theorem 30.43) gives $\dim L_G(q) = 3 + \operatorname{def}_2(G)$, and the lifting space of G_{uw} has dimension $3 + \operatorname{def}_2(G_{uw}) \leq 2 + \operatorname{def}_2(G)$. So the heights of G with equal planes at u and w form a proper subspace of $L_G(q)$. The polynomial of the equality at G_{uw} involves the picture point of x. Fixing that point at one non-root turns it into a nonzero polynomial in the picture coordinates of G, and P is its product with the polynomial of the equality at G. □

Lemma 30.131 (The hub normals of a configuration)**.** *Let G be a finite multigraph with every closed hub-neighbourhood of at most three members. Let q be an admissible picture at which no hub and two of its hub neighbours have collinear picture points, and let $z \in L(q)$. Suppose that the planes of the two ends of every edge joining two hubs differ, and that the planes of any two hubs adjacent to one body that is not a hub differ. Then the configuration (q, z), with the plane normal at every body computed*

from three members of its closed neighbourhood with non-collinear picture points, is a nondegenerate pencil realization.

Proof. The configuration is a pencil realization with adjacent points distinct (Lemma 30.11). A body that is not a hub has at most three members in its closed neighbourhood, and admissibility puts three with non-collinear picture points in it. So the points of its closed neighbourhood are independent.

Each normal is a nonzero multiple of $(h_0, h_1, -1, h_2)$, where $\zeta = h_0\xi + h_1\eta + h_2$ is the plane of the body (Lemma 30.12). So normals are linearly independent exactly when the planes are affinely independent. At a hub v with one hub neighbour this is the first condition. Let v have two hub neighbours w_1 and w_2, and let d_i be the difference of the planes of w_i and v, an affine function. It is nonzero by the first condition, and it vanishes at q_v and at q_{w_i}, where both planes take the height of the body. As q_v, q_{w_1} and q_{w_2} are not collinear, d_2 does not vanish at q_{w_1}, nor d_1 at q_{w_2}. Evaluating a linear relation between d_1 and d_2 at q_{w_1} and at q_{w_2} shows that d_1 and d_2 are independent, so the three planes are affinely independent. The closed hub-neighbourhood of a body that is not a hub consists of at most two of its neighbours, and for two this is the second condition. □

Theorem 30.132 (The general configuration is nondegenerate without a planar--rigid set)**.** *Let K be infinite and let G satisfy the standing hypotheses, be nondegeneracy-feasible, and have $\mathrm{def}_2(G[Y]) \neq 0$ for every set Y of at least two bodies. Then G has a generic pencil realization (Definition 29.25).*

Proof. Every closed hub-neighbourhood has at most three members (Lemma 30.126). Take a picture q off the zero sets of three kinds of nonzero polynomial: one witnessing that the general configuration of G attains (Theorem 30.124), whose non-roots are admissible; for every three distinct bodies, the determinant of their homogeneous picture points; and for every two distinct bodies, the polynomial of Lemma 30.130.

Over q, consider the pairs (z, h) of a height $z \in L(q)$ and, at every body v, an affine function h_v that agrees with z on the closed neighbourhood of v. They form a linear space. The attaining heights contain the non-roots of a polynomial in z that is nonzero at some pair. For two distinct bodies u and w, some coordinate of $h_u - h_w$ is nonzero at the pair that Lemma 30.130 gives. One pair makes all these finitely many polynomials nonzero, by Lemma 30.26 and induction on their number. At it the planes of any two distinct bodies differ, so the configuration (q, z) is a nondegenerate pencil realization (Lemma 30.131). Its height z is an attaining height, so the realization has the deficiency rank (Lemma 30.9). □

Both extensions, across a body of degree two and across a pendant triangle (Lemma 30.134, Lemma 30.135), start from the graph H left when the new bodies are removed. When H is nondegeneracy-feasible, the induction hypothesis gives it a generic realization, and says nothing more about that realization. The extensions need finitely many further conditions on it. For instance, the normals must be independent on every closed hub-neighbourhood in G, which may contain hubs of G that are not hubs of H. A nondegenerate realization of G, restricted to H, satisfies these conditions, but it need not have the deficiency rank. The next lemma combines the two realizations. Up to scalars, the points and the hub normals of

every nondegenerate realization of H are the values of one polynomial map. So the rank of the generic realization, as a lower bound, and each condition hold off the zero sets of polynomials, nonzero at the generic realization and at the restriction respectively. A common non-root gives a generic realization with every condition. Part (a) of the lemma shows that the restriction is nondegenerate, under a condition on the hubs of G that are not hubs of H. So H is nondegeneracy-feasible.

Lemma 30.133 (Nondegenerate and generic realizations of a subgraph)**.** *Let K be infinite.* (a) *Let G be simple and nondegeneracy-feasible, and let H be a subgraph of G. Suppose that whenever a pencil hub h of G is a body of H but not a hub of H, the neighbours of h in G that are not its neighbours in H are not hubs of G. Then G has a nondegenerate realization whose restriction to H is a nondegenerate realization of H. In particular H is nondegeneracy-feasible, by a realization whose normals are independent on the closed hub-neighbourhood in G of every body of G.* (b) *Let H have at least one body. Suppose that H has a generic pencil realization, and a nondegenerate realization with normals n' and points p'. Let $S_1, \dots, S_k \subseteq V(H)$ be sets on which n' is linearly independent, and let $(u_1, w_1), \dots, (u_m, w_m)$ be pairs of bodies of H with $p'_{u_j} \cdot n'_{w_j} \neq 0$. Suppose that every member of every S_i, and every w_j, is a hub of H or has a closed neighbourhood in H of exactly three members. Then H has a generic pencil realization whose normals are independent on every S_i, with $p_{u_j} \cdot n_{w_j} \neq 0$ for every j.*

Proof. A parametrization. Let Γ be a graph whose closed hub-neighbourhoods have at most three members. For $y_1, y_2, y_3 \in K^4$, let $y_1 \times y_2 \times y_3$ be the vector with $(y_1 \times y_2 \times y_3) \cdot t = \det[y_1, y_2, y_3, t]$ for every $t \in K^4$, as for the plane normal of Definition 30.6. It is orthogonal to y_1, y_2 and y_3, its coordinates are polynomials in theirs, and it is nonzero exactly when they are linearly independent. The parameters are a vector $m_v \in K^4$ and three auxiliary vectors at every body v. The normal of a hub v is m_v. The point of a body v is $p_v = y_1 \times y_2 \times y_3$, where y_1, y_2, y_3 are the vectors m_w for w in the closed hub-neighbourhood of v, in a fixed order, completed to three vectors by auxiliary vectors of v. At a body that is not a hub, the closed neighbourhood has at most three members, and the normal is any nonzero vector orthogonal to their points. When it has three members with independent points, this normal is, up to a scalar, the product of their points, a polynomial in the parameters.

Each point is then orthogonal to the normals of the bodies of its closed neighbourhood: to those of hubs by construction, and to the others by the choice of their normals. At parameters where every point is nonzero, adjacent points are independent, and the points of the closed neighbourhood of every body that is not a hub are independent, these points and normals form a nondegenerate realization of Γ. Its hinge at each edge is the line through the points of the ends. The normals of each closed hub-neighbourhood are independent, as factors of a nonzero product. Each of these finitely many conditions says that a polynomial in the parameters is nonzero. If the rank is r at some parameters, it is at least r off the zero set of a polynomial that is nonzero there. Indeed, the hinge of an edge uv is $p_u \vee p_v$ up to a scalar, and the rows of the rigidity matrix at an edge can be taken linear in its hinge, so the minors of the rigidity matrix are polynomials in the parameters.

Conversely, let a nondegenerate realization be given, and let m_v be its normal at every hub v. At each body v, the normals of the closed hub-neighbourhood are independent and orthogonal to p_v. Completing them by auxiliary vectors to a basis of $p_v^\perp$ makes $y_1 \times y_2 \times y_3$ a nonzero multiple of p_v. So the realization comes from some parameters, with its own normals at the bodies that are not hubs, up to a nonzero scalar at each point. These scalars change neither the nondegeneracy nor the line of any hinge, so they do not change the rank.

(a) We parametrize the realizations of G, whose closed hub-neighbourhoods have at most three members (Lemma 30.126). The hubs of H are hubs of G, and each closed neighbourhood or closed hub-neighbourhood in H lies in the one in G. So every condition of nondegeneracy for H is one for G, except at a hub h of G that is a body but not a hub of H. There the points of the closed neighbourhood of h in H must be independent. If h has at most one neighbour in H, these are the points of a body or of the two ends of an edge of G, independent at every nondegenerate realization of G. Let h have two neighbours u and w in H. By hypothesis the closed hub-neighbourhood of h in G lies in $\{h, u, w\}$. Let m_h, m_u and m_w be the first three standard basis vectors, and m_v the fourth at every other body. As no triangle of G contains two hubs (Lemma 30.126), the auxiliary vectors can be chosen among the standard basis vectors so that p_h, p_u and p_w are multiples of the fourth, third and second. So each of these conditions, and each condition of nondegeneracy for G, holds off the zero set of a polynomial that is nonzero at some parameters: those just chosen, or those of a nondegenerate realization of G. As K is infinite, these finitely many polynomials have a common non-root, and it gives the realization.

(b) The graph H is nondegeneracy-feasible, so loopless: the two ends of a loop are one body, whose concurrency point is not projectively distinct from itself. We parametrize the realizations of H, and take parameters s_1 and s_2 that give the generic realization and the second one, up to scalars. The conditions of nondegeneracy, and the rank at s_1 as a lower bound, hold off the zero sets of polynomials nonzero at s_1. Each member of an S_i, and each w_j, is a hub, whose normal is a parameter, or its normal is the product of the three points of its closed neighbourhood. So at s_2 these normals are those of the second realization, up to scalars. Hence the independence on each S_i, and each condition $p_{u_j} \cdot n_{w_j} \neq 0$, holds off the zero set of a polynomial nonzero at s_2. A common non-root of all these polynomials gives a nondegenerate realization with the conditions, of rank at least the deficiency rank. The deficiency rank bounds the rank of every realization (Lemma 20.16), so the realization is generic. □

Lemma 30.134 (The generic realization across a one-body ear)**.** *Let K be infinite and G simple and nondegeneracy-feasible, with a body x of degree two whose neighbours a and b are pencil hubs, and with $\mathrm{def}_3(G-x) \leq \mathrm{def}_3(G)$. If $G-x$ has a generic pencil realization whenever it is nondegeneracy-feasible, then G has a generic pencil realization.*

Proof. Let $H = G - x$. An end of degree three is not a hub of H, and its only neighbour in G outside H is x, which is not a hub. So G has a nondegenerate realization whose restriction to H is nondegenerate (Lemma 30.133(a)), and by hypothesis H has a generic realization. At the realization of G, the normals n_a and

n_b are independent, as a and b lie in the closed hub-neighbourhood of x. Also $p_a \cdot n_b$ and $p_b \cdot n_a$ are not both zero. Otherwise the independent points of x, a and b would all lie in the two-dimensional space $n_a^\perp \cap n_b^\perp$.

We apply Lemma 30.133(b) to H. The sets are the closed hub-neighbourhoods in G of the bodies of H, and $\{a, b\}$. The one pair is (a, b) if $p_a \cdot n_b \neq 0$, and (b, a) otherwise. Every member of these sets is a hub of G. A hub of G that is not a hub of H is an end of degree three, and its closed neighbourhood in H has three members. So H has a generic realization with three properties: the normals of every closed hub-neighbourhood in G are independent, n_a and n_b are independent, and $p_a \cdot n_b$ or $p_b \cdot n_a$ is nonzero.

As $p_a \cdot n_b$ or $p_b \cdot n_a$ is nonzero, p_a and p_b do not both lie on the line $n_a^\perp \cap n_b^\perp$, where the panels of a and b meet. Nor are they proportional: each lies in its own panel, so both products would vanish. So this line is not the line through p_a and p_b. Put p_x on the first line and off the second. Give x the plane through p_a, p_x and p_b, and give the edges ax and xb the lines through the points of their ends. This is a nondegenerate realization of G. The normals of the closed hub-neighbourhoods in G of the bodies of H are independent, and so are those of the closed hub-neighbourhood $\{a, b\}$ of x. The points of the closed neighbourhood $\{x, a, b\}$ of x are independent. A body of H that is not a hub of G is not a or b, so it has the same closed neighbourhood in H.

The new hinges $p_a \vee p_x$ and $p_x \vee p_b$ are independent, so the rank rises by at least six (Lemma 19.32). As $\mathrm{def}_3(G - x) \leq \mathrm{def}_3(G)$, the rank of G reaches its deficiency rank, which bounds it (Lemma 20.16). □

Lemma 30.135 (The generic realization across a pendant triangle)**.** *Let K be infinite and G simple and nondegeneracy-feasible, with two adjacent bodies x_1, x_2 of degree two, both joined to a body c of degree at least four. If $G - \{x_1, x_2\}$ has a generic pencil realization whenever it is nondegeneracy-feasible, then G has a generic pencil realization.*

Proof. Let $H = G - \{x_1, x_2\}$. Of the bodies of H, only c can stop being a hub. If it does, it has degree four in G, and its closed neighbourhood in H has three members. The neighbours it loses, x_1 and x_2, are not hubs. So H is nondegeneracy-feasible, by a realization whose normals are independent on the closed hub-neighbourhood in G of each body of H (Lemma 30.133(a)). By hypothesis H has a generic realization. Every member of these closed hub-neighbourhoods is a hub of G, so H has a generic realization with the same property (Lemma 30.133(b), with these sets).

Put p_{x_1} and p_{x_2} in the panel of c, with p_c, p_{x_1} and p_{x_2} independent. Give x_1 and x_2 the panel of c, and give the three edges of the triangle the lines through the points of their ends. This is a nondegenerate realization of G. The closed hub-neighbourhoods in G of the bodies of H contain neither x_1 nor x_2, which are not hubs, so their normals are independent. That of x_i is $\{c\}$. The closed neighbourhood of x_i is $\{x_1, x_2, c\}$, with independent points. A body of H that is not a hub of G is not c, so it has the same closed neighbourhood in H and is not a hub of H.

Extend p_c, p_{x_1}, p_{x_2} to a basis of K^4. The three hinges of the triangle are three of the six joins of pairs of basis vectors, hence independent. So the triangle is rigid, of rank twelve (Lemma 23.77). The ranks of H and of the triangle add at the cut vertex c (Lemma 19.26). The deficiencies add at c too (Lemma 20.29), and that of

the triangle is nonnegative. Hence the rank reaches the deficiency rank of G, which bounds it (Lemma 20.16). □

We find one of the two structures in the next lemma. We state it for graphs alone, with the two consequences of nondegeneracy-feasibility in Lemma 30.126 as hypotheses.

Lemma 30.136 (A body of degree two with hub neighbours, or a pendant triangle)**.** *Let G satisfy the standing hypotheses (Definition 30.45) and be two-edge-connected on at least four bodies, with every closed hub-neighbourhood of at most three members and no triangle containing two pencil hubs, and suppose* $\operatorname{def}_2(G[Y]) = 0$ *for some set Y of at least two bodies. Then G has a body y of degree two whose neighbours are pencil hubs with* $\operatorname{def}_3(G - y) \leq \operatorname{def}_3(G)$*, or two adjacent bodies of degree two with a common neighbour of degree at least four.*

Proof. Take a planar-rigid set W_0 with as few bodies as possible. Then $s(W_0) \leq 0$, and every $Z \subsetneq W_0$ with $|Z| \geq 2$ has $s(Z) \geq 1$ (Lemma 30.105). Also $|W_0| \geq 3$, since two bodies of a simple graph have singleton value at least one. Adding a body u to a set changes its singleton value by $3 - 2t$, where t is the number of edges from u into the set (Lemma 30.104). So every body of W_0 has at least two neighbours in W_0.

A triangle. Suppose two adjacent bodies y and u of W_0 each have at most one neighbour in $Z = W_0 \setminus \{y, u\}$. Putting y and then u back into Z does not lower the singleton value: y adds at least $3 - 2$, and u, with at most two edges into $Z \cup \{y\}$, at least $3 - 4$. So $s(Z) \leq s(W_0) \leq 0$, and $|Z| \leq 1$. Hence $W_0 = \{y, u, w\}$, and $s(W_0) \leq 0$ makes it a triangle.

A body of degree two. Suppose every body of W_0 is a hub. Then each has at most two hub neighbours, as its closed hub-neighbourhood has at most three members. So each has exactly two neighbours in W_0. The argument above then applies at any edge of $G[W_0]$, and W_0 is a triangle of hubs, which is excluded. So some $y \in W_0$ is not a hub. It has degree two, and both of its neighbours a and b lie in W_0.

The two cases. If a, say, is not a hub, it has degree two too, and the argument above at the edge ya gives a pendant triangle $\{y, a, w\}$. Its third body w has degree at least four. Indeed, at least two edges leave the triangle, as G is two-edge-connected on at least four bodies (Definition 20.20), and all of them are at w. If a and b are hubs, let $X = W_0 \setminus \{y\}$. As y has two edges into X, $s(X) = s(W_0) + 1 \leq 1$. Also $|X| \geq 3$: otherwise $X = \{a, b\}$, $s(X) \leq 1$ needs the edge ab, and $\{y, a, b\}$ would be a triangle with two hubs. So X is rigid (Lemma 30.107). As a and b lie in the rigid set X of $G - y$, we have $\operatorname{def}_3(G - y; a, b) = \operatorname{def}_3(G - y)$ (Lemma 30.106). Putting y into the part of a and b does not change the value of a partition (Lemma 20.32), which gives $\operatorname{def}_3(G - y) \leq \operatorname{def}_3(G)$. □

Theorem 30.137 (The generic conjunct from the induction hypothesis)**.** *Let K be infinite and G simple and two-edge-connected on at least three bodies, nondegeneracy-feasible, and suppose every multigraph on fewer bodies, at least one, satisfies the conditioned pair at $d = 3$ (Definition 29.27). Then G has a generic pencil realization.*

Proof. As a simple two-edge-connected graph on at least three bodies, G satisfies the standing hypotheses (Definition 30.45). If it has no planar-rigid set, it has a generic realization (Theorem 30.132).

On three bodies G is a triangle: each body has degree at least two and, G being simple, at most one edge to each of the other two. A direct construction gives it a generic realization. The three bodies get linearly independent points and the plane through them as their common panel, and each edge gets the line through the points of its ends. No body is a hub, and the three points are independent, so the realization is nondegenerate. The three hinges are linearly independent, so the framework is infinitesimally rigid (Lemma 23.77). Its rank is therefore the deficiency rank, as the triangle is rigid.

Otherwise G has at least four bodies. Its closed hub-neighbourhoods have at most three members, and no triangle contains two hubs (Lemma 30.126). So G has a body of degree two whose neighbours are hubs and whose removal does not raise the deficiency, or a pendant triangle (Lemma 30.136). Removing that body, or the two bodies of degree two of the triangle, leaves a simple graph with fewer bodies, at least one. By hypothesis it satisfies the conditioned pair, whose generic conjunct gives it a generic realization whenever it is nondegeneracy-feasible. So G has a generic realization (Lemma 30.134, Lemma 30.135). □

The hypothesis of Theorem 30.137 asks for the conditioned pair at every multigraph with fewer bodies, not only at those on the whole ambient body set. So we prove it at every multigraph with at least one body.

Theorem 30.138 (The conditioned pair at every nonempty multigraph)**.** *Let K be an infinite field. Every multigraph with at least one body satisfies the conditioned pair at $d = 3$ (Definition 29.27).*

Proof. We prove the contraction and split cases of Theorem 29.30 as in the proof of Theorem 29.33, with the generic statement replaced by the induction hypothesis. Let G be a loopless multigraph on at least three bodies, and suppose that every multigraph on fewer bodies, at least one, satisfies the conditioned pair.

If G is not two-edge-connected, it satisfies the conditioned pair by the cut-edge case of Lemma 29.29. If it is two-edge-connected and not simple, its bare conjunct is Lemma 29.32, and the other two are vacuous. At a simple two-edge-connected multigraph, the distinct conjunct is Lemma 30.125, and the bare conjunct follows from it. The generic conjunct follows from the conditioned pair at the multigraphs with fewer bodies (Theorem 30.137). □

We collect these results in the main theorem of this chapter.

Theorem 30.139 (The general point of the main component attains)**.** *Let K be an infinite field. For every simple connected multigraph G of minimum degree at least two, the general configuration of G attains the rank $6(|V(G)| - 1) - \mathrm{def}(\tilde{G})$ (Definition 30.8; Definition 20.3, at $d = 3$). The two main-component statements (Definition 29.31) hold: every simple two-edge-connected multigraph on at least three bodies has a pencil realization at the deficiency rank with adjacent points distinct*

(Definition 29.26), and, wherever a nondegenerate realization exists at all, a generic one (Definition 29.25).

Proof. The first claim is Theorem 30.124. The distinct statement is Lemma 30.125. The generic statement is not proved from attaining configurations (Lemma 30.127). It is the generic conjunct of the conditioned pair at a simple two-edge-connected multigraph on at least three bodies (Theorem 30.138). □

Appendix A. Notes on the formalization

This chapter records the wrong turns the formalization of Katoh–Tanigawa's Theorem 5.5 [KT11] took on the way to the argument the earlier chapters present: statements whose formal strength diverged from the source's, decompositions that did not survive contact with the assembly they were meant to feed, and obligations left with no artifact tracking them. Three patterns recur across most of the sections below and are named in the next paragraph. Section A.1 is a scaffolding narrative the three-way classification does not apply to at all, and Section A.4 groups two episodes by a shared symptom instead of by one pattern, each episode there self-classifying separately. Within a section, an episode narrates what a check did and did not read at the time a divergence was introduced, and the check that eventually caught it. Because the subject here is the formalization itself, Lean declarations, their types, and short code excerpts appear directly in the prose below, rather than only as parenthetical citations, as they do in the rest of this document.

A *genuine mis-factoring* is an internal decomposition — an abstraction layer, a splice of two arguments, a case split — that did not match the structure the final assembly needed, so the decomposition was replaced without changing the underlying mathematics. A *source-faithfulness correction* is a Lean statement whose formal strength diverged from Katoh–Tanigawa's — weaker, stronger, or differently quantified — found and corrected without invalidating any proof already built on it. A *process failure* is a decision that was sound when made but left with no artifact tracking it, or an error that no check running at the time could have read.

Commits are cited by an eight-character abbreviated SHA linking to the corresponding commit on GitHub; every cited commit postdates the project's 2026-05-13 move to a standalone repository.

A.1. **The $d = 3$-first scaffolding arc.**

The episode below sits outside the three-way classification above: it is not a mis-factoring, a source-faithfulness correction, or a process failure, but a scaffolding choice, refined twice at contact with a build, that left nothing already proved to redo.

Katoh–Tanigawa present Case III (KT §6.4, Theorem 5.5's hardest case) twice: concretely at $d = 3$ (§6.4.1, Lemma 6.10, a fixed dispatch on four vertices v, a, b, c), then at general dimension (§6.4.2, Lemma 6.13, a length-d chain $v_0 v_1 \dots v_d$), with the remark that "the proof strategy is exactly the same as $d = 3$" (p. 692). The formalization took the same order, and made the choice explicit rather than assumed, at Phase 22c's second design-recon pass (commit `6511a01a`, 2026-06-05): $d = 3$ first, following "KT's own §6.4.1-then-§6.4.2 cut," with the general-d lift assigned to its own later phase before any of the concrete case was built.

The same pass staged the $d = 3$ choice again, one level down, rather than attempting Lemma 6.10 — KT's longest proof, roughly twelve printed pages — as a single build: it split the lemma into three difficulty strata (the eq.-(6.12) degenerate placement, giving a rank one short of full; the redundant-row argument of Claim 6.11, named the genuinely new research crux; and Claim 6.12's candidate-completion finish), and committed only the first stratum to the sub-phase then opening (Phase 22c itself). The three strata were then built in turn across Phases 22c–22h, entirely inside the concrete carrier — every panel normal a literal `Fin 4` vector, every screw a literal element of the six-dimensional $\bigwedge^2 \mathbb{R}^4$, arithmetic such as `screwDim 2` $= 6$ closing by `decide` — closing with the $d = 3$ Case-III node itself reaching green, modulo a small number of explicitly carried hypotheses later discharged in subsequent sub-phases (commit `b9a45ea9`, 2026-06-11, closing Phase 22h).

Once the $d = 3$ construction reached this state, a design pass drew a reuse map against it before any general-d Lean was written (commit `036804b2`, 2026-06-07): Claim 6.11's redundant-row mechanism (`exists_redundant_panelRow_ab_of_finrank_eq`) and Lemma 2.1's extensor-span finish were read as reusable verbatim once re-stated at general grade; the exterior-algebra meet API (`complementIso`, `topEquiv`, `pairingDualEquiv`) needed only dimension-parametric re-statement; and the fixed three-candidate dispatch, together with the bespoke $\bigwedge^2 \mathbb{R}^4$ duality, were marked for wholesale replacement by a d-candidate chain dispatch and a $\bigwedge^{d-1}(\mathbb{R}^{d+1})$ duality — the same one-dimensional top-exterior-power fact at a different grade, not the general Hodge-star construction Katoh–Tanigawa never invoke. The same map recorded one further claim: that the realization spine underlying all three cases — the genericity device, `theorem_55`'s skeleton, and Cases I and II — was already general and green, the graph-side combinatorics being dimension-free throughout.

That last claim did not survive Phase 23's own opening design pass (commit `6ea8a691`, 2026-06-17). Re-reading every spine declaration against its landed Lean grade found `theorem_55_d3`, `case_III_realization`, and `case_III_candidate_dispatch` all carrying `ScrewSpace 2` and `Fin 4` literally: dimension-parametricity held for the graph-side bookkeeping the earlier map had checked, not for the panel-geometric spine the induction is actually stated over. The correction split the remaining work along a fault line the earlier map had not drawn — a mechanical lift of the pinned spine to symbolic `screwDim k`, kept apart from the genuinely new chain dispatch and duality — and scheduled the lift first, as sub-phase 23a: not because it was mathematically hard, but because the chain dispatch and the duality both had to be *stated* at general grade, which presupposes a general-grade spine to state their conclusions over, and because the symbolic-k arithmetic underlying `screwDim k` — concrete and decidable at $k = 2$, symbolic thereafter — was the one named risk of a build-time blocker, best surfaced before new mathematics was built on top of it.

The same correction repeated at a finer grain the same day. Lifting the spine's own closing declaration found that four further producers — the base, cut-edge, Case-I, and forgetful-map realizations — were themselves `Fin 4`-pinned through the same $\bigwedge^2 \mathbb{R}^4$ duality lemma, and so were not liftable by the numeral pass that generalized the rest of the spine (commit `ee1674df`, 2026-06-17, closing sub-phase 23a). They

were carried forward as four explicit hypotheses, the project's standing green-modulo idiom, to be discharged once the general chain-dispatch duality became available.

None of these three revisions — the $d = 3$-first cut itself, the reuse map, and its correction at Phase 23's own opening — reopened anything already proved: the $d = 3$ construction stayed green throughout, and Phase 23 closed (commit `623846d0`, 2026-07-02) with Conjecture 1.2 stated as a theorem, along exactly the division the corrected map had drawn. Building the hardest case concretely first did not remove the need to re-scope its generalization; it moved the re-scoping to a point where a whole worked case, not a paper reading alone, was there to check the scope against.

A.2. **Statement-faithfulness episodes.** The episodes below are source-faithfulness corrections in the sense of the introduction: a Lean statement's formal strength diverged from Katoh–Tanigawa's, and the divergence was found and repaired without invalidating proofs already built against it.

A.2.1. *A realization predicate weaker than its prose.* KT Theorem 5.5 asserts that a minimal k-dof graph admits a panel-hinge realization whose rigidity matrix attains the maximal rank. In KT's definition (KT §5.1) each hinge is a line: a nonzero 2-extensor contained in both endpoint panels. The formalization introduced the realization predicate in June 2026 (commit `1cd26cce`) as

```
def HasFullRankRealization (k : ℕ) (G : Graph α β) : Prop :=
  ∃ Q : PanelHingeFramework k α β, Q.graph = G ∧
    Q.toBodyHinge.RankHypothesis 0
```

with no condition on the hinge extensors. The omission makes the predicate satisfiable for every connected graph. The framework assigning every hinge the degenerate endpoint pair (a_0, a_0) has all support extensors zero; the constraint at an edge uv, that the screw difference $S_u - S_v$ lie in the span of the edge's extensor, then reads $S_u = S_v$, so on a connected graph every infinitesimal motion is constant and the rank is the full $D(|V| - 1)$. The realization conjunct of the formalized Theorem 5.5 was satisfied by this witness independently of the theorem's content.

The predicate had been introduced as rank bookkeeping for the induction of KT §5 and was not compared against KT's definition of realization, either at its introduction or at the one later redesign that reworked it (the two-motive split of `e4693d61`'s design pass re-read KT on the simple-versus-non-simple axis and did not pose the question). The blueprint did not expose the divergence: the definition's dependency-graph node was marked formalized beside prose at the paper's strength, and none of the per-commit checks compared a definition's Lean body with its prose — the name-resolution check verifies that pinned declarations exist, and the hypothesis audit reads a theorem's hypotheses, of which a definition has none.

The divergence was found by a pre-build audit of the theorem's hypothesis feeds (commit `e4693d61`, 2026-06-11), before any construction was built against the degenerate witness. It was a weakness of statement selection rather than of proof: the general-position motive introduced at the two-motive split is at KT's strength, and the rank arguments beneath the bare conjunct required no rework. The repair was scheduled at the motive redesign that the same audit had already forced, so the definition surface was reshaped once rather than twice, and landed as (`652ea99f`)

```
def HasPanelRealization (k n : ℕ) (G : Graph α β) : Prop :=
  ∃ (F : BodyHingeFramework k α β) (normal : α → Fin (k + 2) → ℝ),
    F.graph = G ∧
    (∀ v ∈ V(G), normal v ≠ 0) ∧
    (∀ e u v, G.IsLink e u v → F.supportExtensor e ≠ 0 ∧
      ExtensorInPanel (F.supportExtensor e) (normal u) ∧
      ExtensorInPanel (F.supportExtensor e) (normal v)) ∧ …
```

whose nonzero-normal and nonzero-extensor conjuncts are the nondegeneracy conditions of KT's definition. The check this produced is the cheapest-witness audit (`blueprint/CLAUDE.md`, the definition-faithfulness gate): when a definition modeling a paper notion is marked formalized, its blueprint prose is read against the cheapest satisfying witness of the Lean body, not against the paper.

A.2.2. *Katoh–Tanigawa Lemma 4.1's over-quantified statement.* Katoh–Tanigawa's own argument for the degree-of-freedom bookkeeping a splitting-off or a vertex removal induces, on the way to Theorem 4.9, is a forest surgery, [KT11, Lemma 4.1] (p. 660; the 2009 arXiv predecessor, Lemma 5.1, p. 11, is essentially verbatim): "for *any* independent set I of $M(\tilde{G})$ …there exists I' …with $|I'| = |I| - D$." Formalizing this quantifier literally, rather than paraphrasing it into the base-restricted form the induction actually needs, exposed at once that it cannot hold as written: for an independent I with $|I| < D$ — $I = \emptyset$ suffices — the promised $|I'|$ would have to be negative. The disproof was recorded as a Lean witness rather than quietly paraphrased away, in the same commit that added the remark distinguishing the finding's three layers (Remark 21.15) and re-pointed Theorem 4.9's degree-of-freedom tracking off the forest surgery entirely (commit `f0934296`, 2026-06-03). It was introduced as an anonymous `example`; a later cleanup pass promoted it to a named lemma so the corresponding blueprint node could pin it directly, replacing a comment noting it was an anonymous, unpinnable example (commit `bc579c3f`):

```
theorem kt_lemma_41_overquantified (n : ℕ) (hD : 1 ≤ bodyBarDim n) :
    ¬ ∃ I' : Set (β × Fin (bodyHingeMult n)),
      I'.ncard + bodyBarDim n =
        (∅ : Set (β × Fin (bodyHingeMult n))).ncard
```

(`bodyBarDim n` is D; `bodyHingeMult n` the per-edge multiplicity of $\tilde{G}$'s edge fibers.)

A second, subtler layer came with the same finding. Katoh–Tanigawa's proof of the intended, base-quantified reading converts each of the D forests of a chosen packing by removing one edge, silently assuming the packing is *balanced* at the degree-2 vertex v under surgery — every forest already meets one of v's two incident edges. Neither the 2011 paper nor its 2009 predecessor justifies this, and the finding first recorded it as an open question. It resolved positively later the same day: a counting bound (a base of $M(\tilde{G})$ meets at least D of v's $2(D-1)$ incident fibers) together with a redistribution move (a forest avoiding v has v isolated, so it absorbs a free fiber as a pendant without creating a cycle) and a finite descent construct a balanced packing for every base and every $D \geq 2$ (commit `398ce7ac`). By the time this second layer closed, though, Theorem 4.9's tracking no longer needed it: the same finding commit had already re-pointed the induction onto a direct partition-count argument through

the def = corank bridge (Theorem 20.12, `rank_add_deficiency_eq`, Phase 19), discharged within the same session, bypassing the forest surgery. The repaired surgery survives as a sound, off-critical-path formalization of KT 4.1's splitting-off direction; the removal direction, KT 4.2, stays deferred, needed by neither route.

This is a divergence in the source's own statement, not in how the Lean represented it: what the induction actually consumes — the base-quantified deficiency inequality — is exactly Katoh–Tanigawa's intended content, and stating that content precisely, rather than transcribing the paper's literal quantifier, was the formalization's task. Full three-layer statement and repair: Remark 21.15.

A.2.3. *A fixed-candidate disjunction for Claim 6.12's free line choice.* Katoh–Tanigawa's Claim 6.12 [KT11, §6.4.1] (eqs. (6.30)–(6.45)) underlies the $d = 3$ Case-III realization: three candidate placements p_1, p_2, p_3 are built by splitting off along a *freely chosen* $(d-2)$-affine line — "we may choose any $L \subset \Pi(a)$, $L' \subset \Pi(b)$, $L'' \subset \Pi(c)$" — and the claim is that *for some* choice at least one of the corresponding blocks M_1, M_2, M_3 has full rank. The capstone formalizing this, `case_III_claim612`, first modeled the claim not at a free line but at the three *specific* lines the realization producer happens to build (the split edges va, vb, ac), with conclusion $r(C_1) \neq 0 \vee r(C_2) \neq 0 \vee r(C_3) \neq 0$ for the three corresponding fixed extensors, discharged contrapositively through an explicit duality hypothesis (commit `56aef7c4`, 2026-06-07):

```
theorem case_III_claim612
    {r : Module.Dual ℝ (ScrewSpace 2)} (hr : r ≠ 0)
    {C₁ C₂ C₃ : ScrewSpace 2}
    {p : Fin 4 → Fin 3 → ℝ} (hp : AffineIndependent ℝ p)
    (hduality : r C₁ = 0 → r C₂ = 0 → r C₃ = 0 →
      …) :
    r C₁ ≠ 0 ∨ r C₂ ≠ 0 ∨ r C₃ ≠ 0
```

Tracing what discharges `hduality` two days later found the fixed antecedent mathematically unusable, not merely incomplete: C_1, C_2, C_3 are three vectors in the six dimensions of $\bigwedge^2 \mathbb{R}^4$, so "$r \perp C_1, C_2, C_3$" can never force $r = 0$, while Katoh–Tanigawa's own text confirms the three lines are freely chosen, not graph-fixed — so Claim 6.12 is a genuine existential over that continuum, not a disjunction at three fixed points (commit `82b3b50a`). The repair restated the capstone as the existential it actually is, over the six pairwise joins of four affinely-independent points, which do span $\bigwedge^2 \mathbb{R}^4$ (Lemma 18.9, KT Lemma 2.1) even though three fixed lines cannot (commit `d5ff3648`, 2026-06-09):

```
theorem case_III_claim612
    {r : Module.Dual ℝ (ScrewSpace 2)} (hr : r ≠ 0)
    {p : Fin 4 → Fin 3 → ℝ} (hp : AffineIndependent ℝ p) :
    ∃ q : {q : Fin 4 × Fin 4 // q.1 < q.2},
      r ⟨omitTwoExtensor (fun i => homogenize (p i)) (ne_of_lt q.2),
        extensor_mem_exteriorPower _⟩ ≠ 0
```

The realization producer consumes the existential by building its candidate placement so its hinge line *is* the witness join's line. A later generalization to arbitrary

d subsumed this $d = 3$-specific form into `case_III_claim612_gen`, the declaration the blueprint node Lemma 23.191 pins today.

Unlike Lemma 4.1's divergence, which sits in Katoh–Tanigawa's own text, this one originated in the Lean's internal decomposition of the claim: modeling a free line choice as a disjunction over three graph-fixed candidates was a shape choice, not a mistranscription of KT's statement, and the fix corrects that shape — which happened also to be mathematically undischargeable, not merely a looser reading of the source.

A.3. **Abstraction-layer mis-factorings.** The episodes below are genuine mis-factorings in the sense of the introduction: an internal decomposition — an abstraction layer, a splice of two arguments — that did not match the structure the final assembly needed, so the decomposition was replaced without changing the underlying mathematics.

A.3.1. *The Claim 6.12 candidate-row producers.* Katoh–Tanigawa's Claim 6.12 [KT11, §6.4.1], introduced above (Section A.2), supplies the row that completes Case III's rank count: the redundant-row argument of Claim 6.11 falls one row short of the target, and Claim 6.12 supplies the last row from one of three candidate splittings p_1, p_2, p_3. Phase 22e (2026-06-06) formalized the second and third candidates as abstract row-family linear-independence theorems, quantified over an arbitrary `BodyHingeFramework`, not over any concrete construction:

```
theorem linearIndependent_sum_p2_candidateRow (F : BodyHingeFramework k α β) (e : β
    … (hr : ρ (F.supportExtensor e) ≠ 0) :
    LinearIndependent ℝ
      (Sum.elim (Sum.elim rn (fun _ : Unit => hingeRow v b ρ)) ro)
```

(commit `9b8a6c67`, 2026-06-06; the symmetric third candidate, split at the other degree-2 body along ac, landed the same day as `linearIndependent_sum_p3_candidateRow`, commit `87ea1295`). Both are graph-free: `F` is an arbitrary abstract framework, and each theorem concludes independence of the row family the candidate completion assembles, packaged for whichever concrete carrier eventually supplies the surrounding old and new row blocks `rn`/`ro`.

A design pass the following day planned to consume these producers directly: compose the abstract row-family independence with a fixed-framework device-feed brick, so that discharging the witness condition `hr` would close the realization (commit `a893639a`, 2026-06-07). Four days later, reading Katoh–Tanigawa's own eq. (6.29) sketch against that plan found it was not literally transferable: the composed family has to be consumed at the one-parameter shear family Case III's construction actually builds, and the certified family the row-family producers supply is not of that shape (the candidate row, and part of the transported bottom family, are not rows of the sheared candidate at $t \neq 0$). The corrected route instead certifies the target rank at the fixed, unsheared framework F_0, converts the count to a rank lower bound, re-extracts a literal `panelRow` family from that bound, and transfers *that* family along the shear (commit `5aff1320`, 2026-06-11 — "certify-then-rebase"). The rank certification this needs, `case_III_rank_certification`, landed the same day (commit `f6856538`), and the three arm closers that consume it

— `case_III_arm_realization` (M_1), `case_III_arm_realization_M2` (M_2, commit `2981429e`), and `case_III_arm_realization_M3` (M_3) — each read their target rank bound off the certification, not off `linearIndependent_sum_p2_candidateRow` or `_p3_candidateRow`: their own doc-comments record this explicitly, factoring the arm closer as the part that depends only on the certified rank bound and the split/seed data, "not on how that rank was certified".

Neither producer, nor their selector packagings, ever gained a caller outside the file that defines them once the certify-then-rebase route landed: a later audit during the Phase-26 cleanup round found both producers and their selectors dead against the live Lean call chain into the theorem, alongside the $d = 3$ candidate dispatch they were built to feed (commit `fe2f2a73`, 2026-07-07). They were kept for a time — they grounded the worked $d = 3$ case exposition the blueprint's Case III chapter presents, even though the theorem no longer built through them — and a cleanup round deleted them in October 2026, still without a caller; the abstraction itself, correct and graph-free as built, never reached the assembly it was factored to feed. This is a genuine mis-factoring: the row-family layer was mathematically sound at every stage, and what replaced it is a different decomposition of the same rank count, not a correction to its content.

A.3.2. *A motion-space splice for Katoh–Tanigawa's block-triangular rank-addition.* Katoh–Tanigawa's Case I inductive step [KT11, eq. (6.3)] adds the ranks of two separately-placed rigidity matrices: a rigid subgraph H realized at its own generic placement contributes $D(|V(H)|-1)$ rows, and the contraction $G/E(H)$ realized at *its own* generic placement contributes the rest, the two ranks adding because each block occupies disjoint columns once the other is projected away. No step of KT's argument asks for one placement simultaneously realizing both legs.

Phase 21b's first Case-I composer nonetheless took two same-placement rigidity outputs as hypotheses — the rigid subgraph and the contraction both rigid on the *same* witness placement, with no Lean lemma constructing such a placement — flagged the same day by a blueprint-honesty audit as premature (commit `615638e4`, 2026-06-04). The repair that followed built an honest, satisfiable glue lemma in place of assuming the common placement outright:

```
theorem isInfinitesimallyRigidOn_of_splice (F : BodyHingeFramework k α β)
    …
    (hblock : (F.withGraph GH).IsInfinitesimallyRigidOn sH)
    (hcontract : (F.withGraph Gc).IsInfinitesimallyRigidOn sc) :
    F.IsInfinitesimallyRigidOn t
```

(`lem:case-I-splice-seed`, commit `5b6f5a6a`, 2026-06-04). This is a locally sound modelling choice — KT eq. (6.3) read as two overlapping rigid pieces glued along their shared body — and it is genericity-free, exactly as KT's own splice is. But `hblock` and `hcontract` are both stated of the *same* framework `F`: the glue demands one placement simultaneously rigid on both legs, where KT's block-triangular structure places each leg independently and adds the resulting ranks.

Phase 22a's realization producer, built on this glue, then had to supply that common-seed rigidity, and tried to do so across two re-plans on the same day

before the divergence was named, each producing a hypothesis that type-checked but was not dischargeable. The first composer bundled a placement-independent complement-isolation equality — for every placement q, the pinned-motion dimension at the contraction's surviving body set s_c equals $D\cdot|s_c^{\mathbb{C}}|$ — found generically false the same day: off the full vertex set, the contraction leg's interior bodies still carry surviving boundary-edge constraints, so the equality fails (commit `c95d7bc0`, 2026-06-05). An asymmetric-coupling repair deleted that equality and fed the contraction leg's rigidity directly from a universally-quantified general-position hypothesis — for every general-position placement q, the leg is rigid at q — and a second verification pass the same day found this equally undischargeable: general position is pairwise normal independence, strictly weaker than the maximal rank the hypothesis demands (commit `85f5b054`, 2026-06-05). Both passes had taken the glue's shape as fixed and tried to satisfy its obligation; neither questioned the glue itself.

The divergence, once named, was structural rather than a missing hypothesis: the project's rigidity model is a motion-space predicate — every infinitesimal motion constant on a body set — and the glue had translated KT's rank-addition into a motion-space statement that needs a common witness KT's own argument never constructs. The repair replaced the glue's obligation with a row count read off a *single* framework's rigidity-row span directly — independent rows in each block, rather than rigidity of each leg — eliminating the common-seed demand by construction (commit `f504955c`, 2026-06-05, the block-triangular reframe; Lemma 23.89 and Lemma 23.100 record the corrected form). The reframe's own first cut restated the surviving block's obligation with the same universal quantifier over general position again — the same over-quantification recurring one level down — corrected the same day to a rank-polynomial non-root condition, KT's genuine Zariski-open genericity locus (commit `c9750bdf`, 2026-06-05).

This is a genuine mis-factoring: the motion-space model that carried Phases 4–16 without incident re-expressed KT's block-triangular rank-addition as a different argument with a different, harder obligation, and the tell across both re-plans was the same — the arithmetic closed, but each pass needed a fresh hypothesis to bridge a gap Katoh–Tanigawa's own proof does not have.

A.4. **Walls from mis-modelling.** The two episodes below share a symptom rather than one cause: each is a point at which the formalization's chosen approach turned out to be unable, in principle, to produce the fact the next step needed — a wall, in the sense that no further building along the same route would have closed it — and each was diagnosed only once the target quantity was traced directly through the source construction, rather than by a check of the surrounding declarations' types. The two do not share one classification from the introduction's three-way taxonomy; each states its own below.

A.4.1. *The member-mapping wall.* Katoh–Tanigawa's proof of Case III at general dimension (KT §6.4.2, Lemma 6.13, the length-d chain $v_0v_1 \dots v_d$) certifies the rank added at each induction step i by a $D \times D$ corner block M_i, whose rank adds to a base block's to reach the full $D + D(|V|-2) = D(|V|-1)$ (eqs. (6.64)–(6.65)). The corner's second row is one fixed vector $r \in \mathbb{R}^D$ at every step i — held fixed by a

degree-two cancellation, not transported across the induction (eq. (6.66)) — and the construction shows some choice of the corner's first row makes M_i invertible against r, over a free line each step is built from (eq. (6.67)).

The formalization's first approach to the general-d step (Phase 23b, opened 2026-06-17) modeled this certification as a transported span membership instead: hold one witness row's membership in the span of already-certified rows fixed, and carry it forward across the induction. Several distinct routes were tried in turn — a per-step fold along the chain, a fresh re-derivation of the redundancy at each candidate, a re-shaping of the base realization consuming Katoh–Tanigawa's column operation directly — and each, checked against what that operation actually does, reached the same obstruction: Katoh–Tanigawa's own column operation (eq. (6.62)) relabels which of the base graph's rows plays the corner's redundant-row role as i advances, rather than fixing one witness row whose membership a single argument keeps certifying; a route asking for one witness row's membership to survive past the first step therefore has, in the text, nothing to derive it from (commit `591cf306`, 2026-06-21, the last of these routes to be refuted this way).

Reshaping the certification itself to Katoh–Tanigawa's own block form — rather than transporting a membership at all — was sketched in outline that same afternoon (commit `50fb3229`, 2026-06-21): certify the rank as a block matrix whose corner is M_i and whose bottom is the base rows with the one redundant row removed, holding r fixed only as an equation between two vectors in $\mathbb{R}^D$, never as a transported membership. Confirming that this was the *only* escape took three more days: Phase 23c, opened that same afternoon, spent them checking whether the existing certification could instead be fed the needed fact some other way — a different base block, a different re-derivation of the redundancy — and found every such feed reduced to the identical obstruction. It closed at what the design notes call the member-mapping-wall STOP (commit `4efcabf5`, 2026-06-24): the wall was confirmed intrinsic to Katoh–Tanigawa's own row bookkeeping, not an artifact of one Lean encoding, and the block-form reshape was confirmed the only route around it.

Turning that reshape into a literal Lean certificate took two more false starts once building resumed. A bottom block built from the split-off graph with the reduced vertex removed failed, because that graph's rows are not literally contained in the candidate's own rigidity-row span (the candidate graph lacks the corresponding edge); and a construction following the project's column operation directly failed for the same underlying reason approached from the other side, zeroing the wrong block and stranding the fill row the bottom needs in the block the certificate discards. Both were, on inspection, one obstruction restated: a certificate whose corner and bottom occupy disjoint row blocks of the candidate graph's own edges cannot express Katoh–Tanigawa's argument, which shares one row's data between the corner's $\pm r$ entry and the bottom's redundant-row fill. The shape that dodges it — zero in the *other* corner, so a single row operation on the corner side, not the bottom, produces the needed block-triangular form — was found and landed axiom-clean, closing Phase 23e (commits `a02b8c58` and `f4c7ae57`, both 2026-06-26):

`theorem finrank_span_rigidityRows_ge_of_edge_submatrix_fromBlocks_zero`$_{12}$

```
(F : BodyHingeFramework k α β) …
(Lrow : Matrix … ℝ) (hLrow : IsUnit Lrow.det)
(U : Matrix … ℝ) (hU : IsUnit U.det)
(re : m₁ ⊕ m₂ → …)
(en : (n₁ ⊕ n₂) ≃ …)
(hblock : (Lrow * F.rigidityMatrixEdge ends hgp * U).submatrix re en
  = Matrix.fromBlocks A 0 C D)
(hA : LinearIndependent ℝ A.row) (hD : LinearIndependent ℝ D.row) :
Fintype.card m₁ + Fintype.card m₂
  ≤ Module.finrank ℝ (Submodule.span ℝ F.rigidityRows)
```

This is a genuine mis-factoring: the transported-membership certificate was a locally sound internal decomposition of the same rank count Katoh–Tanigawa's own certificate proves, and it was replaced — without changing the target rank or reopening any already-certified step — by a decomposition matching the source's own block structure. The formal proof later made these counts with spans of rows rather than with matrices, and a cleanup round deleted the matrix certificate in October 2026, when nothing used it any longer.

A.4.2. *The eq.-(6.12) $+(D-1)$ vs $+D$ shortfall.* Katoh–Tanigawa's Lemma 6.8 (KT §6.3, eq. (6.12)) realizes a k-reducible degree-two vertex v by a degenerate placement: with (G_v^{ab}, q) the inductive realization, v's hinge to b is placed at the very hinge $q(ab)$ the split-off edge already occupies. Column operations then exhibit $R(G_v^{ab}, q)$ as a submatrix of $R(G, p_1)$, with the vb-row reproducing $R(G_v^{ab}, q)$'s ab-row, so the rank adds: $(D-1)$ new rows plus whatever rank $R(G_v^{ab}, q)$ already has.

The project's degree-two split producer (Phase 21b, planned under the name "Case II") took this construction as its route to the full rank $D(|V|-1)$ the theorem's $k = 0$ split needs. Several re-plans circled the one row the construction seemed to be missing without first checking whether eq. (6.12) reaches the target at all; two, both on 2026-06-04, are representative. The first was a row-side attempt to extract that row directly from G_v^{ab}'s own old block, avoiding the split edge (commit `524e741c`), found geometrically impossible ($G - v$ is not itself rigid, so no such row exists to extract). The second was a motion-side reframing, pinning the missing freedom directly rather than adding a row (commit `7d205752`), which assumed a motion of G constant off v — not something a rigid G actually forces, since $G - v$ need not itself be rigid.

A math-first pass against Katoh–Tanigawa's own text, rather than against either re-plan's declarations, found the target unreachable by eq. (6.12) alone at $k = 0$ regardless of which row is added: by KT Lemma 4.8(i), G_v^{ab} is minimal 0-dof at this split, so $R(G_v^{ab}, q)$ is already full rank, $D(|V \setminus \{v\}| - 1)$, and eq. (6.12) gives rank at least $(D-1) + D(|V \setminus \{v\}| - 1)$, i.e. $D(|V| - 1) - 1$ — exactly one row short of the $D(|V|-1)$ target. Katoh–Tanigawa's own text states the same shortfall directly, illustrated at $d = 3$ ($D = 6$) in the sketch introducing Lemma 6.10 (KT p. 680): "we only have rank $R(G, p_1) \geq 6(|V|-1)-1$ ... which does not complete the proof." Lemma 6.8's $+(D-1)$ bound suffices only for the $k > 0$ split it is stated for; the project's split is the $k = 0$ case, whose missing row is supplied not by Lemma 6.8 but

by the redundant-edge argument of Lemma 6.10/6.13 (§6.4, Section A.1) — Case III, not Case II (commit `db1bfca5`, 2026-06-04, closing Phase 21b and re-scoping the producer to Phases 22–23).

The shortfall was visible in the arithmetic above from the day the producer was scoped, but sat under these re-plans because each reasoned about which named construction could supply the missing row rather than tracing the target count through Katoh–Tanigawa's own construction by hand; the checks the project runs on a new declaration — name resolution, the audit of a theorem's hypotheses — read types and hypotheses, not the numeral a rank inequality closes at, so neither re-plan's declarations tripped anything before the math-first pass traced the count directly.

This is a process failure: the arithmetic gap was real from the day the producer was scoped, and no check running at the time — type-correctness, hypothesis-honesty — was positioned to read it; only tracing the target quantity through the source construction by hand could.

A.5. **Process/tracking failures.** The two episodes below are process failures in the introduction's sense: each is a call that was reasonable given what was read at the time, but left no artifact for a later check to find, or read a signal that looked decisive and was not.

A.5.1. *The Case-I dispatch's untracked Lemma-6.5 arm.* Katoh–Tanigawa's Case I (KT §6.2, pp. 673, 676–677) trifurcates on the simplicity of the contraction $G/E(H)$ of a proper rigid subgraph H: Lemma 6.2 handles a non-simple G directly; Lemma 6.3 handles a simple G with some simple contraction, adding the rigid block's rank to the contraction's own inductive rank by KT's block-triangular structure (eq. (6.3), Section A.3); and Lemma 6.5 handles what remains — every contraction non-simple — via Claim 6.6, which extracts a degree-two vertex v from a maximal proper rigid subgraph and re-attaches it after inducting on the smaller graph $G - v$.

A design recon opening the Case-I sub-phase read this trifurcation correctly against Katoh–Tanigawa's text and named the right tracking idiom for its harder leg: carry the Lemma-6.3-versus-6.5 dispatch as an explicit hypothesis rather than build it outright (commit `ac958152`, 2026-06-04). The composer that landed the next day discharged only the Lemma-6.3 leg, taking the case hypothesis `hcSimple` — that $G/E(H)$ is simple — as a fixed input, and recorded the alternative directly in its own doc-comment rather than tracking it: "the `hcontract` non-simple branch …and full-`theorem_55` dispatch are the coordinator's wiring" (commit `c1ef55a0`, 2026-06-05). The blueprint flip that landed the same day minted a red node for the one hypothesis the composer's own signature carried as content — KT Claim 6.4, the surviving block's generic rank (Lemma 23.98) — and none for the Lemma-6.3-versus-6.5 dispatch or the Lemma-6.5 arm (commit `8fa5b02a`); the sub-phase's own close then recorded Claim 6.4 as "the single deferred obligation" (commit `0e10b5ed`). Neither the dispatch nor the Lemma-6.5 arm appeared on any forward list from this point.

Both resurfaced five sub-phases later, in the same audit of the realization layer's hypothesis feeds against the landed Lean that also found the vacuous bare realization

predicate (Section A.2): tracing what discharges `hcontractGP` found no Lemma-6.3-versus-6.5 dispatch anywhere in the tree, and the Lemma-6.5 arm genuinely missing rather than merely deferred (commit `e4693d61`, 2026-06-11). The dispatch itself was formalized that same day as a classical case split on whether some proper rigid subgraph has a simple contraction, with the Lemma-6.5 branch carried forward as a new named hypothesis `h65` rather than built (commit `06763ffb`). Building Claim 6.6 itself — the vertex-extraction argument the branch needed — took five more days, landing as `case_I_realization_h65` (Lemma 23.106; its form for every $d \geq 2$, `case_I_realization_h65_gen`, later replaced it) and completing the dispatch (Lemma 23.107), which now cases on all three of KT's Lemmas 6.2, 6.3, and 6.5 (commit `0f40c44d`, 2026-06-16).

This is a process failure: the recon that opened the sub-phase read Katoh–Tanigawa's case split correctly and named the idiom for tracking its harder leg, but the commit that built the easier leg described the rest as coordination rather than mathematics instead of using that idiom, and the blueprint flip that followed took the composer's own carried hypotheses as the complete list of what remained open. A hypothesis with no corresponding node is invisible to any check that reads only `\uses` edges or a lemma's hypothesis list against its own dependencies; it surfaced only once a scheduled audit read the Lean's actual case coverage against Katoh–Tanigawa's own case split.

A.5.2. *The $d = 3$ Claim-6.12 "dead island" misread.* Katoh–Tanigawa's Claim 6.12 at general dimension (KT §6.4.2, Lemma 6.13) subsumes the $d = 3$-specific development described in Section A.1: once the general chain dispatch landed, the $d = 3$-only entry point `case_III_candidate_dispatch` kept no caller in the live proof. A cleanup-round audit flagged this the same week the general lift closed: `case_III_candidate_dispatch` has zero callers, so the chain running through it to `case_III_claim612` "may be describing dead code as if live," and the flag was carried forward, unresolved, as the next round's first task (commit `582ed642`, 2026-07-05).

The cleanup round that inherited the flag opened by reading it as settled fact rather than as the hedge it had been recorded as: its task list stated outright that "the live all-`k` path routes through `chainData_dispatch` for every grade incl. $d = 3$, so the island is bypassed," proposing to retire the whole section. A `lean_references` reachability pass the same morning found this partly right and partly wrong: exactly three declarations were genuinely dead — the `(k:=2)` wrapper `case_III_claim612` itself, a self-obsolete predecessor, and the abstract row-family cluster of Section A.3's first episode — while the rest of the concrete chain remained reachable from within itself (commit `fc5deb3f`, 2026-07-07). The two clearly-dead declarations were cut and `case_III_claim612`'s node was re-pinned to the general `case_III_claim612_gen` the same morning, still under a plan to demote the whole section as secondary (commit `78404289`).

Tracing the actual Lean call chain behind the discriminator node (Lemma 23.200) reversed that plan the same afternoon: `chainData_fire_discriminator` calls `exists_shared_redundancy_and_matched_candidate`, then `exists_chainData_discriminator_pick`,

then `exists_complementIso_ne_zero_of_homogeneousIncidence_gen`, which invokes `case_III_claim612_gen` directly — so the re-pinned node was not dead at all, but the general Claim 6.12 the general chain itself calls. Its zero incoming `\uses` edges, the signal that had made it look orphaned from the start, was a blueprint wiring gap, not a fact about the Lean (commit `fe2f2a73`, 2026-07-07); the missing edges, from the discriminator's proof to Lemma 23.191 and Lemma 22.16, were added the same day (commit `a528e227`). Only the concrete $d = 3$ dispatch itself is genuinely bypassed by the general chain, and it was kept, not cut, as the worked $d = 3$ case Section A.1 grounds.

This is a process failure, in the same audit as the candidate-row episode of Section A.3, but with the opposite tell: two signals that each look decisive on their own — a dep-graph node with no incoming `\uses` edge, a declaration with no grep-visible caller — agreed on a section that was in fact load-bearing, because the caller of a general-purpose declaration is a fact about the Lean call chain, not about either static signal, and a hedge recorded at one audit read as settled fact by the time a later one inherited it.

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